University of Windsor Mathematics Contest Practice Problems
Principle of Inclusion-Exclusion and Pigeonhole Principle
Solutions

1. Suppose there are 200 first-year students, 100 of which are taking Calculus and 70 of which are taking Algebra. If there are 50 first-year students that are taking both Calculus and Algebra, how many first-year students are taking neither course?

   Solution:
   Let c_1 be the property that the student is taking Calculus.
   Let c_2 be the property that the student is taking Algebra.
   Then $N = 200$, $N(c_1) = 100$, $N(c_2) = 70$ and $N(c_1c_2) = 50$. Then,

   $$\overline{N} = N - [N(c_1) + N(c_2)] + N(c_1c_2)$$

   $$\overline{N} = 200 - [100 + 70] + 50 = 80$$

   Therefore there are 80 first-year students that are not taking Calculus and are not taking Algebra.

2. Determine the number of integers between 1 and 10000 that are not divisible by 6, 7, or 8.

   Solution:
   Let c_1 be the property that n is divisible by 6.
   Let c_2 be the property that n is divisible by 7.
   Let c_3 be the property that n is divisible by 8. Then
   $N(c_1) = \lfloor \frac{10000}{6} \rfloor = 1666$,
   $N(c_2) = \lfloor \frac{10000}{7} \rfloor = 1428$,
   $N(c_3) = \lfloor \frac{10000}{8} \rfloor = 1250$,
   $N(c_1c_2) = \lfloor \frac{10000}{(6)(7)} \rfloor = 238$,
   $N(c_1c_3) = \lfloor \frac{10000}{lcm(6,8)} \rfloor = \lfloor \frac{10000}{24} \rfloor = 416$,
   $N(c_2c_3) = \lfloor \frac{10000}{(7)(8)} \rfloor = 178$
   $N(c_1c_2c_3) = \lfloor \frac{10000}{lcm(6,7,8)} \rfloor = \lfloor \frac{10000}{168} \rfloor = 59$
   Therefore

   $$\overline{N} = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

   $$\overline{N} = 10000 - [1666 + 1428 + 1250] + [238 + 416 + 178] - 59 = 6429$$

   Therefore there are 6429 integers between 1 and 10000 that are not divisible by 6, 7, or 8.

3. In how many ways can a poker hand (5 cards) be selected from a regular deck (52 cards) such that the hand contains at least one card in each suit?

   Solution:

Let c_i be the property that the hand does not have any card of suit $i, i = 1, 2, 3, 4$.
$N = 52C5$
$N(c_i) = 39C5$
$N(c_i c_j) = 26C5$
$N(c_i c_j c_k) = 13C5$
$N(c_i c_j c_k c_l) = 0$

$\overline{N} = N - (4C1)N(c_i) + (4C2)N(c_i c_j) - (4C3)N(c_i c_j c_k) + (4C4)N(c_i c_j c_k c_l)$

$$\overline{N} = 52C5 - 4(39C5) + 6(26C5) - 4(13C5) + 0$$

4. How many permutations of the 26 letters do not contain any of the following sequences:
PUTNAM, EXAM, DEC, FIRST ?

Solution:
Let c_1 be the property that $PUTNAM$ occurs.
Let c_2 be the property that $EXAM$ occurs.
Let c_3 be the property that DEC occurs.
Let c_4 be the property that $FIRST$ occurs.
$N = 26!$
$N(c_1) = 21!$
$N(c_2) = 23!$
$N(c_3) = 24!$
$N(c_4) = 22!$
$N(c_1 c_2) = 0$, since both words contain A and M and cannot overlap.
$N(c_1 c_3) = 19!$
$N(c_1 c_4) = 0$, since the T cannot overlap.
$N(c_2 c_3) = 0$, since the E cannot overlap.
$N(c_2 c_4) = 19!$
$N(c_3 c_4) = 20!$
$N(c_1 c_2 c_3) = 0$
$N(c_1 c_2 c_4) = 0$
$N(c_2 c_3 c_4) = 0$
$N(c_1 c_2 c_3 c_4) = 0$

$$\overline{N} = 26! - [21! + 23! + 24! + 22!] + [19! + 19! + 20!]$$

5. 5 friends run a race everyday for four months (excluding February). If no race ends in a tie, show that there are at least 2 races with identical outcomes.

Solution:
There are $5! = 120$ possible outcomes for the races. Assuming that two of the months contain 30 days and the other two months contain 31 days there are a total of 122 days on which a race occurs. Therefore, by Pigeonhole Principle, at least 2 of the outcomes will be the same.

6. Given any 5 points in a unit square, show that two of these points must be within $\frac{\sqrt{2}}{2}$ of each other.

   Solution:
   Divide the unit square into 4 equal quadrants, each sub-square having side length 0.5. Notice the diagonal length of each sub-square is $\frac{\sqrt{2}}{2}$. Place a point in each of the four sub-square, hence forcing the fifth point to be within $\frac{\sqrt{2}}{2}$ of one of the other points.

7. The capacity of an arena is 800 people. How many people must there be to ensure that at least 2 people have the same first and last initial?

   Solution:
   There are $26C1$ ways to choose the first initial and $26C1$ ways to choose the second. Since they are independent, multiply to find that there are 26^2 possible ways to choose a first and last initial. Therefore, to ensure that at least to have the same first and last initial we need at least $26^2 + 1 = 677$ people in the arena.

8. A bag contains 100 apples, 100 oranges, 100 bananas and 100 pears. Every minute you choose one fruit from the bag. How long will it take to ensure that you have at least a dozen fruit of the same kind?

   Solution:
   Suppose we have already chosen 11 apples, 11 oranges, 11 bananas and 11 pears. The next choice will ensure that we have a dozen of one type of fruit. Therefore it takes $4(11) + 1 = 45$ minutes to ensure that we have at least on dozen of one type of fruit.