OPTIMAL CONTROL RELATING TO THE EXCESS OF ONE POISSON PROCESS OVER ANOTHER

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ABSTRACT. This paper deals with optimal control in a double-ended queue that arises as the difference of two independent Poisson processes over a finite time period. The implementation of the control has the effect of changing the arrival rates of one or both processes, and represents a strategy on the part of an observer whose object is to maximize a specified objective function at the end of the time period. For each strategy within a specified set of strategies, we study the transient behaviour of the resulting nonhomogeneous state-dependent double-ended queue and determine the optimal time to implement the control.

1. Introduction.

A great deal of attention in the stochastic processes literature has centered on the development of optimal control policies for the arrival and service patterns of queues (e.g. Crabill et al. [**1**], Jo and Shaler [**5**], Miller [**7**], Scott and Hsia [**9**].) Concerning double-ended queues, work has been done by Dobbie [**2**], Kendall [**6**], Srivastava and Kashyap [**10**], for example.

Our aim in this paper is the derivation of an optimal control policy for the arrival mechanism in double-ended queues with a view to maximizing a specified objective function.

One application of this work is to two-person games. We shall present the theory in the context of an ice hockey game. For other possible applications, see Section 5.

In hockey games, it is very common for a team which is losing near the end of the game, to "pull" its goalie. That is, the team which is losing replaces its goalkeeper by an extra offensive player. The effect of this action is to change the rates of scoring of one or both teams. In fact, the rates of both teams usually increase, with the greater increase going to the team which is leading.

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Implementation of this potentially dangerous strategy implies desperation on the part of the team which implements it. It is of interest, therefore, to determine the optimal times at which a team should have its goalkeeper off the ice. The optimal strategy for a team can be defined as that which maximizes the probability that the game ends in a win or a tie for that team.

The salient features involved in this paper are:

(i) a double-ended queue arises as the difference of the scores of the teams over time;

(ii) the time period is *finite*;

(iii) the distribution of the final score in the game depends on the chosen strategy;

(iv) each strategy considered depends on the score of the game at each time point.

Section 2 analyses the score in the game, for each strategy, as a nonhomogeneous state-dependent Markov process. This results in an infinite system of differential-difference equations. Section 3, in effect, solves these equations for the score in the game at each time point. In Section 4, we express the optimal strategy as a maximum value of a certain integral expression. In addition, a limited simulation is performed. Section 5 suggests generalizations, extensions, variations, and applications.

2. Differential-difference equations for the game.

Suppose that the hockey game between teams White and Black will be played over the finite time interval $[0, t_0]$. Let $X(t)$ and $Y(t)$ denote the number of goals scored up to time t by White and Black, respectively, and let $K(t) = X(t) - Y(t)$. The process $\{K(t)\}$ has index set $[0, t_0]$ and state space the set of integers Z. For each t, the following assumptions hold throughout the paper:

(A1) exactly one of two mutually exclusive situations holds at t, namely,
Situation 1: neither team has its goalkeeper off the ice,
Situation 2: team Black has its goalkeeper off the ice;

(A2) team Black will consider removing its goalkeeper from the ice (that is, passage from situation 1 to situation 2) if and only if $K(t) = 1$;

(A3) if situation 2 holds at t, then for any s satisfying $t \leq s \leq t_0$, situation 2 holds also at s if and only if $K(s) = 1$;

(A4) $X(t)$ and $Y(t)$ are independent Poisson random variables with rates w_1 and b_1, respectively, if situation 1 holds, and rates w_2 and b_2, respectively, if situation 2 holds.

Before proceeding, we comment briefly on each of the above assumptions (see also Section 5). (A1) implies that team White is never allowed to pull its goalkeeper. While there do exist teams that will under no circumstances pull their goalkeepers, a more general theory could be presented by allowing team White the option to pull also. In practise, we have never known of a hockey game in which both teams pulled their goalkeepers. In addition, in many other applications it is either impossible or infeasible to implement a control on each of the two competing queues–for example, one may attempt to increase the arrival rate of taxis, but not of passengers, to a particular location. (A2) stipulates that Black will consider pulling its goalkeeper at time t if and only if it is exactly one goal down at time t. This assumption is quite reasonable, since no team will pull its goalkeeper if it is winning or even, and rarely does a team have its goalkeeper out of the game if it is losing by two or more goals, perhaps because of a "lost cause" attitude. (A3) implies that if Black's goalkeeper is off the ice at time t, he should also be off the ice at any later time point s for which $K(s) = 1$; but, in accord with (A2), if Black is anything other than one goal down at time s, its goalkeeper should be back on the ice at that time.

Support for both the distributional and independence assumptions in (A4) is given by the statistical analysis of National Hockey League data in Mullet [**8**]. The statistical problem of estimating the parameters w_1, b_1, w_2, b_2 will not be dealt with in this paper.

We define an optimal strategy for Black as one that maximizes the probability that the game ends in a tie or a win for Black. Specifically, if

$$F_k(t) = P(K(t) = k), \qquad 0 \le t \le t_0, \quad k \in Z, \tag{2.1}$$

we wish to find a strategy (within the class of all strategies for Black that are permitted by (A2) and (A3)) that maximizes

$$P(\text{game ends in a tie or win for Black}) = \sum_{k=-\infty}^{0} F_k(t_0). \tag{2.2}$$

Any given strategy for Black can be expressed in terms of an indicator function $B : Z \times [0, t_0] \to \{0, 1\}$, with the following interpretation:
if $B(k, t) = 1$, then Black has his goalkeeper off the ice at time t provided

$$K(t) = k;$$

if $B(k, t) = 0$, then Black has his goalkeeper on the ice at time t provided

$$K(t) = k.$$

Thus any given $B(\cdot,\cdot)$ is a rule that, conditional on the score in the game at time t, tells Black if situation 1 or situation 2 should be in effect. Our aim is to choose the optimal $B(\cdot,\cdot)$ in the sense of maximizing (2.2). Now, by (A2), for each t, $B(k,t) = 0$ if $k \neq 1$ so our problem is reduced to finding the optimal value (0 or 1) of $B(1,t)$, for each t. We will write $B(t) = B(1,t)$ for brevity. By (A3), $B(t) = 1 \Rightarrow B(s) = 1$ for all s, $t \leq s \leq t_0$. Thus, if we define

$$t_1 = \begin{cases} \inf\{t : B(t) = 1\} & \text{if } \{t : B(t) = 1\} \text{ is non-empty,} \\ t_0 & \text{otherwise,} \end{cases} \tag{2.3}$$

we have

$$B(t) = \begin{cases} 0 & t < t_1, \\ 1 & t \geq t_1. \end{cases} \tag{2.4}$$

Any given strategy is therefore specified by t_1, and our objective is to choose the optimal t_1.

For any given stategy $B(t)$, $0 \leq t \leq t_0$, it is easy to see from (A4) that on taking the difference-quotient $(F_k(t+\Delta t) - F_k(t))/\Delta t$ and then the limit as $\Delta t \to 0$, we have

$$\begin{aligned} \frac{d}{dt} F_k(t) &= \alpha_k(t)F_k(t) + \beta_{k-1}(t)F_{k-1}(t) + \gamma_{k+1}(t)F_{k+1}(t) \\ &\quad \text{with } F_0(0) = 1, \quad F_k(0) = 0 \quad \text{for } k \neq 0, \end{aligned} \tag{2.5}$$

where

$$\left.\begin{aligned} \alpha_k(t) &= \begin{cases} (-w_2-b_2)B(t)+(-w_1-b_1)(1-B(t)) & \text{if } k = 1, \\ -w_1 - b_1 & \text{if } k \neq 1, \end{cases} \\ \beta_k(t) &= \begin{cases} w_2B(t) + w_1(1 - B(t)) & \text{if } k = 1, \\ w_1 & \text{if } k \neq 1, \end{cases} \\ \gamma_k(t) &= \begin{cases} b_2B(t) + b_1(1 - B(t)) & \text{if } k = 1, \\ b_1 & \text{if } k \neq 1, \end{cases} \end{aligned}\right\} \tag{2.6}$$

and we have

$$-\alpha_k(t) = \beta_k(t) + \gamma_k(t). \tag{2.7}$$

(2.5) is valid provided $t \neq t_1$, where t_1 is given by (2.3). When $t = t_1$, the right-hand and left-hand derivatives do not coincide unless $w_1 = w_2$ and $b_1 = b_2$. This does not affect the analysis, however. We shall simply interpret (2.5) as the left-hand derivative when $t = t_1$.

In the following section, we examine the behaviour of $F_k(t)$. Note that the rates in (2.6) are both time- and state-dependent. However, (2.5) is analytically tractable, as we now demonstrate.

3. Solution of the system (2.5).

Define the generating functions:

$$G_1(z,t) = \sum_{k=-\infty}^{0} F_k(t)z^k \quad \text{and} \tag{3.1}$$

$$G_2(z,t) = \sum_{k=1}^{\infty} F_k(t)z^k. \tag{3.2}$$

From (2.5) and (2.6), we get

$$\begin{aligned}
\frac{\delta}{\delta t} G_1(z,t) &= \sum_{k=-\infty}^{0} \frac{d}{dt} F_k(t)z^k \\
&= \sum_{k=-\infty}^{0} \left[\alpha_k(t)F_k(t) + \beta_{k-1}F_{k-1}(t) + \gamma_{k+1}(t)F_{k+1}(t)\right] z^k \\
&= \sum_{k=-\infty}^{0} \alpha_k(t)F_k(t)z^k + z \sum_{k=-\infty}^{-1} \beta_k(t)F_k(t)z^k \\
&\quad + z^{-1} \sum_{k=-\infty}^{1} \gamma_k(t)F_k(t)z^k \\
&= \left(\alpha_0 + \beta_0 + \gamma_0 z^{-1}\right) \sum_{k=-\infty}^{0} F_k(t)z^k - z\beta_0F_0(t) + \gamma_1(t)F_1(t),
\end{aligned}$$

that is

$$\begin{aligned}
\frac{\delta}{\delta t} G_1(z,t) = &\left[-w_1 - b_1 + w_1 z + b_1 z^{-1}\right] G_1(z,t) - z w_1 F_0(t) \\
&+ \left[(b_2 - b_1)B(t) + b_1\right] F_1(t).
\end{aligned} \tag{3.3}$$

Similarly we find that

$$\frac{\delta}{\delta t} G_2(z,t) = \left[-w_1 - b_1 + w_1 z + b_1 z^{-1}\right] G_2(z,t) + \left[(-w_2 - b_2 + w_1 + b_1) z B(t) + (w_2 - w_1) z^2 B(t) - b_1\right] \times F_1(t) + w_1 z F_0(t). \tag{3.4}$$

If we write $\mathcal{L}[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$ for the Laplace transform (LT) of a function $g(t)$, we see from (3.3) and (3.4), along with $G_1(z,0) = 1$, $G_2(z,0) = 0$, that

$$s\mathcal{L}[G_1(z,t)] - 1 = [-w_1 - b_1 + w_1 z + b_1 z^{-1}]\mathcal{L}[G_1(z,t)] - z w_1 \mathcal{L}[F_0(t)] + (b_2 - b_1)\mathcal{L}[B(t)F_1(t)] + b_1 \mathcal{L}[F_1(t)] \tag{3.5}$$

and

$$s\mathcal{L}[G_2(z,t)] = [-w_1 - b_1 + w_1 z + b_1 z^{-1}]\mathcal{L}[G_2(z,t)] + [(-w_2 - b_2 + w_1 + b_1)z + (w_2 - w_1)z^2] \times \mathcal{L}[B(t)F_1(t)] + w_1 z \mathcal{L}[F_0(t)] - b_1 \mathcal{L}[F_1(t)]. \tag{3.6}$$

We rewrite (3.5) and (3.6) as

$$\mathcal{L}[G_1(z,t)] = \frac{z\{1 - z w_1 \mathcal{L}[F_0(t)] + (b_2 - b_1)\mathcal{L}[B(t)F_1(t)] + b_1 \mathcal{L}[F_1(t)]\}}{z\{s + w_1 + b_1 - w_1 z - b_1 z^{-1}\}} \tag{3.7}$$

and

$$\mathcal{L}[G_2(z,t)] = \left[z\{z w_1 \mathcal{L}[F_0(t)] + [(-w_2 - b_2 + w_1 + b_1)z + (w_2 - w_1)z^2] \times \mathcal{L}[B(t)F_1(t)] - b_1 \mathcal{L}[F_1(t)]\}\right] \div \left[z\left\{s + w_1 + b_1 - w_1 z - b_1 z^{-1}\right\}\right]. \tag{3.8}$$

The right-hand sides of (3.7) and (3.8) involve the unknown functions $\mathcal{L}[F_0(t)]$, $\mathcal{L}[B(t)F_1(t)]$ and $\mathcal{L}[F_1(t)]$ of s. In order to solve for these functions, we employ the following consideration, which is a standard analytical argument in stochastic processes (see, e.g. Gross and Harris [**4**, page 77]). The numerators of the right-hand members of (3.7) and (3.8) must vanish at any zero of the denominator which lies within the region of convergance of $\mathcal{L}[G_1(z,t)]$ and $\mathcal{L}[G_2(z,t)]$, respectively.

The denominator of the right-hand members of (3.7) and (3.8) has two roots which are, as functions of s,

$$z_1 = \frac{w_1 + b_1 + s - \sqrt{(w_1 + b_1 + s)^2 - 4w_1 b_1}}{2w_1} \tag{3.9}$$

and

$$z_2 = \frac{w_1 + b_1 + s + \sqrt{(w_1 + b_1 + s)^2 - 4w_1 b_1}}{2w_1}. \tag{3.10}$$

Note that

$$z_1 z_2 = \frac{b_1}{w_1}. \tag{3.11}$$

Use of Rouche's Theorem (see Gross and Harris [**4**, page 78]) shows that z_1 lies within the unit circle and z_2 lies outside the unit circle. Now $\mathcal{L}[G_1(z,t)]$ converges outside the unit circle and $\mathcal{L}[G_2(z,t)]$ converges within the unit circle. We can therefore equate the numerators of the right-hand members of (3.7) and (3.8) to zero for $z = z_2$ and $z = z_1$, respectively. We then obtain

$$\begin{aligned} &1 - w_1 z_2 \mathcal{L}[F_0(t)] + (b_2 - b_1)\mathcal{L}[B(t)F_1(t)] + b_1 \mathcal{L}[F_1(t)] = 0, \\ &w_1 z_1 \mathcal{L}[F_0(t)] + [(-w_2 - b_2 + w_1 + b_1)z_1 + (w_2 - w_1)z_1^2] \\ &\qquad \times \mathcal{L}[B(t)F_1(t)] - \mathcal{L}[F_1(t)] = 0. \end{aligned} \tag{3.12}$$

Define

$$f_{t_1}(s) = \int_0^{t_1} e^{-st} F_1(t)\, dt \tag{3.13}$$

where t_1 is given by (2.3). Then we see from (2.4) and (3.13) that

$$\mathcal{L}[F_1(t)] = \mathcal{L}[B(t)F_1(t)] + f_{t_1}(s). \tag{3.14}$$

Using (3.14), we can rewrite the equations (3.12) as

$$\begin{aligned} &1 - w_1 z_2 \mathcal{L}[F_0(t)] + b_2 \mathcal{L}[B(t)F_1(t)] + b_1 f_{t_1}(s) = 0, \\ &w_1 z_1 \mathcal{L}[F_0(t)] + [(-w_2 - b_2 + w_1 + b_1)z_1 + (w_2 - w_1)z_1^2 - b_1] \\ &\qquad \times \mathcal{L}[B(t)F_1(t)] - b_1 f_{t_1}(s) = 0. \end{aligned} \tag{3.15}$$

LEMMA 3.1. *If $t < t_1$, then $F_k(t) = e^{-(w_1+b_1)t}(w_1/b_1)^{k/2} I_k(2\sqrt{w_1 b_1}\, t)$ for all $k \in Z$, where $I_k(u) = \sum_{j=0}^{\infty} \frac{(u/2)^{2j+k}}{j!(j+k)!}$, $k > -1$, is the modified Bessel function of the first kind of order k.*

PROOF: The proof is elementary since, for any $t < t_1$, $K(t)$ is the difference of two independent Poisson variables with rates w_1 and b_1. □

The result of this lemma is a special case of the work of Dobbie [**2**]. Note, however, that Dobbie's results, which are for time dependent double-ended queues cannot be used in this paper to evaluate $F_k(t)$ when $t > t_1$, because our problem is, in addition, state-dependent.

COROLLARY 3.1.

$$\mathcal{L}^{-1}[f_{t_1}(s)] = \begin{cases} e^{-(w_1+b_1)t}\sqrt{\frac{w_1}{b_1}} I_1\left(2\sqrt{w_1 b_1}t\right) & \text{if } t \le t_1, \\ 0 & \text{if } t > t_1. \end{cases}$$

PROOF: By (3.13) and Lemma 3.1, we have

$$\begin{aligned} f_{t_1} &= \left(\frac{w_1}{b_1}\right)^{1/2} \int_0^{t_1} e^{-st} e^{-(w_1+b_1)t} I_1\left(2\sqrt{w_1 b_1}\, t\right) dt \\ &= \left(\frac{w_1}{b_1}\right)^{1/2} \int_0^{\infty} e^{-st}(1-B(t)) e^{-(w_1+b_1)t} I_1\left(2\sqrt{w_1 b_1}\, t\right) dt, \text{ by (2.4)} \\ &= \left(\frac{w_1}{b_1}\right)^{1/2} \mathcal{L}\left[(1-B(t)) e^{-(w_1+b_1)t} I_1\left(2\sqrt{w_1 b_1}\, t\right)\right], \end{aligned}$$

and the corollary follows upon inversion. □

For brevity in the remainder of the paper, we define constants

$$\begin{aligned} c_1 &= \frac{1}{2w_1}\left[w_1 - w_2 + b_1 - b_2 + \sqrt{(w_2 - w_1 + b_1 - b_2)^2 + 4w_2 b_2}\right] \\ c_2 &= \frac{1}{2w_1}\left[w_1 - w_2 + b_1 - b_2 - \sqrt{(w_2 - w_1 + b_1 - b_2)^2 + 4w_2 b_2}\right]. \end{aligned} \tag{3.16}$$

THEOREM 3.1. *For all t, we have*

$$\begin{aligned} F_1(t) &= e^{-(w_1+b_1)t}\sqrt{\frac{w_1}{b_1}} I_1\left[2\sqrt{w_1 b_1}\, t\right] - e^{-(w_1+b_1)t}(c_1 - c_2)^{-1} \\ &\times \sum_{k=0}^{\infty}\left[\frac{b_1}{w_1}(c_1^k - c_2^k) - c_1^{k+2} + c_2^{k+2}\right](k+1)\left(\frac{w_1}{b_1}\right)^{\frac{k}{2}+1} \\ &\times \int_0^t B(\xi)(t-\xi)^{-1} I_1\left[2\sqrt{w_1 b_1}\, \xi\right] I_{k+1}\left[2\sqrt{w_1 b_1}\,(t-\xi)\right] d\xi. \end{aligned}$$

PROOF: If we multiply the first equation of (3.15) by z_1, multiply the second equation of (3.15) by z_2, add the resulting equations, and then solve for $\mathcal{L}[B(t)F_1(t)]$ using (3.11), we get

$$\begin{aligned} \mathcal{L}[B(t)F_1(t)] &= \left(-z_2^2 + \frac{b_1}{w_1}\right) f_{t_1}(s) + \frac{1}{w_1} \\ &\times \left\{z_2^2 + z_2\left(\frac{w_2}{w_1} + \frac{b_2}{w_1} - \frac{b_1}{w_1} - \frac{w_1}{w_1}\right) + \left(\frac{b_1}{w_1} - \frac{b_2}{w_1} - \frac{b_1 w_2}{w_1^2}\right)\right\}^{-1}. \end{aligned} \tag{3.17}$$

The denominator on the right-hand side of (3.17) is a quadratic in z_2 with roots given by (3.16). Noting that

$$\frac{1}{[(z_2 - c_1)(z_2 - c_2)]} = \frac{1}{c_1 - c_2}\left[\frac{1}{z_2 - c_1} - \frac{1}{z_2 - c_2}\right],$$

we can rewrite (3.17) as

$$\begin{aligned}\mathcal{L}[B(t)F_1(t)] = &\left[\left(-z_2^2 + \frac{b_1}{w_1}\right) f_{t_1}(s) + \frac{1}{w_1}\right](c_1 - c_2)^{-1} \\ &\times \left[\frac{1}{z_2 - c_1} - \frac{1}{z_2 - c_2}\right].\end{aligned} \tag{3.18}$$

For s sufficiently large, we can write

$$\frac{1}{z_2 - c_1} = \frac{1}{z_2}\sum_{k=0}^{\infty}\left(\frac{c_1}{z_2}\right)^k \quad \text{and} \quad \frac{1}{z_2 - c_2} = \frac{1}{z_2}\sum_{k=0}^{\infty}\left(\frac{c_2}{z_2}\right)^k.$$

Hence (3.18) can be written as

$$\begin{aligned}\mathcal{L}[B(t)F_1(t)] &= (c_1 - c_2)^{-1}\sum_{k=1}^{\infty}(c_1^k - c_2^k) \\ &\quad \times \left[\frac{1}{w_1 z_2^{k+1}} + \frac{b_1}{w_1 z_2^{k+1}} f_{t_1}(s) - \frac{1}{z_2^{k-1}} f_{t_1}(s)\right] \\ &= (c_1 - c_2)^{-1}\sum_{k=1}^{\infty}(c_1^k - c_2^k)\frac{1}{w_1 z_2^{k+1}} \\ &\quad + (c_1 - c_2)^{-1}\sum_{k=0}^{\infty}\left[\frac{b_1}{w_1}(c_1^k - c_2^k) - (c_1^{k+2} - c_2^{k+2})\right] \\ &\quad \times \frac{1}{z_2^{k+1}} f_{t_1}(s) - f_{t_1}(s).\end{aligned} \tag{3.19}$$

Using (3.14), standard results on Bessel functions, and Corollary (3.1), we get from (3.19) and a little manipulation,

$$
\begin{aligned}
F_1(t) &= (c_1 - c_2)^{-1} \frac{1}{w_1} e^{-(w_1+b_1)t} \sum_{k=1}^{\infty} (c_1^k - c_2^k) \sqrt{\left(\frac{w_1}{b_1}\right)^{k+1}} \sqrt{w_1 b_1} \\
&\times \left(I_k \left[2\sqrt{w_1 b_1}\, t \right] - I_{k+2} \left[2\sqrt{w_1 b_1}\, t \right] \right) + (c_1 - c_2)^{-1} e^{-(w_1+b_1)t} \\
&\times \sum_{k=0}^{\infty} \left[\frac{b_1}{w_1} (c_1^k - c_2^k) - (c_1^{k+2} - c_2^{k+2}) \right] \left(\frac{w_1}{b_1} \right)^{\frac{k}{2}+1} \\
&\times I_{k+2} \left[2\sqrt{w_1 b_1}\, t \right] - (c_1 - c_2)^{-1} e^{-(w_1+b_1)t} \\
&\times \sum_{k=0}^{\infty} \left[\frac{b_1}{w_1} (c_1^k - c_2^k) - (c_1^{k+2} - c_2^{k+2}) \right] \left(\frac{w_1}{b_1} \right)^{\frac{k}{2}+1} \\
&\times \int_0^t (t-\xi)^{-1} B(\xi) I_1 \left[2\sqrt{w_1 b_1}\, \xi \right] I_{k+1} \left[2\sqrt{w_1 b_1}\, (t-\xi) \right] d\xi,
\end{aligned}
$$

and it is easy to see that this is identical to the expression for $F_1(t)$ given in the statement of the theorem. □

THEOREM 3.2. *For all t, we have*

$$
\begin{aligned}
F_0(t) &= e^{-(w_1+b_1)t} I_0 \left[2\sqrt{w_1 b_1}\, t \right] - e^{-(w_1+b_1)t} (c_1 - c_2)^{-1} \frac{b_2}{w_1} \\
&\times \sum_{k=0}^{\infty} \left[\frac{b_1}{w_1} (c_1^k - c_2^k) - (c_1^{k+2} - c_2^{k+2}) \right] (k+2) \sqrt{\left(\frac{w_1}{b_1} \right)^{k+3}} \\
&\times \int_0^t B(\xi)(t-\xi)^{-1} I_1 \left[2\sqrt{w_1 b_1}\, \xi \right] I_{k+2} \left[2\sqrt{w_1 b_1}\, (t-\xi) \right] d\xi \\
&+ e^{-(w_1+b_1)t} \left(\frac{b_2 - b_1}{b_1} \right) \\
&\times \int_0^t B(\xi)(t-\xi)^{-1} I_1 \left[2\sqrt{w_1 b_1}\, \xi \right] I_1 \left[2\sqrt{w_1 b_1}\, (t-\xi) \right] d\xi.
\end{aligned}
$$

PROOF: The proof is omitted since it is similar to that shown for Theorem 3.1. □

By Theorems 3.1 and 3.2, we now have explicit expressions for $F_1(t)$ and $F_0(t)$. The evaluation of $F_k(t)$, for all k, is now straightforward. For example, when $k \leq 0$, $F_k(t)$ is just the inverse Laplace transform of the coefficient of z^k in (3.7). We shall not, however, carry out the routine, but extremely tedious, calculations involved. Indeed we shall see in the following section that knowledge of only $F_1(t)$ and $F_0(t)$ suffices to tackle the optimization problem in which we are interested.

4. Optimal control and simulation.

We recall from Section 2 that we wish to choose a strategy for Black that maximizes

$$\begin{aligned} P(\text{game ends in a tie or win for Black}) &= \sum_{k=-\infty}^{0} F_k(t_0) \\ &= G_1(z, t_0)\big|_{z=1} \end{aligned} \tag{4.1}$$

and we saw that this is equivalent to the optimal choice of t_1, where t_1 is given by (2.3).

By (3.7) and (3.14),

$$\mathcal{L}[G_1(1,t_0)] = s^{-1}\{1 - w_1\mathcal{L}[F_0(t)] + b_2\mathcal{L}[F_1(t)] - (b_2 - b_1)f_{t_1}(s)\}. \tag{4.2}$$

If we invert (4.2) with the aid of Corollary (3.1), we thus get

$$\begin{aligned} &P(\text{game ends in a tie or a win for Black}) = \\ &\quad 1 - w_1\int_0^{t_0} F_0(t)\,dt + b_2\int_0^{t_0} F_1(t)\,dt \\ &\quad - (b_2 - b_1)\sqrt{\left(\frac{w_1}{b_1}\right)}\int_0^{t_1} e^{-(w_1+b_1)t} I_1[2\sqrt{w_1 b_1}t]\,dt, \end{aligned} \tag{4.3}$$

where $F_0(t)$ and $F_1(t)$ are functions of t_1 and are given by Theorems 3.2 and 3.1 respectively.

Now (4.3) is clearly a continuous function of t_1 over the compact set $[0, t_0]$ and this function thus takes on a maximum at some point t_1^*. In view of the fact that the first and second derivatives of (4.3) with respect to t_1 over the interval $(0, t_0)$ are rather messy expressions, we shall not here discuss constructive (iterative) procedures for obtaining t_1^*. The practical interpretation of the number t_1^* is as follows (see also (2.3)): if Black is losing by one goal at any fixed time point t, then Black's goalkeeper should be off the ice (i.e. situation 2 should hold) at time t if and only if $t \geq t_1^*$. Of course, in accordance with (A2), situation 1 should hold at time t if Black is not losing by exactly one goal at that time.

Clearly from (4.3), t_1^* depends on the rates w_1, b_1, w_2 and b_2, and on the duration t_0 of the game. In order to assess how t_1^* varies with these rates, a limited simulation by computer was performed. Corresponding to various rates (expressed as expected number of goals per minute), pseudo exponential random values were generated and these were used to simulate a large number of games played in accordance with assumptions (A1) through (A4). The duration of the game was $t_0 = 60$ minutes in all simulations.

The results are presented in Table I below. The values of $t_0 - t_1^*$ are measured to within 1/4 minute accuracy and hence are somewhat crude. Nevertheless the results do form patterns that conform to our intuition.

TABLE I

$w_1 + b_1$	$b_1/(w_1 + b_1)$	$w_2 + b_2$	$b_2/(w_2 + b_2)$	$t_0 - t_1^*$
.166	.4	1	.1	.5
.166	.4	1	.2	1.75
.166	.4	1	.3	2.75
.166	.4	1	.4	> 4
.166	.5	1	.1	.25
.166	.5	1	.2	1.25
.166	.5	1	.3	2.00
.166	.5	1	.4	> 4
.166	.6	1	.1	0
.166	.6	1	.2	1.25
.166	.6	1	.3	2.00
.166	.6	1	.4	2.75
.166	.4	.8	.1	.5
.166	.4	.8	.2	1.5
.166	.4	.8	.3	2.75
.166	.4	.8	.4	> 4
.166	.5	.8	.1	0
.166	.5	.8	.2	1.25
.166	.5	.8	.3	2.00
.166	.5	.8	.4	3.25
.166	.6	.8	.1	0
.166	.6	.8	.2	1.0
.166	.6	.8	.3	1.75
.166	.6	.8	.4	2.50

5. Concluding remarks.

The theory developed in this paper has several variations, extensions and generalizations, some of which will be dealt with in later work. At the cost of additional complications in the analysis, one could, for example, develop a theory that would allow a control to be applied to *both* queues. In other practical situations, it is desirable to maximize some objective function other than the function maximized in this paper. For example, if our two competing queues were traffic arrivals from right angle directions at a traffic light, we may wish to maximize the probability that both queues have the same size at a certain time of the day. On the other hand, the administration in a blood bank may wish to maximize the probability that there will be at least as many units of blood donated as are required. Here a control on the rate of blood donation may correspond to a publicity campaign, and the objective function should incorporate a cost factor. We note finally that assumption (A2) in this paper, while suitable for hockey games, is highly restrictive in many applications of interest, for example in production control. Indeed in many practical situations, it is desirable to ensure that with high probability one queue does not get too large relative to the other. In these situations, an operator will consider implementing the control if and only if $K(t) \geq k$, for some preassigned nonnegative integer k.

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