Control of Queues by Laplace Transform

by

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Abstract

In this paper we apply Laplace transform approach to queueing model applied to admission control system. We first present some relevant properties and important theorems of Laplace transforms and Catastrophe Process. We determine the optimal time that the gatekeeper should wait after a rejected customer, before admitting another customer and related expected time measurements by Laplace transform approach. This optimal time depends on penalty costs and on expected interadmit times, interaccept times and other time measurements. We verify our obtained expected times with Tang (2005) who found the expected times by setting up complex linear systems of equations. We also obtain the complete distribution of the relevant times. This Laplace transforms approach to this problem is completely new and adds additional information to that obtained by Tang (2005). Moreover Tang's comments on the $M/E_k/1$ system do not completely generalize her $M/E_2/1$ result, whereas we give a more lengthy discussion of this case. Finally our results confirm Tang's result and Tang's result act as a partial verification that our results are correct.

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Contents

Chapter 1

Introduction

The following description is taken from Tang (2005).

"According to Lin and Ross (2003), admission control has been widely used to improve the performance of a system which has problems occurring due to overload. Consider a web site consisting of web servers and HTTP GET requests sent by the users. When the arrival rate of new requests exceeds the servers' capacity, queues build up and the consequence is long response times when users are visiting the web site. There is a tendency for users to leave and never come back to sites that perform poorly. This could lead to profit loss if the site is commercial. Such performance problems can be solved by implementing admission control mechanisms in the sites. The main idea is to admit only a certain number of requests coming into the server and to block others whenever the arrival rate is too high.

"To solve an admission problem, the fundamental thing to consider is the formulation of the objective function that associates a reward for serving a job, a cost when blocking customers in queue and a cost for rejecting a request. Optimal policies are considered in many different models. Under the assumption that the number of customers in the system is unknown, Altman and Koole (1995) considered a service control problem with noisy delayed information. Lin and Ross (2003) considered a multiple-server loss model. They showed that, in the case of a single server with exponential service, a threshold-type policy is optimal and they proposed two types of heuristic policies when there are multiple servers. Cao and Nyberg (2004) considered the problem of admission control to an M/M/1 queue under periodic observations with average cost criterion."

1.1 Objective

The following is also taken from Tang (2005).

"In this report, we consider a model similar to that of Lin and Ross (2003) with only one server. However we allow Erlang service times rather than only allowing exponential service times. Customers arrive to a gatekeeper according to a Poisson process with rate λ. When a new customer arrives, the gatekeeper has to decide whether to admit or to block that customer. The gatekeeper will be informed if an admitted customer finds the server busy, and that customer leaves the system immediately without being served. If an admitted customer finds the server empty, that customer is accepted and begins to receive service. However, the gatekeeper has no information on the departure time of the last accepted customer. We assume that a smaller penalty cost c will be incurred if an arrival is blocked by the gatekeeper from entering the system and a larger penalty cost K will be incurred if an admitted customer is rejected by the server. We also have a benefit occurring as a result of an accepted customer that is admitted by the server. By means of translation, we set the benefit to zero. Further, without loss of generality, we let $K = 1$ and assume $0 < c < 1$."

The goal of Tang (2005)'s paper was to determine the optimal time that the gatekeeper should wait after a rejected customer, before admitting another customer. This optimal time depended on penalty costs and on expected interadmit times, interaccept times and other time measurements. These expected times were found by Tang (2005) by setting up somewhat complex linear systems of equations involving the expected times and solving these equations. Rather than using Tang (2005)'s approach, we will obtain the complete distribution of the relevant times in terms of Laplace transforms which will give Tang (2005)'s expected time results (by evaluating the derivative of the Laplace transforms at 0) and give much more information as well. This Laplace transform approach to the problem is completely new and adds additional information to that obtained by Tang (2005). In addition, Tang (2005)'s comments on the $M/E_k/1$ system do not completely generalize her $M/E_2/1$ results, whereas we give a more lengthy discussion of this case. Finally our results confirm Tang (2005)'s results and Tang (2005)'s results act as a partial verification that our results are correct.

1.2 Organization of the Report

In chapter 2, we present some general results about the probabilistic interpretation of Laplace transforms. In chapter 3, we consider the problem of admission control to an M/M/1 queue. We use a Laplace transform approach to show a relationship between the blocking cost and the optimal blocking time. In chapter 4, we consider the case of an M/E_2/1 queue, where we assume a two-stage Erlang service time. A longer blocking time is used if an admitted customer is blocked at the first stage and a shorter blocking time is applied if that customer is blocked at the second stage. We analytically derive the cost function in terms of the two blocking times. In chapter 5, we present an extension to an M/E_k/1 queue.

Chapter 2

Laplace Transforms and Catastrophe Processes

The following description is taken from Roy (1997).

"The Laplace transform is an often used integral transform that is employed in many diverse fields of mathematics. It is particularly well known for its use in solving linear differential equations with constant coefficients. The study of stochastic processes also utilizes Laplace transforms in areas such as risk theory, renewal theory and queueing theory. In fact, many well-known results for M/G/l queues are stated in terms of Laplace transforms. We will restrict our study of Laplace transforms to queueing applications."

2.1 Laplace Transforms in Queueing Applications

We are interested in transforms of probability density functions (p.d.f.'s) of waiting times in queues. In this case, there is a probabilistic interpretation of the Laplace transform. The Laplace transform of a p.d.f. is the probability that the corresponding random variable is smaller than an exponential random variable with a particular rate where the random variables are independent. This interpretation can be employed to compute transforms of certain p.d.f.'s and prove relationships between quantities of interest in queueing theory without the standard computational and integration techniques.

The probabilistic interpretation of the Laplace transform was first introduced in 1949 by van Dantzig whose original purpose was to give an interpretation of the z-transform (probability generating function). van Dantzig (1949)'s interpretation (which he called "the theory of collective marks") and its associated techniques were described by Runnenburg (1965). In these papers, applications to queueing theory were emphasized. Rade (1972) also utilized these interpretations to solve problems in applied probability from a practical point of view, that would be understandable by both the technician and the theoretician.

The following description is taken from Roy (1997).

"Cong published several articles (Cong 1994*a*, Cong 1994*b*, Cong 1995) on queueing theory and collective marks. In these papers, Cong derives results for queueing systems with complicated restrictions. Cong's results are more general and have shorter, more efficient proofs than previous results regarding the same queueing models.

"It is worth noting that van Dantzig (1949), Runnenburg (1965), Rade (1972) and Cong are all associated with the University of Amsterdam. While the probabilistic interpretation of Laplace transforms is known outside of the Netherlands, it does not seem to be well known and is definitely under-utilized as a tool in the analysis of queues. For instance, Lipsky (1992) mentions the interpretation of Laplace transforms as does Haight (1981), but they do not use these insights to prove any results. Kleinrock (1975) also notes the interpretation and derives some renewal theory results, but fails to utilize it in situations where the proofs could be made more efficient and intuitive. Most standard queueing texts ignore this subject completely."

For the reasons above, the focus of this major paper is to bring attention to the probabilistic interpretation of Laplace transforms and build upon this interpretation to provide a framework for the analysis of queues. See Hlynka (2003) and Roy (1997) for detailed description.

Definition 2.1.1. *The Laplace transform of a function $f(x)$ with positive support is denoted by $f^*(s)$ and is given by*

$$f^*(s) = \int_0^\infty e^{-sx} f(x)dx,$$

where $s > 0$.

The following properties of the Laplace transform are used in this report.

Proposition 2.1.1. *Let m_i denote the i^{th} moment of X where the p.d.f. of X is $f(x)$, $x > 0$. Then*

$$m_i = (-1)^i \frac{d^i}{ds^i} f^*(s)|_{s=0}.$$

This result is related to the fact that the Laplace transform of a pdf $f(x)$ is its moment generating function, $M(t)(= \int_0^\infty e^{xt} f(x)dx)$ evaluated at $t = -s$.

Proposition 2.1.2. *Let X be a random variable with positive support and with first two moments $E(X)$ and $E(X^2)$. Then*

$$E(X) = -L'(s)|_{s=0},$$

$$E(X^2) = L''(s)|_{s=0}.$$

Proposition 2.1.3. *The exponential distribution is "memoryless," that is, if X is an exponential random variable, then $P(X > t + s | X > s) = P(X > t)$.*

Proposition 2.1.4. *If $X_1 \sim exp(\lambda_1), X_2 \sim exp(\lambda_2)$ are independent, then $P(X_1 < X_2) = P(X_1 \text{occurs first}) = \frac{\lambda_1}{\lambda_1+\lambda_2}$.*

Proof.

$$\begin{aligned} P(X_1 < X_2) &= \int_0^\infty \left(\int_0^{x_2} f(x_1)dx_1 \right) f(x_2)dx_2 \\ &= \int_0^\infty F_{X_1}(x_2) f(x_2) dx_2 \\ &= \int_0^\infty \left(1 - e^{-\lambda_1 x_2}\right) \lambda_2 e^{-\lambda_2 x_2} dx_2 \\ &= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2} dx_2 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_0^\infty (\lambda_1 + \lambda_2)\, e^{(-\lambda_1 + \lambda_2) x_2} dx_2 \\ &= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2}. \end{aligned}$$

Proposition 2.1.5. *Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with rates $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$P(X_j \text{ occurs first }) = \frac{\lambda_j}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}, \text{ for all } j = 1, \ldots, n.$$

Proof. This is a straight forward generalization of the Proposition 2.1.4.

2.2 Catastrophe Process

Theorem 1. *Let X and Y be independent random variables with positive support. Further, suppose that $Y \sim exp(s)$ and the p.d.f. of X is $f(x)$. Then,*

$$f^*(s) = P(X < Y).$$

Proof.

$$\begin{aligned} P(X < Y) &= \int_0^\infty \int_x^\infty f(x) s e^{-sy} dy dx \\ &= \int_0^\infty f(x) e^{-sy}|_{y=\infty}^{y=x} dx \\ &= \int_0^\infty f(x) e^{-sx} dx \\ &= f^*(s). \end{aligned}$$

The exponential random variable Y is called the catastrophe. The Laplace transform of a random variable X is the probability that X occurs before the catastrophe. More precisely, the Laplace transform of a probability density function $f(x)$ of a random variable X can be interpreted as the probability that X precedes a catastrophe where the time to the catastrophe is an exponentially distributed random variable Y with rate s, independent of X.

Another way of describing the process is in terms of a race. The Laplace transform of a random variable X is the probability that X wins a race against an exponential opponent Y.

Proposition 2.2.1. *Let X be an exponential random variable with rate λ. Then,*

$$L_X(s) = \frac{\lambda}{\lambda + s}.$$

Proof. Let Y be exponential with rate s and independent of X. Then, by Proposition 2.1.4,

$$L_X(s) = P(X < Y) = \frac{\lambda}{\lambda + s}.$$

In this section, we interpret the Laplace transform of probability density functions as the probability that the corresponding random variable "wins a race" against an exponentially distributed catastrophe. We also use this interpretation to give intuitive explanations of some of the properties of the Laplace transform.

Our interest is in a process which generates events where the time until the next event has pdf $f(x)$. To calculate the Laplace transform of $f(x)$, consider a process that generates "catastrophes" (a catastrophe is simply another type of event). If the time between catastrophes is distributed as an independent exponential random variable with rate s then Theorem 1 states that the Laplace transform of the distribution of the time until the next event, $f^*(s)$, is simply the long-term proportion of time that the event occurs before the catastrophe, for a large number of situations with the same initial position.

Theorem 2. *Let $X_1, X_2, \ldots, X_n$ be a sequence of n independent random variables where each X_i has p.d.f, $f_i(x_i)$. If $X = \sum_{i=1}^{n} X_i$ and the pdf of X is $f(x)$, then*

$$f^*(s) = \prod_{i=1}^{n} f_i^*(s).$$

Since the catastrophe process is memoryless, if we are given that k events have occurred before the catastrophe, we simply reset the "race" to be between the length of time for the remaining $n - k$ events to occur and the catastrophe.

Our probabilistic interpretation also allows us to numerically compute a Laplace transform using simulation (or real data), provided that we are able to simulate (or obtain) random values from the density function in question. To compute $f^*(s)$ for particular values for s, we can simulate a series of exponential values $\{y_1, y_2, \ldots, y_n\}$, at rate s and a series of values from the density in question, $f(x), \{x_1, x_2, \ldots, x_n\}$ and count the proportion of pairs (x_i, y_i) such that $x_i < y_i$. As $n \to \infty$, this is exactly the value of the Laplace transform at point s, namely $f^*(s)$.

Chapter 3

M/M/1 Queue Control Model

In this chapter we consider the problem of admission control to an M/M/1 queue with unknown completion time.

3.1 $M_\lambda/M_\mu/1$ Queue

As stated in the objective of this study, we want to determine the optimal time that the gatekeeper should wait after a rejected customer, before admitting another customer. This optimal time is a function of the penalty costs and depends on expected interadmit times, interaccept times and other time measurements. Tang (2005) derived these optimal times. We will obtain the complete distribution of the relevant times in terms of Laplace Transforms which will produce Tang (2005)'s results but also give much more information as well. See Figure 3.1.

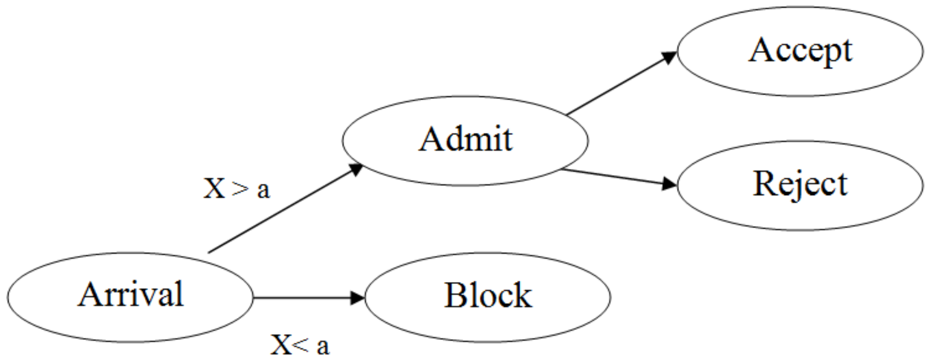

Figure 3.1: arrival rate = accept rate + reject rate + block rate.

This description is taken from Tang (2005).

"Lin and Ross (2003) show that a threshold-type policy that blocks for a certain amount of time after an admitted arrival is optimal. So in our problem, the gatekeeper blocks all the arrivals for a time units whenever an admission occurs. Because of the memoryless property of the exponential service time, no matter when an admitted customer is accepted or is rejected, the gatekeeper blocks new arrivals for the next a time units. Upon acceptance of a customer, we want to find the expected time to the next acceptance.

"Suppose that there is a customer which has just been accepted. Then after a time units, a new arrival has to face one of two possible situations. One possibility is that the last accepted customer has completed service and thus the new arrival could enter service immediately. The other situation is that the server is still busy so the new arrival is rejected, has to leave the system and the gatekeeper blocks all arrivals for another a time units. Then the next customer arriving a time units later will encounter the same problem."

3.2 Expected Value Using Laplace Transform

In this section we will find the expected interaccept time and interadmit time which will determine the optimal time that the gatekeeper should wait after rejecting a customer and before admitting another customer.

Let T_1 be the interaccept time and T_2 be the interadmit time. Let Y be the exponential catastrophe variable with rate s. Thus $f_Y(y) = se^{-sy},\ y > 0$. Then we can define

$$L_{T_1}(s) = P(T_1 < Y) \equiv \text{ Laplace transform of the the interaccept time.}$$
$$L_{T_2}(s) = P(T_2 < Y) \equiv \text{ Laplace transform of the interadmit time.}$$

The following equation is explained below.

$$\begin{aligned} L_{T_1}(s) &= P\left(T_1 < Y\right) \\ &= e^{-sa}\left(1 - e^{-\mu a}\right)\left(\frac{\lambda}{\lambda + s}\right) \\ &+ e^{-sa}\left(e^{-\mu a}\right)\left(\frac{\mu}{\lambda + s + \mu}\right)\left(\frac{\lambda}{\lambda + s}\right) \\ &+ e^{-sa}\left(e^{-\mu a}\right)\left(\frac{\lambda}{\lambda + s + \mu}\right) L_{T_1}(s). \end{aligned}$$

Consider the first of the three summands. There is no catastrophe during the initial wait a, which has probability $P(Y > a) = \int_a^\infty se^{-sy}dy = e^{-sa}$. During the wait a, the probability of service completion is $\int_0^a \mu e^{-\mu x}dx = 1 - e^{-\mu a}$. Upon completion of the service, the customer will no longer be in the system. Therefore, the next event is going to be either an arrival or a catastrophe. The probability of an arrival before a catastrophe is $\lambda/(\lambda + s)$.

Consider the second summand. During the wait of length a, the probability of no service completion is $\left(1-\int_0^a \mu e^{-\mu x}dx\right) = e^{-\mu a}$. If there is no service completion then the next step might be a service completion for the customer already in the system. The probability of a service completion before a catastrophe and an arrival is $\left(\frac{\mu}{\lambda+s+\mu}\right)$. After a service is completed, the system will be empty. Therefore, the next event will be either an arrival or a catastrophe. The probability of an arrival is $\left(\frac{\lambda}{\lambda+s}\right)$.

In the third summand, the term $\left(\frac{\lambda}{\lambda+s+\mu}\right)$ represents the probability of an arrival before a catastrophe and service completion. The other two terms are defined earlier. If there is an arrival then it will be admitted and then rejected. Thus we must wait an additional time a before another admission. This means that we are at the same situation as the original position. So we multiply by the probability $L_{T_1}(s)$.

Taking the first derivative of $L_{T_1}(s)$ we get (Maple codes are given in Appendix A.1)

$$
\begin{aligned}
L'_{T_1}(s) = -\left[(\lambda+s)^2\left(\lambda+\mu+s-\lambda\,e^{-a(\mu+s)}\right)^2\right]^{-1} & \left[-ae^{-a(\mu+s)}s^3 - ae^{-a(\mu+s)}\lambda^3 - \right.\\
& \mu\,e^{-a(2\,s+\mu)}\lambda - 2\,e^{-a(\mu+s)}\lambda\,s + 2\,e^{-sa}\lambda\,s + ae^{-sa}\lambda^3 + 2\,e^{-sa}\mu\,\lambda \\
& +ae^{-sa}s^3 + 2\,e^{-sa}\mu\,s + e^{-sa}\lambda^2 + e^{-sa}\mu^2 + e^{-sa}s^2 + 2\,ae^{-sa}\mu\,\lambda^2 \\
& +3\,ae^{-sa}\lambda^2 s + ae^{-sa}\lambda\,\mu^2 + 3\,ae^{-sa}\lambda\,s^2 + ae^{-sa}s\mu^2 + 2\,ae^{-sa}\mu\,s^2 \\
& -\mu\,ae^{-a(\mu+s)}\lambda^2 - \mu\,ae^{-a(\mu+s)}s^2 + 4\,ae^{-sa}\lambda\,\mu\,s - 2\,\mu\,ae^{-a(\mu+s)}\lambda\,s \\
& \left. -e^{-a(\mu+s)}s^2 - e^{-a(\mu+s)}\lambda^2 - 3\,ae^{-a(\mu+s)}\lambda^2 s - 3\,ae^{-a(\mu+s)}\lambda\,s^2\right]\lambda.
\end{aligned}
$$

Now putting $s=0$ into the above equation and simplifying we get

$$L'_{T_1}(0) = -\frac{(\lambda+\mu)\,(\lambda\,a+1)}{\lambda\,\left(\lambda+\mu-e^{-\mu\,a}\lambda\right)}.$$

Then using Proposition 2.1.2, the expected interaccept time is,

$$\begin{aligned} E(T_1) = -L'_{T_1}(0) &= \frac{(\lambda+\mu)(a\lambda+1)}{\lambda(\lambda+\mu-\lambda e^{-\mu a})} \\ &= \left(a+\frac{1}{\lambda}\right)\left(1-\frac{\lambda e^{-\mu a}}{\lambda+\mu}\right)^{-1}. \end{aligned}$$

This result agrees with Tang (2005). But she used a different more complex derivation. Further her method did not give the extra information contained in the Laplace transform, but only worked with the expected value.

Let us denote the Laplace transform of the interadmit time by L_{T_2}. The interadmit time depends on the probability of the next arrival occuring before a catastrophe. Thus

$$\begin{aligned} L_{T_2}(s) &= P(T_2 < Y) \\ &= e^{-sa}P(\text{next arrival before the catastrophe}) \\ &= e^{-sa}\left(\frac{\lambda}{\lambda+s}\right). \end{aligned}$$

Taking the first derivative of L_{T_2} we get,

$$L'_{T_2}(s) = -ae^{sa}\left(\frac{\lambda}{\lambda+s}\right) - \frac{e^{-sa}(\lambda)}{(\lambda+s)^2}.$$

Then the expected interadmit time is $E(T_2) = -L'_{T_2}(0)$, where

$$L'_{T_2}(0) = -(a+\frac{1}{\lambda}).$$

Therefore

$$E(T_2) = \left(a + \frac{1}{\lambda}\right).$$

This result agrees with that of Tang (2005) though we used a different derivation.

The arrival rate consists of accept rate, reject rate and block rate. Let $\lambda_1, \lambda_2, \lambda_3$ be the acceptance, rejection and block rates respectively, so

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3,$$

where

$$\begin{aligned}
\lambda_1 &= \frac{1}{E(T_1)}, \\
\lambda_2 &= \frac{1}{E(T_2)} - \frac{1}{E(T_1)}, \\
\lambda_3 &= \lambda - \frac{1}{E(T_2)}.
\end{aligned}$$

Then the expected penalty cost per unit time is

$$\begin{aligned}
\lambda_1\,(0) + \lambda_2(1) + \lambda_3(c) &= \frac{1}{a+\frac{1}{\lambda}} - \frac{\left(1 - \frac{\lambda e^{-\mu a}}{\lambda+\mu}\right)}{a+\frac{1}{\lambda}} + \lambda c - \frac{c}{a+\frac{1}{\lambda}} \\
&= \left(\frac{\lambda}{a\lambda+1}\right) e^{-\mu a} \left(\frac{\lambda}{\lambda+\mu}\right) + \left(\lambda - \frac{\lambda}{a\lambda+1}\right) c.
\end{aligned}$$

For fixed c, λ, μ, this is a function of a which we call $f(a)$. Tang (2005) showed that a unique minimum exists. To minimize $f(a)$, we take the derivative w.r.t. a and set it to 0, yielding

$$\frac{-\lambda^2}{(a\lambda+1)^2}e^{-\mu a}\frac{\lambda}{\lambda+\mu}+\frac{\lambda}{a\lambda+1}\left(-\mu e^{-\mu a}\right)\frac{\lambda}{\lambda+\mu}+\frac{\lambda^2 c}{(a\lambda+1)^2}=0.$$

$$\Rightarrow c = e^{-\mu a^*}\frac{\lambda}{\lambda+\mu}\left(\lambda+\mu+a^*\lambda\mu\right)=e^{-\mu a^*}\left(1+\frac{a^*\mu\lambda}{\lambda+\mu}\right),$$

where a^* is an optimal blocking time. Intuitively, when the blocking cost is small, we are likely to increase our blocking time to avoid the possible rejection cost 1. When the service rate is high, the probability of an admitted customer being rejected is small, so the blocking time a decreases as μ increases with everything else fixed.

Chapter 4

M/E$_2$/1 Queue Control Model

In this chapter we extend the model of Lin and Ross (2003) to the problem of admission control for an M/E$_2$/1 queue with unknown completion time.

4.1 M/E$_2$/1 Queue

Motivated by the policy we used on an M/M/1 queue, we employ a similar strategy for an M/E$_2$/1 queue. First we assume that when an admitted customer is rejected, the gatekeeper will be informed immediately at which stage the last accepted customer is being served. Our threshold policy states that upon rejection of an admitted customer,

1. the gatekeeper blocks any arrival for the next b time units if the last accepted customer is being served at the first stage,

2. the gatekeeper blocks any arrival for the next a time units if the last accepted customer is being served at the second stage,

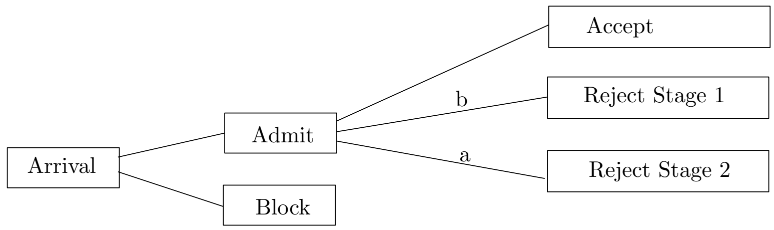

Figure 4.1: arrival rate=accept rate+reject rate at stage 1+reject rate at stage 2+block rate

where $a < b$. Note that at the moment the server accepts an admitted customer, the gatekeeper will block all the new arrivals for the next b time units. See Figure 4.1.

4.2 Expected Value Using Laplace Transforms

The arrival rate consists of accept rate, reject rate and block rate. We are going to find the expected interaccept time first (Maple codes are given in Appendix A.2).

We define

$$E_{AA} \equiv \text{expected interaccept time},$$
$$E_{1A} \equiv \text{expected time from stage 1 rejection to the next acceptance},$$
$$E_{2A} \equiv \text{expected time from stage 2 rejection to the next acceptance}.$$

To find expected value using Laplace transforms we know, (using Proposition 2.1.2)

$$E_{AA} = -L'_{AA}(s)|_{s=0},$$
$$E_{1A} = -L'_{1A}(s)|_{s=0},$$
$$E_{2A} = -L'_{2A}(s)|_{s=0},$$

where $L_{AA}(s)$= Laplace transform of the interaccept time=$P_A(T_A < Y)$, and the stages are $1, 2, A$ with A indicating an acceptance has just occurred or will occur. T_A means the time to the next acceptance at stage A.

$$L_{1A}(s) = P_1(T_A < Y),$$

where P_1 is the probability given that a customer has just been rejected at stage 1.

$$L_{2A}(s) = P_2(T_A < Y),$$

where P_2 is the probability given that a customer has just been rejected at stage 2.

Note that $L_{AA}(s) = L_{1A}(s)$, because regardless of whether the last admitted customer is accepted or is rejected at the first stage, the gatekeeper has to block all arrivals within the next b time units. The two-stage Erlang service time is the sum of two exponentials. We could interpret the system as having two servers in series each with an exponential service time but with the condition that a new customer cannot enter the first stage if the second stage is nonempty. Thus at any time, the status of an accepted customer can be obtained by counting the number of service completions which occur according to a Poisson process, i.e. during a time interval, an accepted customer remains at first stage if there is no arrival in the relevant time interval. So,

$$
\begin{aligned}
L_{AA}(s) = {} & e^{-b(\mu+s)}\left(\frac{\lambda}{\lambda+\mu+s}\right)L_{1A}(s) \\
& + e^{-b(\mu+s)}\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+\mu+s}\right)L_{2A}(s) \\
& + e^{-b(\mu+s)}\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+s}\right) \\
& + e^{-bs}\frac{(\mu b)^1 e^{-\mu b}}{1!}\left(\frac{\lambda}{\lambda+\mu+s}\right)L_{2A}(s) \\
& + e^{-bs}\frac{(\mu b)^1 e^{-\mu b}}{1!}\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+s}\right) \\
& + e^{-bs}\left(1-e^{-\mu b}-\frac{(\mu b)^1 e^{-\mu b}}{1!}\right)\left(\frac{\lambda}{\lambda+s}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
L_{2A}(s) = {} & e^{-a(\mu+s)}\left(\frac{\lambda}{\lambda+\mu+s}\right)L_{2A}(s) \\
& + e^{-a(\mu+s)}\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+s}\right) \\
& + e^{-as}\left(1-e^{-a\mu}\right)\left(\frac{\lambda}{\lambda+s}\right).
\end{aligned}
$$

Taking the first derivative of $L_{AA}(s)$ we get

$$
L'_{AA}(s) = L'_{q1}(s) + L'_{q2}(s) + L'_{q3}(s) + L'_{q4}(s) + L'_{q5}(s),
$$

where

$$
\begin{aligned}
L'_{q1}(s) = & -be^{-b(\mu+s)}\mu\,\lambda^2\left(e^{-a(\mu+s)}\mu + e^{-as}\lambda + e^{-as}\mu + e^{-as}s\right. \\
& \left. -e^{-as}\lambda\,e^{-a\mu} - e^{-as}e^{-a\mu}\mu - e^{-as}e^{-a\mu}s\right)(\lambda+\mu+s)^{-2} \\
& \left(\lambda^2 + \lambda\,\mu + 2\,\lambda\,s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-1} \\
& \left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - 2\,e^{-b(\mu+s)}\mu\,\lambda^2\left(e^{-a(\mu+s)}\mu\right. \\
& \left. +e^{-as}\lambda + e^{-as}\mu + e^{-as}s - e^{-as}\lambda\,e^{-a\mu} - e^{-as}e^{-a\mu}\mu - e^{-as}e^{-a\mu}s\right) \\
& (\lambda+\mu+s)^{-3}\left(\lambda^2 + \lambda\,\mu + 2\,\lambda\,s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-1} \\
& \left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} + e^{-b(\mu+s)}\mu\,\lambda^2\left(-ae^{-a(\mu+s)}\mu - ae^{-as}\lambda - ae^{-as}\mu\right. \\
& \left. -ae^{-as}s + e^{-as} + ae^{-as}\lambda\,e^{-a\mu} + ae^{-as}e^{-a\mu}\mu + ae^{-as}e^{-a\mu}s - e^{-as}e^{-a\mu}\right) \\
& (\lambda+\mu+s)^{-2}\left(\lambda^2 + \lambda\,\mu + 2\,\lambda\,s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-1} \\
& \left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - e^{-b(\mu+s)}\mu\,\lambda^2\left(e^{-a(\mu+s)}\mu\right. \\
& \left. +e^{-as}\lambda + e^{-as}\mu + e^{-as}s - e^{-as}\lambda\,e^{-a\mu} - e^{-as}e^{-a\mu}\mu - e^{-as}e^{-a\mu}s\right) \\
& \left(2\,\lambda + ae^{-a(\mu+s)}\lambda^2 + \mu + 2\,s - e^{-a(\mu+s)}\lambda + sae^{-a(\mu+s)}\lambda\right) \\
& (\lambda+\mu+s)^{-2}\left(\lambda^2 + \lambda\,\mu + 2\,\lambda\,s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-2} \\
& \left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - e^{-b(\mu+s)}\mu\,\lambda^2\left(e^{-a(\mu+s)}\mu\right. \\
& \left. +e^{-as}\lambda + e^{-as}\mu + e^{-as}s - e^{-as}\lambda\,e^{-a\mu} - e^{-as}e^{-a\mu}\mu - e^{-as}e^{-a\mu}s\right) \\
& \left(\frac{be^{-b(\mu+s)}\lambda}{\lambda+\mu+s} + \frac{e^{-b(\mu+s)}\lambda}{(\lambda+\mu+s)^2}\right)(\lambda+\mu+s)^{-2}\left(\lambda^2 + \lambda\,\mu\right. \\
& \left. +2\,\lambda\,s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-1}\left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
L'_{q2}(s) = &-be^{-b(\mu+s)}\mu^2\lambda\left(\lambda+\mu+s\right)^{-2}\left(\lambda+s\right)^{-1}\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} \\
&-2\,e^{-b(\mu+s)}\mu^2\lambda\left(\lambda+\mu+s\right)^{-3}\left(\lambda+s\right)^{-1}\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} \\
&-e^{-b(\mu+s)}\mu^2\lambda\left(\lambda+\mu+s\right)^{-2}\left(\lambda+s\right)^{-2}\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} \\
&-e^{-b(\mu+s)}\mu^2\lambda\,\left(\frac{be^{-b(\mu+s)}\lambda}{\lambda+\mu+s}+\frac{e^{-b(\mu+s)}\lambda}{\left(\lambda+\mu+s\right)^2}\right)\left(\lambda+\mu+s\right)^{-2} \\
&\left(\lambda+s\right)^{-1}\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
L'_{q3}(s) = & -b^2 e^{-bs}\mu\, e^{-b\mu}\lambda^2 \left(e^{-a(\mu+s)}\mu + e^{-as}\lambda + e^{-as}\mu + e^{-as}s\right. \\
& \left. -e^{-as}\lambda\, e^{-a\mu} - e^{-as}e^{-a\mu}\mu - e^{-as}e^{-a\mu}s\right)(\lambda+\mu+s)^{-1} \\
& \left(\lambda^2 + \lambda\,\mu + 2\,\lambda\, s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-1} \\
& \left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - e^{-bs}\mu\, be^{-b\mu}\lambda^2 \left(e^{-a(\mu+s)}\mu + e^{-as}\lambda\right. \\
& \left. + e^{-as}\mu + e^{-as}s - e^{-as}\lambda\, e^{-a\mu} - e^{-as}e^{-a\mu}\mu - e^{-as}e^{-a\mu}s\right) \\
& (\lambda+\mu+s)^{-2}\left(\lambda^2 + \lambda\,\mu + 2\,\lambda\, s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-1} \\
& \left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} + e^{-bs}\mu\, be^{-b\mu}\lambda^2 \left(-ae^{-a(\mu+s)}\mu - ae^{-as}\lambda\right. \\
& -ae^{-as}\mu - ae^{-as}s + e^{-as} + ae^{-as}\lambda\, e^{-a\mu} + ae^{-as}e^{-a\mu}\mu \\
& \left. + ae^{-as}e^{-a\mu}s - e^{-as}e^{-a\mu}\right)(\lambda+\mu+s)^{-1} \\
& \left(\lambda^2 + \lambda\,\mu + 2\,\lambda\, s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-1} \\
& \left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - e^{-bs}\mu\, be^{-b\mu}\lambda^2 \left(e^{-a(\mu+s)}\mu + e^{-as}\lambda\right. \\
& \left. + e^{-as}\mu + e^{-as}s - e^{-as}\lambda\, e^{-a\mu} - e^{-as}e^{-a\mu}\mu - e^{-as}e^{-a\mu}s\right) \\
& \left(2\,\lambda + ae^{-a(\mu+s)}\lambda^2 + \mu + 2\, s - e^{-a(\mu+s)}\lambda + sae^{-a(\mu+s)}\lambda\right) \\
& (\lambda+\mu+s)^{-1}\left(\lambda^2 + \lambda\,\mu + 2\,\lambda\, s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-2} \\
& \left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - e^{-bs}\mu\, be^{-b\mu}\lambda^2 \left(e^{-a(\mu+s)}\mu + e^{-as}\lambda\right. \\
& \left. + e^{-as}\mu + e^{-as}s - e^{-as}\lambda\, e^{-a\mu} - e^{-as}e^{-a\mu}\mu - e^{-as}e^{-a\mu}s\right) \\
& \left(\frac{be^{-b(\mu+s)}\lambda}{\lambda+\mu+s} + \frac{e^{-b(\mu+s)}\lambda}{(\lambda+\mu+s)^2}\right)(\lambda+\mu+s)^{-1} \\
& \left(\lambda^2 + \lambda\,\mu + 2\,\lambda\, s - e^{-a(\mu+s)}\lambda^2 + s\mu + s^2 - se^{-a(\mu+s)}\lambda\right)^{-1} \\
& \left(1 - \frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-2},
\end{aligned}
$$

$$L'_{q4}(s) = -b^2 e^{-bs}\mu^2 e^{-b\mu}\lambda\,(\lambda+\mu+s)^{-1}$$

$$(\lambda+s)^{-1}\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - e^{-bs}\mu^2 b e^{-b\mu}\lambda$$

$$(\lambda+\mu+s)^{-2}(\lambda+s)^{-1}\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1}$$

$$- e^{-bs}\mu^2 b e^{-b\mu}\lambda\,(\lambda+\mu+s)^{-1}(\lambda+s)^{-2}$$

$$\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - e^{-bs}\mu^2 b e^{-b\mu}\lambda$$

$$\left(\frac{be^{-b(\mu+s)}\lambda}{\lambda+\mu+s}+\frac{e^{-b(\mu+s)}\lambda}{(\lambda+\mu+s)^2}\right)(\lambda+\mu+s)^{-1}$$

$$(\lambda+s)^{-1}\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-2},$$

$$L'_{q5}(s) = -be^{-bs}\left(1-e^{-b\mu}-\mu\,be^{-b\mu}\right)\lambda\,(\lambda+s)^{-1}$$

$$\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - e^{-bs}\left(1-e^{-b\mu}-\mu\,be^{-b\mu}\right)$$

$$\lambda\,(\lambda+s)^{-2}\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-1} - e^{-bs}\left(1-e^{-b\mu}-\mu\,be^{-b\mu}\right)$$

$$\lambda\left(\frac{be^{-b(\mu+s)}\lambda}{\lambda+\mu+s}+\frac{e^{-b(\mu+s)}\lambda}{(\lambda+\mu+s)^2}\right)(\lambda+s)^{-1}\left(1-\frac{e^{-b(\mu+s)}\lambda}{\lambda+\mu+s}\right)^{-2}.$$

Now for $s = 0$ we get

$$L'_{AA}(0) = L'_{q1}(0) + L'_{q2}(0) + L'_{q3}(0) + L'_{q4}(0) + L'_{q5}(0).$$

where

$$
\begin{aligned}
L'_{q1}(0) = & -\mu\, be^{-b\mu}\lambda^2 \left(\lambda+\mu-e^{-a\mu}\lambda\right)(\lambda+\mu)^{-2} \\
& \left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -2\,e^{-b\mu}\mu\,\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)(\lambda+\mu)^{-3} \\
& \left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& +e^{-b\mu}\mu\,\lambda^2\left(-a\lambda-a\mu+1+a\lambda\,e^{-a\mu}-e^{-a\mu}\right) \\
& (\lambda+\mu)^{-2}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -e^{-b\mu}\mu\,\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)\left(2\,\lambda+\mu+ae^{-a\mu}\lambda^2-e^{-a\mu}\lambda\right) \\
& (\lambda+\mu)^{-2}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-2}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -e^{-b\mu}\mu\,\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right) \\
& (\lambda+\mu)^{-2}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
L'_{q2}(0) = & -be^{-b\mu}\mu^2(\lambda+\mu)^{-2}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -2\,e^{-b\mu}\mu^2(\lambda+\mu)^{-3}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -e^{-b\mu}\mu^2(\lambda+\mu)^{-2}\lambda^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -e^{-b\mu}\mu^2\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right) \\
& (\lambda+\mu)^{-2}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
L'_{q3}(0) = &-b^2\mu\, e^{-b\mu}\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)(\lambda+\mu)^{-1}\\
&\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}\\
&-\mu\, be^{-b\mu}\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)(\lambda+\mu)^{-2}\\
&\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}\\
&+\mu\, be^{-b\mu}\lambda^2\left(-a\lambda-a\mu+1+a\lambda\, e^{-a\mu}-e^{-a\mu}\right)\\
&(\lambda+\mu)^{-1}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}\\
&-\mu\, be^{-b\mu}\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)\left(2\,\lambda+\mu+ae^{-a\mu}\lambda^2-e^{-a\mu}\lambda\right)\\
&(\lambda+\mu)^{-1}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-2}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}\\
&-\mu\, be^{-b\mu}\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right)\\
&(\lambda+\mu)^{-1}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
L'_{q4}(0) = &-b^2\mu^2e^{-b\mu}(\lambda+\mu)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}\\
&-be^{-b\mu}\mu^2(\lambda+\mu)^{-2}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}\\
&-be^{-b\mu}\mu^2(\lambda+\mu)^{-1}\lambda^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}\\
&-be^{-b\mu}\mu^2\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right)(\lambda+\mu)^{-1}\\
&\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
L'_{q5}(0) = & -b\left(1-e^{-b\mu}-\mu\, be^{-b\mu}\right)\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -\left(1-e^{-b\mu}-\mu\, be^{-b\mu}\right)\lambda^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -\left(1-e^{-b\mu}-\mu\, be^{-b\mu}\right)\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right) \\
& \left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2}.
\end{aligned}
$$

Then $L'_{AA}(0)$ will be,

$$
\begin{aligned}
L'_{AA}(0) = {} & -2\,\mu\, be^{-b\mu}\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)(\lambda+\mu)^{-2} \\
& \left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -2\,e^{-b\mu}\mu\,\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)(\lambda+\mu)^{-3} \\
& \left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& +e^{-b\mu}\mu\,\lambda^2\left(-a\lambda-a\mu+1+a\lambda\, e^{-a\mu}-e^{-a\mu}\right) \\
& (\lambda+\mu)^{-2}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -e^{-b\mu}\mu\,\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)\left(2\,\lambda+\mu+ae^{-a\mu}\lambda^2-e^{-a\mu}\lambda\right) \\
& (\lambda+\mu)^{-2}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-2}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -e^{-b\mu}\mu\,\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right) \\
& (\lambda+\mu)^{-2}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2} \\
& -2\,be^{-b\mu}\mu^2(\lambda+\mu)^{-2}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -2\,e^{-b\mu}\mu^2(\lambda+\mu)^{-3}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -e^{-b\mu}\mu^2(\lambda+\mu)^{-2}\lambda^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& -e^{-b\mu}\mu^2\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right)(\lambda+\mu)^{-2} \\
& \left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2}-b^2\mu\, e^{-b\mu}\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right) \\
& (\lambda+\mu)^{-1}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1} \\
& +\mu\, be^{-b\mu}\lambda^2\left(-a\lambda-a\mu+1+a\lambda\, e^{-a\mu}-e^{-a\mu}\right)
\end{aligned}
$$

$$(\lambda+\mu)^{-1}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}$$

$$-\mu\,be^{-b\mu}\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)\left(2\,\lambda+\mu+ae^{-a\mu}\lambda^2-e^{-a\mu}\lambda\right)$$

$$(\lambda+\mu)^{-1}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-2}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}$$

$$\mu\,be^{-b\mu}\lambda^2\left(\lambda+\mu-e^{-a\mu}\lambda\right)\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right)$$

$$(\lambda+\mu)^{-1}\left(\lambda^2+\mu\,\lambda-e^{-a\mu}\lambda^2\right)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2}$$

$$-b^2\mu^2e^{-b\mu}(\lambda+\mu)^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}$$

$$-\mu^2be^{-b\mu}(\lambda+\mu)^{-1}\lambda^{-1}\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}$$

$$-\mu^2be^{-b\mu}\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right)(\lambda+\mu)^{-1}$$

$$\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2}-b\left(1-e^{-b\mu}-\mu\,be^{-b\mu}\right)$$

$$\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}-\left(1-e^{-b\mu}-\mu\,be^{-b\mu}\right)\lambda^{-1}$$

$$\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-1}-\left(1-e^{-b\mu}-\mu\,be^{-b\mu}\right)$$

$$\left(\frac{be^{-b\mu}\lambda}{\lambda+\mu}+\frac{e^{-b\mu}\lambda}{(\lambda+\mu)^2}\right)\left(1-\frac{e^{-b\mu}\lambda}{\lambda+\mu}\right)^{-2}$$

$$=-(\lambda^2+2\,\mu\,\lambda-e^{-a\mu}\lambda^2+\mu^2+2\,b\mu\,\lambda^2+\mu^2b\lambda+b\lambda^3+\mu\,e^{-b\mu}\lambda$$

$$-e^{-a\mu}\mu\,\lambda+\mu\,be^{-b\mu}\lambda^2+\mu\,e^{-b\mu}\lambda^2a-\mu\,b\lambda^2e^{-a\mu}$$

$$+\mu^2be^{-b\mu}\lambda-b\lambda^3e^{-a\mu}+\mu\,be^{-b\mu}\lambda^3a+\mu^2be^{-b\mu}\lambda^2a)$$

$$\left(\lambda^2+\mu^2+2\,\mu\,\lambda-e^{-a\mu}\lambda^2-e^{-b\mu}\lambda^2+e^{-\mu\,(b+a)}\lambda^2-e^{-a\mu}\mu\,\lambda-\mu\,e^{-b\mu}\lambda\right)^{-1}\lambda^{-1}.$$

so

$$-L'_{AA}(0) = \left(b + \frac{1}{\lambda}\right)\left(1 - \frac{\lambda e^{-\mu b}}{\lambda + \mu}\right)^{-1} + \frac{\left(a + \frac{1}{\lambda}\right)\left(\lambda \mu e^{-\mu b}\right)\left(1 + (\lambda + \mu)\, b\right)}{\left(\lambda + \mu - \lambda e^{-\mu a}\right)\left(\lambda + \mu - \lambda e^{-\mu b}\right)}.$$

Then the expected interaccept time

$$E_{AA} = -L'_{AA}(0)$$

which exactly matches with the result of E_{AA} from Tang (2005).

In the following, we compute the Laplace transform of the time between two consecutive rejections which occur at the same stage(Maple codes are given in Appendix A.2). Let us define the following terms

$L_{11}(s) = P_1(T_1 < Y) \equiv$ Laplace transform of the time between two consecutive stage 1 rejections,

$L_{A1}(s) = P_A(T_1 < Y) \equiv$ Laplace transform of the time between an acceptance and the next stage 1 rejection,

$L_{21}(s) = P_2(T_1 < Y) \equiv$ Laplace transform of the time between a stage 2 rejection and the next stage 1 rejection,

$L_{22}(s) = P_2(T_2 < Y) \equiv$ Laplace transform of the time between two consecutive stage 2 rejections,

$L_{A2}(s) = P_A(T_2 < Y) \equiv$ Laplace transform of the time between an acceptance and the next stage 2 rejection,

$L_{12}(s) = P_1(T_2 < Y) \equiv$ Laplace transform of the time between a stage 1 rejection and the next stage 2 rejection.

Note that $L_{A1}(s) = L_{11}(s), L_{A2}(s) = L_{12}(s)$. Then the expected time between two consecutive stage 1 rejections is given by

$$
\begin{aligned}
L_{A1}(s) = {} & e^{-b(\mu+s)}\left(\frac{\lambda}{\lambda+\mu+s}\right) \\
& + e^{-b(\mu+s)}\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+\mu+s}\right)L_{21}(s) \\
& + e^{-b(\mu+s)}\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+s}\right)L_{A1}(s) \\
& + e^{-bs}\frac{(\mu b)^1 e^{-\mu b}}{1!}\left(\frac{\lambda}{\lambda+\mu+s}\right)L_{21}(s) \\
& + e^{-bs}\frac{(\mu b)^1 e^{-\mu b}}{1!}\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+s}\right)L_{A1}(s) \\
& + e^{-bs}\left(1-e^{-\mu b}-\frac{(\mu b)^1 e^{-\mu b}}{1!}\right)\left(\frac{\lambda}{\lambda+s}\right)L_{A1}(s).
\end{aligned}
$$

where

$$
\begin{aligned}
L_{21}(s) = {} & e^{-a(\mu+s)}\left(\frac{\lambda}{\lambda+\mu+s}\right)L_{21}(s) \\
& + e^{-a(\mu+s)}\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+s}\right)L_{A1}(s) \\
& + e^{-as}\left(1-e^{-a\mu}\right)\left(\frac{\lambda}{\lambda+s}\right)L_{A1}(s).
\end{aligned}
$$

Taking the first derivative of $L_{A1}(s)$ and putting $s=0$ we get,

$$
\begin{aligned}
L'_{A1}(0) = {} & -(\mu\,\lambda + b\mu\,\lambda^3 a + \mu^2 b\lambda^2 a + \mu\,\lambda^2 a + e^{b\mu} b\lambda^3 + 2\,e^{b\mu}\mu\,b\lambda^2 + e^{b\mu}\mu^2 b\lambda \\
& + 2\,e^{b\mu}\mu\,\lambda + e^{b\mu}\lambda^2 + e^{b\mu}\mu^2 - e^{\mu\,(b-a)} b\lambda^3 - e^{\mu\,(b-a)}\lambda^2 - e^{\mu\,(b-a)}\mu\,b\lambda^2 \\
& - e^{\mu\,(b-a)}\mu\,\lambda + \mu\,b\lambda^2 + \mu^2 b\lambda)[\lambda^2\left(\lambda - e^{-a\mu}\lambda + \mu\right)]^{-1}
\end{aligned}
$$

Then the expected time between an acceptance and the next stage 1 rejection is

$$\begin{aligned}
E_{A1} &= -L'_{A1}(0) \\
&= (\mu\,\lambda + b\mu\,\lambda^3 a + \mu^2 b\lambda^2 a + \mu\,\lambda^2 a + e^{b\mu} b\lambda^3 + 2\,e^{b\mu}\mu\,b\lambda^2 + e^{b\mu}\mu^2 b\lambda \\
&\quad + 2\,e^{b\mu}\mu\,\lambda + e^{b\mu}\lambda^2 + e^{b\mu}\mu^2 - e^{\mu\,(b-a)} b\lambda^3 - e^{\mu\,(b-a)}\lambda^2 - e^{\mu\,(b-a)}\mu\,b\lambda^2 \\
&\quad - e^{\mu\,(b-a)}\mu\,\lambda + \mu\,b\lambda^2 + \mu^2 b\lambda)(\lambda^2\left(\lambda - e^{-a\mu}\lambda + \mu\right))^{-1},
\end{aligned}$$

which exactly matches with the result of E_{A1} from Tang (2005).

Again,

$$\begin{aligned}
L_{22}(s) &= e^{-a(\mu+s)}\left(\frac{\lambda}{\lambda+\mu+s}\right) \\
&\quad + e^{-a(\mu+s)}\left(\frac{\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+s}\right)L_{A2}(s) \\
&\quad + e^{-as}\left(1-e^{-a\mu}\right)\left(\frac{\lambda}{\lambda+s}\right)L_{A2}(s),
\end{aligned}$$

where

$$
\begin{aligned}
L_{A2}(s) = {} & e^{-b(\mu+s)} \left(\frac{\lambda}{\lambda+\mu+s}\right) L_{A2}(s) \\
& + e^{-b(\mu+s)} \left(\frac{\mu}{\lambda+\mu+s}\right) \left(\frac{\lambda}{\lambda+\mu+s}\right) \\
& + e^{-b(\mu+s)} \left(\frac{\mu}{\lambda+\mu+s}\right) \left(\frac{\mu}{\lambda+\mu+s}\right) \left(\frac{\lambda}{\lambda+s}\right) L_{A2}(s) \\
& + e^{-bs} \frac{(\mu b)^1 e^{-\mu b}}{1!} \left(\frac{\lambda}{\lambda+\mu+s}\right) \\
& + e^{-bs} \frac{(\mu b)^1 e^{-\mu b}}{1!} \left(\frac{\mu}{\lambda+\mu+s}\right) \left(\frac{\lambda}{\lambda+s}\right) L_{A2}(s) \\
& + e^{-bs} \left(1 - e^{-\mu b} - \frac{(\mu b)^1 e^{-\mu b}}{1!}\right) \left(\frac{\lambda}{\lambda+s}\right) L_{A2}(s).
\end{aligned}
$$

Then the expected time between two consecutive stage 2 rejections will be, $E_{22} = -L'_{22}(0)$ (from Proposition 2.1.2) where

$$
\begin{aligned}
L'_{22}(0) = {} & \lambda \left(\mu\, e^{-a\mu} \lambda\, b + \mu\, e^{-a\mu} + e^{-a\mu} \lambda - \mu^2 a \lambda\, b e^{-b\mu}\right. \\
& -\mu^2 \lambda\, b^2 e^{-b\mu} - \mu\, a \lambda\, e^{-b\mu} + e^{-a\mu} \lambda^2 b \\
& \left.-\mu\, a \lambda^2 b e^{-b\mu} - \mu\, \lambda^2 b^2 e^{-b\mu}\right) / (e^{-b\mu} \mu\, \lambda^2 + \mu\, b e^{-b\mu} \lambda^3 + \mu^2 b e^{-b\mu} \lambda^2) \\
& - \left(\lambda \left(e^{-b\mu} \mu\, \lambda + b e^{-b\mu} \mu^2 \lambda + b e^{-b\mu} \mu\, \lambda^2\right) \left(\mu^2 + \lambda^2 + 2\, \lambda\, \mu + b \lambda^3\right.\right. \\
& + e^{-b\mu} \mu\, \lambda - b^2 \mu\, e^{-b\mu} \lambda^3 - b^2 \mu^2 e^{-b\mu} \lambda^2 + b e^{-b\mu} \mu\, \lambda^2 + 2\, b \lambda^2 \mu \\
& \left.\left. + b \lambda\, \mu^2 + b e^{-b\mu} \mu^2 \lambda\right)\right) / \left(e^{-b\mu} \mu\, \lambda^2 + \mu\, b e^{-b\mu} \lambda^3 + \mu^2 b e^{-b\mu} \lambda^2\right)^2 \\
= {} & -\left(-\mu\, e^{-a\mu} \lambda^2 b - \mu\, e^{-a\mu} \lambda - e^{-a\mu} \lambda^2 + \mu^2 a \lambda^2 b e^{-b\mu} + \mu\, a \lambda^2 e^{-b\mu}\right. \\
& - e^{-a\mu} \lambda^3 b + \mu\, a \lambda^3 b e^{-b\mu} + \mu^2 + \lambda^2 + 2\, \lambda\, \mu + b \lambda^3 + e^{-b\mu} \mu\, \lambda \\
& \left. + b e^{-b\mu} \mu\, \lambda^2 + 2\, b \lambda^2 \mu + b \lambda\, \mu^2 + b e^{-b\mu} \mu^2 \lambda\right) e^{b\mu} / (\mu\, \lambda^2\, (1 + b\lambda + b\mu))
\end{aligned}
$$

Then

$$\begin{aligned}
E_{22} &= -L'_{22}(0) \\
&= \big(-\mu\, e^{-a\mu}\lambda^2 b - \mu\, e^{-a\mu}\lambda - e^{-a\mu}\lambda^2 + \mu^2 a\lambda^2 b e^{-b\mu} + \mu\, a\lambda^2 e^{-b\mu} \\
&\quad -e^{-a\mu}\lambda^3 b + \mu\, a\lambda^3 b e^{-b\mu} + \mu^2 + \lambda^2 + 2\,\lambda\,\mu + b\lambda^3 + e^{-b\mu}\mu\,\lambda \\
&\quad +be^{-b\mu}\mu\,\lambda^2 + 2\,b\lambda^2\mu + b\lambda\,\mu^2 + be^{-b\mu}\mu^2\lambda\big)\, e^{b\mu}/(\mu\,\lambda^2\,(1 + b\lambda + b\mu)),
\end{aligned}$$

which exactly matches with the value of E_{22} from Tang (2005).

We denote $\lambda_a, \lambda_1, \lambda_2, \lambda_b$ as the accept rate, reject stage 1 rate, reject stage 2 rate and block rate respectively. Then

$$\lambda = \lambda_a + (\lambda_1 + \lambda_2) + \lambda_b.$$

where $\lambda_a = 1/E_{AA}, \lambda_1 = 1/E_{11}, \lambda_2 = 1/E_{22}, \lambda_b = \lambda - \lambda_a - \lambda_1 - \lambda_2$.

The expected penalty cost per unit time yields,

$$\lambda_a(0) + (\lambda_1 + \lambda_2)(1) + \lambda_b(c).$$

For a given blocking cost c and given λ, μ, one can find a pair (a^*, b^*) that minimizes the above function which was already done by Tang (2005).

Chapter 5

$M/E_k/1$ Queue Control Model

5.1 An $M/E_k/1$ Queue

A two-stage Erlang service time can be extended to a k-stage Erlang service time. Let a_i be the gap time from a rejection at stage i (i.e. $i-1$ stages have been completed) until next allowable admission. See Figure 5.1.

Let

$$L_{ij}(s) = P_i(T_j < Y),$$

where T_j means the time to next rejection at stage j and i subscript on P_i refers to a customer having just been rejected where the customer in service has completed exactly $i-1$ stages, $i = 1, 2, \ldots, k$ and Y is a catastrophe random variable at rate s.

Let

$$L_{Aj} = P_A(T_j < Y)$$

where j means the customer in service has completed exactly $j-1$ stages, $j =$

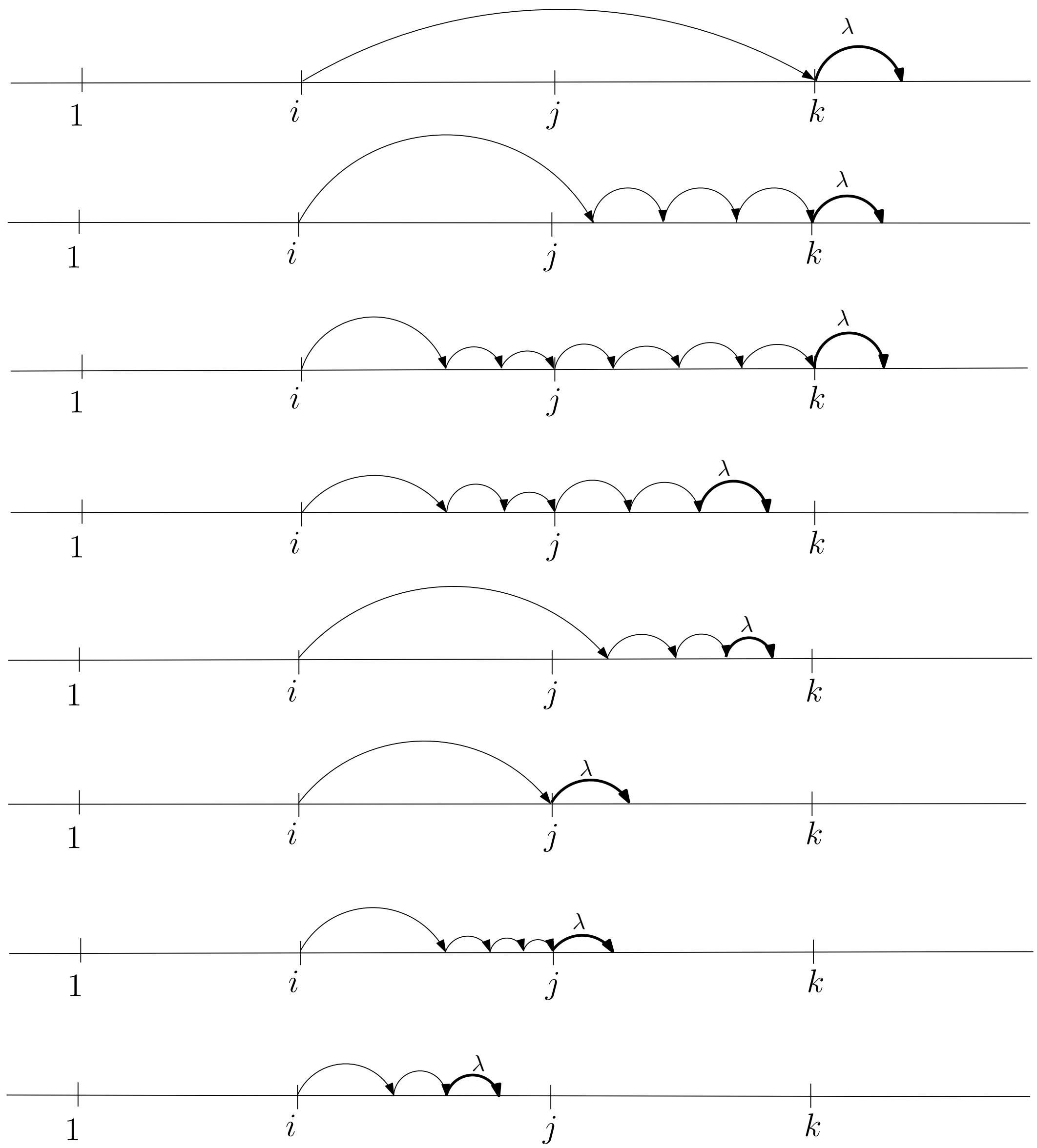

Figure 5.1: Arrival pattern for $M/E_k/1$ queue control model.

$1, 2 \ldots, k$ and A means that an acceptance has just occurred or will occur. First note that $L_{Aj}(s) = L_{1j}(s)$.

Proposition 5.1.1. *If* $1 \le i \le j \le k$ *then*

$$\begin{aligned}
L_{ij}(s) = {} & e^{-a_i s}\left(1 - \sum_{m=0}^{k-i} \frac{(a_i\mu)^m e^{-a_i\mu}}{m!}\right)\left(\frac{\lambda}{\lambda+s}\right) L_{Aj}(s) \\
& + e^{-a_i s}\left(\sum_{m=0}^{k-i} \frac{(a_i\mu)^m e^{-a_i\mu}}{m!}\right)\left(\frac{\mu}{\lambda+\mu+s}\right)^{k-i-m+1}\left(\frac{\lambda}{\lambda+s}\right) L_{Aj}(s) \\
& + e^{-a_i s}\left(\sum_{\substack{m=0\\ i+m+n\neq j}}^{k-i} \frac{(a_i\mu)^m e^{-a_i\mu}}{m!}\right)\sum_{n=0}^{k-i-m}\left(\frac{\mu}{\lambda+\mu+s}\right)^{n}\left(\frac{\lambda}{\lambda+\mu+s}\right) L_{i+n+m,j}(s) \\
& + e^{-a_i s}\left(\sum_{m=0}^{j-i} \frac{(a_i\mu)^m e^{-a_i\mu}}{m!}\right)\left(\frac{\mu}{\lambda+\mu+s}\right)^{j-i-m}\left(\frac{\lambda}{\lambda+\mu+s}\right).
\end{aligned}$$

where,

k=*Number of stages.*

m=*Number of service completions that happen at the beginning.*

n=*Number of service completions before an arrival.*

Proof. Similar to E_2 case.

Proposition 5.1.2. *If* $1 \le j < i \le k$ *then*

$$L_{ij}(s) = L_{iA}(s) L_{Aj}(s),$$

Where

$$L_{iA}(s) = P_i(T_A < Y),$$

Y is a catastrophe random variable at rate s, T_A means the time to the next acceptance at stage A and the i subscript refers to a customer having just been rejected where the customer in service has completed exactly $i-1$ stages, $i = 1, 2, \ldots, k$.

Proof. There must be an acceptance between i and j. The result follows.

Proposition 5.1.3. *For $i = 1, \ldots, k$,*

$$\begin{aligned}
L_{iA}(s) &= e^{-a_i s}\left(1-\sum_{m=0}^{k-i}\frac{(a_i\mu)^m e^{-a_i\mu}}{m!}\right)\left(\frac{\lambda}{\lambda+s}\right)L_{Aj}(s) \\
&+ e^{-a_i s}\left(\sum_{m=0}^{k-i}\frac{(a_i\mu)^m e^{-a_i\mu}}{m!}\right)\left(\frac{\mu}{\lambda+\mu+s}\right)^{k-i-m+1}\left(\frac{\lambda}{\lambda+s}\right)L_{Aj}(s) \\
&+ e^{-a_i s}\left(\sum_{m=0}^{k-i}\frac{(a_i\mu)^m e^{-a_i\mu}}{m!}\right)\sum_{n=0}^{k-i-m}\left(\frac{\mu}{\lambda+\mu+s}\right)^{n}\left(\frac{\lambda}{\lambda+\mu+s}\right)L_{i+n+m,A}(s).
\end{aligned}$$

Proof. The argument is similar to those used previously.

The expected penalty cost per unit time is $\lambda_a(0)+(\lambda_1+\lambda_2+\ldots+\lambda_k)(1)+\lambda_b(c)$, where $\lambda_a = \frac{1}{E_{AA}}, \lambda_1 = \frac{1}{E_{11}}, \ldots, \lambda_k = \frac{1}{E_{kk}}, \lambda_b = \lambda-\lambda_a-(\lambda_1+\ldots+\lambda_k)$. Expressions for $E_{AA}, E_{11}, \ldots, E_{kk}$ can be obtained from $L_{AA}(s), L_{11}(s), \ldots, L_{kk}(s)$. Note that $L_{AA} = L_{1A}$ and this is given in Proposition 5.1.3. From all the equations in this section (in Propositions 5.1.1, 5.1.2, 5.1.3), we can solve for $L_{AA}(s), L_{11}(s), \ldots, L_{kk}(s)$ in terms of $a_i (i = 1, \ldots, k), \lambda, \mu$. As in chapter 4, for given blocking cost c and given λ, μ, we can find numerically $(a_i^*, \ldots, a_k^*)$ to minimize the expected penalty cost.

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Appendix A

Maple Code for Various Queue Control Models

A.1 M/M/1 Queue Control Model

A.1.1 Maple Code for Finding $E(T_1)$

```
Ls1 := (1-(lambda*exp(-mu*a)*exp(-s*a))/(lambda+mu+s))^(-1);
Ls2 := (lambda*exp(-s*a)*(1-exp(-mu*a)))/(lambda+s);
Ls3 := (mu*lambda*exp(-s*a)*exp(-mu*a))/((lambda+mu+s)*(lambda+s));
Ls := Ls1*(Ls2+Ls3);

df := diff(Ls, s);

df0 :=subs(s=0,df);

simplify(df);
simplify(df0);

evalf(df0);

latex(simplify(df));

latex(simplify(df0));

simplify(subs(s=0,df));
```

A.1.2 Maple Code for $E(T_2)$

```
LT2:=exp(-s*a)(lambda)/(lambda+s);
df:=diff(LT2,s);
latex(df);
df0:=subs(s=0,df);
simplify(df0);
```

A.2 M/E$_2$/1 Queue Control Model

A.2.1 Maple Code for Finding $E(AA)$

```
ebmus := exp(-b*(mu+s));
eamus := exp(-a*(mu+s));
ebs := exp(-b*s);
eas := exp(-a*s);
eamu := exp(-a *mu);
ebmu := exp(-b*mu);
esb := exp(-s*b);
lmus := lambda + mu +s;
ls := lambda+s;


eq1:=L2A=(eamus*(mu/lmus)*(lambda/ls))*(1-eamus*(lambda/lmus))^(-1)
            +eas*(1-eamu)*(lambda/ls)*(1-eamus*(lambda/lmus))^(-1);

eq2:=LAA=   (ebmus*(mu/lmus)*(lambda/lmus))*L2A*(1- ebmus*(lambda/lmus))^(-1)
            +(ebmus*(mu/lmus)*(mu/lmus)*(lambda/ls))*(1- ebmus*(lambda/lmus))^(-1)
            +(ebs*mu*b*ebmu*(lambda/lmus))*L2A*(1- ebmus*(lambda/lmus))^(-1)
            +(ebs*mu*b*ebmu*(mu/lmus)*(lambda/ls))*(1- ebmus*(lambda/lmus))^(-1)
            +(ebs*(1-ebmu-mu*b*ebmu)*(lambda/ls))*(1- ebmus*(lambda/lmus))^(-1);

solL2A:=solve(eq1,L2A);

Lq1:=(ebmus*(mu/lmus)*(lambda/lmus))*solL2A*(1- ebmus*(lambda/lmus))^(-1);
Lq2:=(ebmus*(mu/lmus)*(mu/lmus)*(lambda/ls))*(1- ebmus*(lambda/lmus))^(-1);
Lq3:=(ebs*mu*b*ebmu*(lambda/lmus))*solL2A*(1- ebmus*(lambda/lmus))^(-1);
Lq4:=(ebs*mu*b*ebmu*(mu/lmus)*(lambda/ls))*(1- ebmus*(lambda/lmus))^(-1);
Lq5:=(ebs*(1-ebmu-mu*b*ebmu)*(lambda/ls))*(1- ebmus*(lambda/lmus))^(-1);

df1:=diff(Lq1,s);
latex(df1);

df2:=diff(Lq2,s);
latex(df2);
```

```
df3:=diff(Lq3,s);
latex(df3);

df4:=diff(Lq4,s);
latex(df4);

df5:=diff(Lq5,s);
latex(df5);

df10:=subs(s=0,df1);
latex(df10);

df20:=subs(s=0,df2);
latex(df20);

df30:=subs(s=0,df3);
latex(df30);

df40:=subs(s=0,df4);
latex(df40);

df50:=subs(s=0,df5);
latex(df50);

LAAprimeso:=df10+df20+df30+df40+df50;
simplify(LAAprimeso);
latex(simplify(LAAprimeso));
```

A.2.2 Maple Code for Finding $E(A1)$

```
ebmus := exp(-b*(mu+s));
eamus := exp(-a*(mu+s));
ebs := exp(-b*s);
eas := exp(-a*s);
eamu := exp(-a *mu);
ebmu := exp(-b*mu);
esb := exp(-s*b);
lmus := lambda + mu +s;
ls := lambda+s;

e21 := L21 =  eamus * lambda * (L21/lmus) +
(eamus * mu * lambda * LA1)/(lmus * ls ) + eas * (1-eamu) * lambda * LA1/ls ;

solL21 := solve(e21, L21);

ea1 := LA1 =  ebmus * (lambda /lmus) +
(ebmus * mu * lambda * solL21/(lmus * lmus)) +
(ebmus * mu * mu * lambda * LA1) /(lmus* lmus* ls) +
```

```
(ebs * mu*b * ebmu * lambda * solL21)/lmus +
(ebs * mu*b * ebmu * mu*lambda * LA1)/(lmus * ls) +
ebs * (1-ebmu - (mu*b * ebmu))*lambda * LA1 /ls;
solLA1:= solve(ea1, LA1);

df := diff(solLA1, s);
simplify(df);
latex(df);

df0 :=subs(s=0, df);

latex(df0);

simplify(df0);

latex(simplify(df0));

latex(-df0);

simplify(-df0);
latex(simplify(-df0));
```

A.2.3 Maple Code for Finding $E(22)$

```
ebmus := exp(-b*(mu+s));
eamus := exp(-a*(mu+s));
ebs := exp(-b*s);
eas := exp(-a*s);
eamu := exp(-a *mu);
ebmu := exp(-b*mu);
esb := exp(-s*b);
lmus := lambda + mu +s;
ls := lambda+s;

eqA2:=LA2=  ebmus*(lambda/lmus)*LA2
            +(ebmus*(mu/lmus)*(lambda/lmus))
            +(ebmus*(mu/lmus)*(mu/lmus)*(lambda/ls))*LA2
            +(ebs*mu*b*ebmu*(lambda/lmus))
            +(ebs*mu*b*ebmu*(mu/lmus)*(lambda/ls))*LA2
            +(ebs*(1-mu*b*ebmu-ebmu))*(lambda/ls)*LA2;




solLA2:= solve(eqA2,LA2);

eq22:=L22= eamus*(lambda/lmus)+(eamus*(mu/lmus)*(lambda/ls))*solLA2
           +(eas*(1-eamu)*(lambda/ls))*solLA2;
```

```
solL22:= solve(eq22,L22);

df:=diff(solL22,s);
simplify(df);
latex(df);

df0:=subs(s=0,df);
latex(df0);

simplify(df0);
latex(simplify(df0));

latex(-df0);
latex(simplify(-df0));
```