# Connexions and the Gumbel Distribution

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# Outline

- Definition of Gumbel Distribution
- Riemann zeta function
- proof that 0=1
- coupon collector's problem
- Integer partitions

# Definition of Gumbel Distribution

DEFINITION: $X$ has a Gumbel distribution (max) if the pdf of $X$ is $f(x) = e^{-x}e^{-e^{-x}}$.

## Proposition

*(a) The cdf (max) is* $F(x) = e^{-e^{-x}}$*.*
*(b)* $E(X) = \gamma$*, where* $\gamma$ *is Euler's constant,*
$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n\right)$
*(c) Suppose that* $(X_1, X_2, \ldots)$ *is a sequence of independent random variables, each with the standard exponential distribution. The distribution of* $Y_n = max\{X_1, X_2, \ldots, X_n\} - ln(n)$ *converges to the standard Gumbel distribution as* $n \to \infty$*.*

# Definition of Laplace Transform

DEFINITION: The Laplace transform $L(s)$ of a pdf $f(x)$ with positive support is given by

$$L_X(s) = \int_0^\infty e^{-sx} f(x) dx$$

where $s > 0$.

# Catastrophe Process

THEOREM: Let $X$ be a r.v. with positive support and with pdf $f(x)$. Let $Y$ be a r.v. independent of $X$, such that $Y \sim$ exponential with rate $s$. Then

$$L_X(s) = P(X < Y).$$

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- The exponential random variable $Y$ is called the catastrophe.
- The Laplace transform of a p.d.f of a random variable $X$ is the probability that $X$ occurs before the catastrophe.
- More precisely, the Laplace transform of a probability density function $f(x)$ of a random variable $X$ can be interpreted as the probability that $X$ precedes a catastrophe where the time to the catastrophe is an exponentially distributed random variable $Y$ with rate $s$, independent of $X$.

# Laplace Transform of max(exp)

Let $X_1, \ldots, X_n$ be iid exponential r.v with common rate $\lambda = 1$.
Let $M = max X_i$.
Then $L_M(s) = P(M < Y)$ where $Y$ is exp rate $s$, so

$$L(s) = (\frac{n}{n+s})(\frac{n-1}{n-1+s})\ldots(\frac{1}{1+s})$$

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But $prod(LT) = LT(sum)$ so $M = M_1 + \cdots + M_n$ where $M_i$ are indep with $M_i$ exp with rates $n, n-1, \ldots, 1$ respectively.

# $1 + 1/2 + 1/3 + \dots$

$E(M) = E(M_1) + E(M_2) + \cdots + E(M_n) = 1/n + 1/(n-1) + \cdots + 1$
so if $X$ is Gumbel, then
$E(X) = lim(E(M) - ln(n))$
$= lim((1 + 1/2 + 1/3 + ... + 1/n) - ln(n)) = \gamma$

# More on $1 + 1/2 + 1/3 + \ldots$

(a) $1 + 1/2 + 1/3 + ... + 1/n$ is the sum of the first $n$ terms of the harmonic series (which diverges).
(b) Although $1 + 1/2 + 1/3 + ...$ diverges, the partial sums (other than the first one) are never equal to an integer.
(b) The Riemann zeta function is $\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s}$.

# Poem on Riemann

Where are the zeros of zeta of s?
...
https://www.math.upenn.edu/ pemantle/songs/zeta.new
Poem by Tom Apostol on the Riemann hypothesis.

# Proof that 0=1

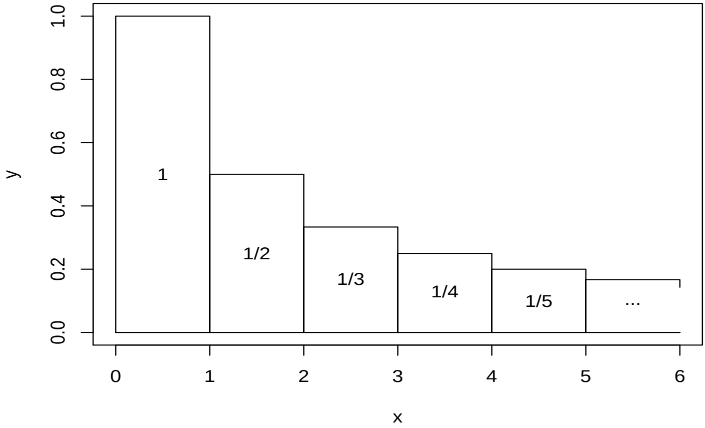
We next give a fallacious proof that $0 = 1$ based on a geometric consideration of the harmonic series

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

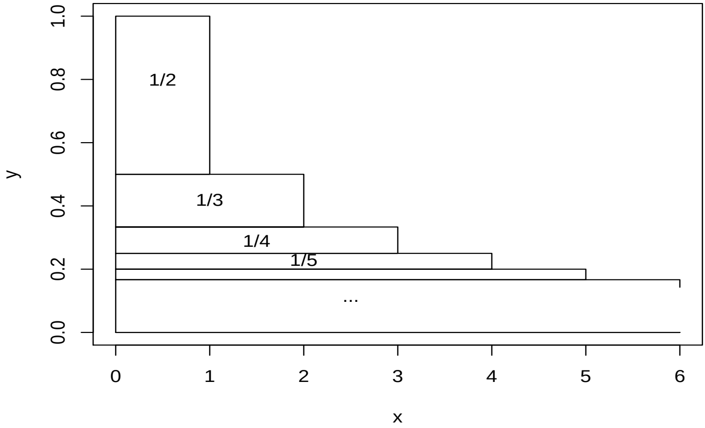
Let $1 + S = 1 + (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots)$

The area $1 + S$ appears in the following plot with vertical strips, followed by another with horizontal strips.

# 0=1



# 0=1



From the diagram , we have that

$$1 + S = Area(Figure1) = Area(Figure2) = \frac{1}{2} + \frac{1}{3} + \cdots = S$$

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Since this is false, it follows that the harmonic series diverges.

# coupon collector's problem

Let $\{X_i | i \in N\}$ be iid random variables uniformly distributed on $\{1, 2, 3, .., N\}$. Let $T_N$ be the smallest index $n$ such that $\{1, ..., N\} \subset \{X_1, \ldots, X_n\}$ , i.e. the least number of trials such that all $N$ coupons have been obtained. Then $\dfrac{T_N - Nlog(N)}{N}$ converges to the Gumbel distribution.

-from http://math.stackexchange.com/questions/563797

# Coupon collector

There are N different coupons to be collected, uniformly distributed, one per package.
What is the expected number of packages that must be opened in order to collect all N coupons?
SOLUTION:
The first package, with prob 1, gives a new coupon. The 2nd package, with prob (n-1)/1 gives a new coupon. So expected number of extra steps to get second coupon is the reciprocal, n/(n-1).
Similarly get n/(n-2) extra steps for 3rd coupon.
Expected total number of steps is $1 + n/(n-1) + n/(n-2) + \cdots + n/1$
$E(number) = n(1 + 1/2 + 1/3 + \cdots + 1/n)$

# Partitions and R

Let $p(n)$ be the number of partitions of $n$.
e.g.
$5 = 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$
so $p(5) = 7$
Let $p_N(n)$ be the number of partitions of $n$ with all parts $\leq N$
e.g. $p_3(5) = 5$
The following is from J.Roccia, P. Leboeuf.
$\frac{p_N(n)}{p(n)} \approx e^{-e^{-g}}$ where $g = \sqrt{\pi^2/(6n)}N + 1/2 ln(\pi^2/(6n))$ and
$N = o(n^{1/3})$

From Wagner

Theorem 2. Let $R_{r,n}$ be the largest integer in a random partition of $n$ that occurs at least $r$ times. Then the normalised random variable $n^{-1/2}(R_{r,n} - \frac{\sqrt{6n}}{2\pi r} logn)$ tends to a Gumbel distribution with mean $\frac{\gamma - log(\pi r/\sqrt{6})}{\pi r/\sqrt{6}}$ and variance $1/r^2$.

# Generating Random Partitions and R

```
a=rbinom(9,1,.5)
a
1 0 1 0 0 1 1 1 0
b=which(a==1)
b
1 3 6 7 8
d=c(0,b)
d
0 1 3 6 7 8
e=c(b,10)
e
1 3 6 7 8 10
f=sort(e-d)
f
1 1 1 2 2 3
```

```
g=as.data.frame(table(f))
g
f Freq
1 1 3
2 2 2
3 3 1
h=g[,2]
h
3 2 1
k=factorial(sum(h))/prod(factorial(h))
k
60
```

```
q=c();for(i in 1:100000)
{ a=rbinom(19,1,.5);b=which(a==1);d=c(0,b);e=c(b,20);
f=sort(e-d);g=as.data.frame(table(f));h=g[,2];
k=factorial(sum(h))/prod(factorial(h));
if(runif(1)<1/k) {q=c(q,1+sum(a))}}; hist(q)
```

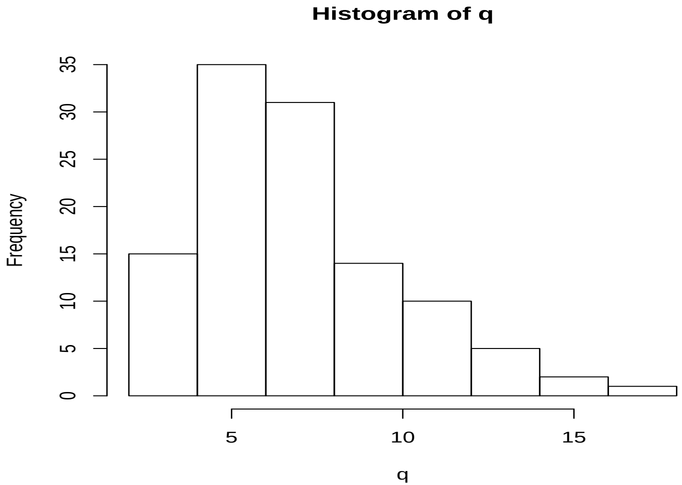


Figure: histogram of number of parts

# A better algorithm

```
n=20
x=exp(-pi/sqrt(6*n))
for(i in 1:1000) {z=rgeom(n,x);if (sum((1:n)*z)==n){print(z)}}
3 1 0 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0
1 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0
.
```

Source: Fristedt, The Structure of Random Partitions of Large Integer (1993).

# Partitions and Maple

Countng partitions is easy with generating functions.

$f(n) := \prod_{k=1}^{n} \left(1 + \sum_{i=1}^{\text{floor}(n/k)} x^{ki}\right)$

f(75);

simplify(%);

expand(%);

$... + 8118264x^{75} + 7089500x^{74} + ...7 * x^5 + 5 * x^4 + 3 * x^3 + 2 * x^2 + x + 1$

.

Thus $p(75) = 8117264$

.

We can also obtain partitions restricted to parts less than or equal to $m$ for $m < n$. $f(n) := \prod_{k=1}^{m} \left(1 + \sum_{i=1}^{\text{floor}(n/k)} x^{ki}\right)$

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4. P.Erdos and J.Lehner. The distribution of the number of summands in the partitions of a positive integer. Duke Math. J., 8:335345, 1941.
5. S. Wagner. Limit Distributions of smallest gap and largest repeated part in integer partitions. The Ramanujan Journal 25/2 (2011), 229-246.
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