A Queueing Theorist Looks at MCMC

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Outline

- Usual MarkovChainMonteCarlo
- Fibonacci Distribution
- Birth and Death process
- Markov chain matrix
- Output, Graphs and Comments

Usual MCMC

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- We can use P to simulate values of $\vec{\pi}$

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- I would not have thought of R-H.

Fibonacci Distribution

The Fibonacci distribution has values $f(3) = f(2) = 1/4$.
In general $f(n) = F_{n-1}/2^n$ for $n = 2, 3, \ldots,$
where $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}$
so $\{F_i\} = 1, 1, 2, 3, 5, 8, 13, \ldots$
It turns out that $f(n)$ is the probability that exactly n steps are required for a random walk to reach state 2 from state 0 if the system moves to the right with probability .5 on each step and moves left (or remains the same at 0) with probability .5

Fibonacci and MCMC I

Suppose we wish to simulate observations from the Fibonacci distribution using MCMC. First we compute the ratio of consecutive Fibonacci probabilities.

$$\frac{f(x+1)}{f(x)} = \frac{F_x}{2^{x+1}} \Big/ \frac{F_{x-1}}{2^x} = \frac{F_x}{2F_{x-1}}. \tag{1}$$

Fibonacci and MCMC II

$$2F_{x-1}f(x+1) = F_x f(x).$$

We take this equation to be the balance equation for a continuous time Markov process. The LHS is the rate from state $x+1$ to state x and is the product of the limiting probability $f(x+1)$ of being in state $x+1$ times the rate $2F_{x-1}$ of moving to state x given that the system is in state $x+1$. The RHS is the rate from state x to state $x+1$ which is the product of the limiting probability $f(x)$ of being in state x times the rate F_x of moving to state $x+1$ given that the system is in state x.

Fibonacci and MCMC III

Our birth and death transition diagram looks like

$$\begin{array}{ccccccccc} & F_2 & & F_3 & & F_4 & & F_5 & \\ & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & \\ 2 & & 3 & & 4 & & 5 & & 6 \cdots \\ & \leftarrow & & \leftarrow & & \leftarrow & & \leftarrow & \\ & 2F_1 & & 2F_2 & & 2F_3 & & 2F_4 & \end{array}$$

Fibonacci and MCMC IV

$$2 \underset{\frac{2F_1}{F_2+2F_1}}{\overset{\frac{F_2}{F_2+2F_1}}{\rightleftarrows}} 3 \underset{\frac{2F_2}{F_3+2F_2}}{\overset{\frac{F_3}{F_3+2F_2}}{\rightleftarrows}} 4 \underset{\frac{2F_3}{F_4+2F_3}}{\overset{\frac{F_4}{F_4+2F_3}}{\rightleftarrows}} 5 \underset{\frac{2F_4}{F_5+2F_4}}{\overset{\frac{F_5}{F_5+2F_4}}{\rightleftarrows}} 6 \cdots$$

Fibonacci and MCMC V

The corresponding infinitesimal generator matrix for states $2, 3, \ldots$ (with the pairs of rates appearing in the off-diagonal positions) is

$$Q = \begin{bmatrix} a & \frac{F_2}{F_2 + 2F_1} & 0 & 0 & 0 & \ldots \\ \frac{2F_1}{F_2 + 2F_1} & b & \frac{F_3}{F_3 + 2F_2} & 0 & 0 & \ldots \\ 0 & \frac{2F_2}{F_3 + 2F_2} & c & \frac{F_4}{F_4 + 2F_3} & 0 & \ldots \\ 0 & 0 & \frac{2F_3}{F_4 + 2F_3} & d & \frac{F_5}{F_5 + 2F_4} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $a, b, c, d, \ldots$ are negative values chosen so that each row sums to 0.

Fibonacci and MCMC VI

$\vec{0} = \vec{\pi}Q$. Let $Q^* = Q/2$. Then $\vec{0} = \vec{\pi}Q^*$. All entries of Q^*, excluding the diagonal entries, are less than .5 in absolute value.
Next we add $\vec{\pi}$ to both sides to get P satisfying $\vec{\pi} = \vec{\pi}(I + Q^*) = \vec{\pi}P$.
Here $P = I + Q^*$ so

$$P = \begin{bmatrix} 1+.5a & \frac{.5F_2}{F_2+2F_1} & 0 & 0 & 0 & \dots \\ \frac{F_1}{F_2+2F_1} & 1+.5b & \frac{.5F_3}{F_3+2F_2} & 0 & 0 & \dots \\ 0 & \frac{F_2}{F_3+2F_2} & 1+.5c & \frac{.5F_4}{F_4+2F_3} & 0 & \dots \\ 0 & 0 & \frac{F_3}{F_4+2F_3} & 1+.5d & \frac{.5F_5}{F_5+2F_4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Fibonacci and MCMC VII

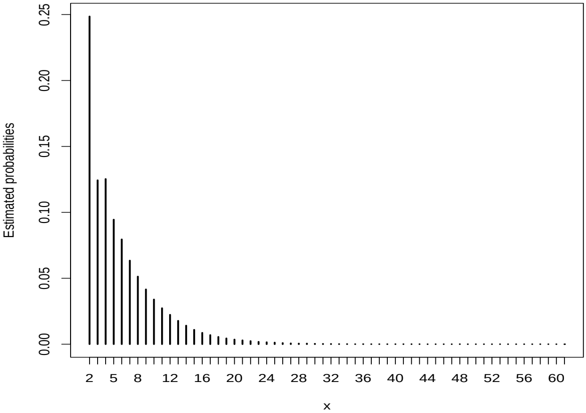
$$P \approx \begin{bmatrix} .833 & .167 & 0 & 0 & 0 & \dots \\ .333 & .417 & .25 & 0 & 0 & \dots \\ 0 & .25 & .536 & .214 & 0 & \dots \\ 0 & 0 & .286 & .48 & .227 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

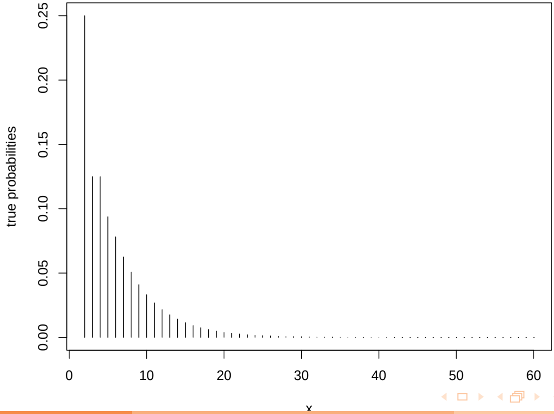
R program

```
x=rep(2,1000000);
#This is the start vector with all  values 2
u=runif(1000001);
#This generates 1000001 random uniform (0,1) values.
F=rep(1,150)
for (i in 4:150) {F[i]=F[i-1]+F[i-2]}
#generate first 150 Fibonacci numbers
for (i in 1:1000000)
{
a=(x[i]>2)*(u[i+1]<F[x[i]-1]/(F[x[i]]+2*F[x[i]-1]))
b=+1*(u[i+1]>(1-(.5*F[x[i]+1]/(F[x[i]+1]+2*F[x[i]]))))
x[i+1]=x[i]-a+b
            }
```

R output

```
x[1:100]
  [1] 2 2 2 2 2 2 2 3 4 3 3 3 2 2 2 2 2 3 3 2 2 2 2 2
 [28] 2 2 2 2 3 2 3 3 4 4 3 3 2 2 2 2 2 2 2 2 2 2 2 2
 [55] 4 4 4 5 5 4 5 5 6 5 6 6 5 5 4 4 5 5 5 4 4 5 5 5
 [82] 4 4 4 3 2 2 2 2 3 4 4 4 4 3 3 2 2 2 2
```


Estimated probabilities
0.00
0.05
0.10
0.15
0.20
0.25
2
5
8
12
16
20
24
28
32
36
40
44
48
52
56
60
x



true probabilities
0.00
0.05
0.10
0.15
0.20
0.25
0
10
20
30
40
50
60
x

The End.

Thank you.