Busy Period M/M/*/* Laplace Transforms

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Outline

- Probabilistic Interpretation of Laplace Transform
- Busy Period

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- M/M/c models

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Probabilistic interpretation of Laplace Transforms

DEFINITION: The Laplace transform $L(s)$ of a function $f(x)$ with positive support is given by

$$L_X(s) = \int_0^\infty e^{-sx} f(x)dx \text{ where } s > 0.$$

THEOREM: Let X be a r.v. with positive support and with pdf $f(x)$. Let Y be a r.v. independent of X, such that $Y \sim$ exponential with rate s. Then

$$L_X(s) = P(X < Y).$$

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- The exponential random variable Y is called the catastrophe.
- The Laplace transform of a p.d.f of a random variable X is the probability that X occurs before the catastrophe.

Gross and Harris

Define an i channel busy period for an $M/M/c$ system ($1 \leq i \leq c$) to begin with an arrival to a system with $i - 1$ and end at the next point in time when the system dips to $i - 1$.
Let $T_{b,i}$ be the time length of the i-channel busy period.
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”Proceeding further at this point would get us bogged down...”
“Any resultant cdf would be in terms of modified Bessel functions ...”

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MH: The most interesting cases are $i = 1$ and $i = c$.
We focus on the $i = 1$ case.

M/M/c case

$M/M/c$ represents a system where arrivals folllow a Poisson process at rate λ and form a single queue, there are c servers and service times per server are exponentially distributed at rate μ.

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M/M/1 model

Let $L_1(s)$ be the Laplace transform for the busy period of an M/M/1 queueing system. Then

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s}(L_1(s))^2$$

M/M/2 model

Let $L_1(s)$ be the Laplace transform for the busy period of an $M/M/2$ queueing system. Let $L_2(s)$ be the probability that a busy period of an $M/M/2$ system, which begins with two customers, will end (reach 0) before a catastrophe. Let $M_{2,1}(s)$ be the probability that the $M/M/2$ system drops from 2 customers to 1 customer before a catastrophe. Let $M_{3,2}(s)$ be the probability that the $M/M/2$ system drops from 3 customers to 2 customer before a catastrophe. Let λ and μ be the arrival and service rates. Then

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$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_2(s) \quad (1)$$

$$L_2(s) = M_{2,1}(s) L_1(s) \quad (2)$$

M/M/2 continued

$$M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{3,2}(s) M_{2,1}(s) \quad (3)$$

$$M_{3,2}(s) = M_{2,1}(s) \quad (4)$$

So we can get:

$$M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{2,1}(s)^2 \quad (5)$$

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{21}(s) L_1(s) \quad (6)$$

M/M/c, $c > 2$

From above we can find the Laplace transform of the $M/M/c$ busy period:

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_2(s) \tag{7}$$

$$L_2(s) = M_{2,1}(s) L_1(s) \tag{8}$$

$$M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{3,2}(s) M_{2,1}(s) \tag{9}$$

$$M_{3,2}(s) = M_{2,1}(s) \text{ is now false for } c \geq 3 \tag{10}$$

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$$M_{3,2}(s) = \frac{3\mu}{\lambda + 3\mu + s} + \frac{\lambda}{\lambda + 3\mu + s} M_{4,3}(s) M_{3,2}(s) \tag{11}$$

M/M/c, $c > 2$

$$M_{c+1,c}(s) = M_{c,c-1}(s)$$

$$M_{c,c-1}(s) = \frac{c\mu}{\lambda + c\mu + s} + \frac{\lambda}{\lambda + c\mu + s} M_{c+1,c}(s) M_{c,c-1}(s)$$

$$M_{c-1,c-2}(s) = \frac{(c-1)\mu}{\lambda + (c-1)\mu + s} + \frac{\lambda}{\lambda + (c-1)\mu + s} M_{c,c-1}(s) M_{c-1,c-2}(s)$$

$$\cdots$$

M/M/c, $c > 2$

$$M_{c,c-1}(s) = \frac{\lambda + c\mu + s - \sqrt{(\lambda + c\mu + s)^2 - 4c\lambda\mu}}{2\lambda}$$

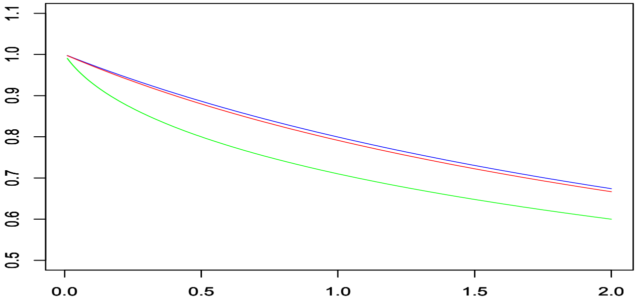
$$M_{c-1,c-2}(s) = \frac{(c-1)\mu}{\lambda + (c-1)\mu + s - \lambda M_{c,c-1}(s)}$$

$$M_{c-2,c-3}(s) = \frac{(c-2)\mu}{\lambda + (c-2)\mu + s - \lambda M_{c-1,c-2}(s)}$$

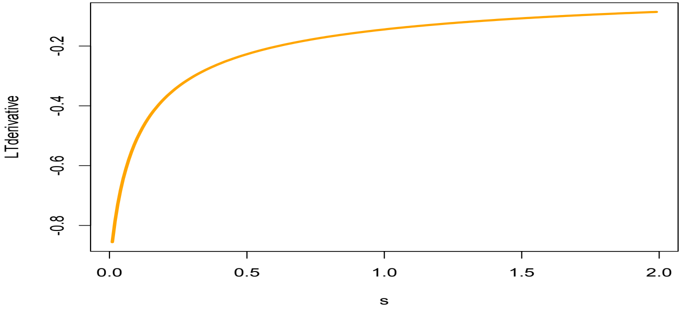
$$\vdots$$

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}$$

LT plots, lambda=5, mu=6, c=1,2,3



LT derivative, lambda=5, mu=6, c=1



The slope of LT at 0 gives the negative of the expected busy period time length. $E(B)$ The slope of the derivative of LT at 0 gives the second moment $E(B^2)$.

M/M/1/k

For M/M/1/k, arrivals follow a Poisson process at rate λ, there is one server, service times are exponentially distributed with rate μ, and the maxmum number of customers in the system, including customer in service, is k. We seek the LT of the busy period.

M/M/1/1

$L_B(s) = P(BusyPeriodEndsBeforeCatastrophe) =$
$P(ServiceEndsBeforeCatastrophe) = \mu/(\mu + s)$.

M/M/1/k

$$L_1(s) = \frac{\mu}{\lambda+\mu+s} + \frac{\lambda}{\lambda+\mu+s} M_{21}(s) L_1(s)$$

$$M_{21}(s) = \frac{\mu}{\lambda+\mu+s} + \frac{\lambda}{\lambda+\mu+s} M_{32}(s) M_{21}(s)$$

$$\vdots$$

$$M_{k-1,k-2}(s) = \frac{\mu}{\lambda+\mu+s} + \frac{\lambda}{\lambda+\mu+s} M_{k,k-1}(s) M_{k-1,k-2}(s)$$

$$M_{k,k-1}(s) = \frac{\mu}{\mu+s}$$

M/M/1/k

Solving

$$M_{k,k-1}(s) = \frac{\mu}{\mu + s}$$

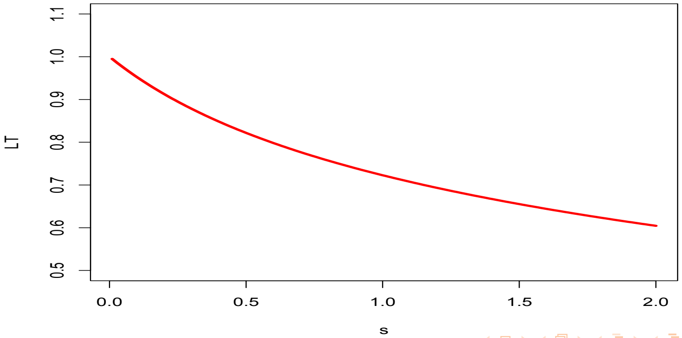
$$M_{k-1,k-2}(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{k,k-1}(s)}$$

$$\vdots$$

$$M_{21}(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{3,2}(s)}$$

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}$$

LT Busy Period M/M/1/4, lambda=5, mu=6



M/M/c/c

For M/M/c/c, arrivals follow a Poisson process at rate λ, there are c servers, service times are exponentially distributed with rate μ per server, and the maxmum number of customers in the system, including customer in service, is c. We seek the LT of the busy period.

M/M/1/1

$L_B(s) = \mu/(\mu + s)$.

M/M/c/c, $c \geq 3$

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{21}(s) L_1(s)$$

$$M_{21}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{32}(s) M_{21}(s)$$

$$\vdots$$

$$M_{c-1,c-2}(s) = \frac{(c-1)\mu}{\lambda + (c-1)\mu + s} + \frac{\lambda}{\lambda + (c-1)\mu + s} M_{c,c-1}(s) M_{c-1,c-2}(s)$$

$$M_{c,c-1}(s) = \frac{c\mu}{c\mu + s}$$

M/M/c/c

Solving

$$M_{c,c-1}(s) = \frac{c\mu}{c\mu + s}$$

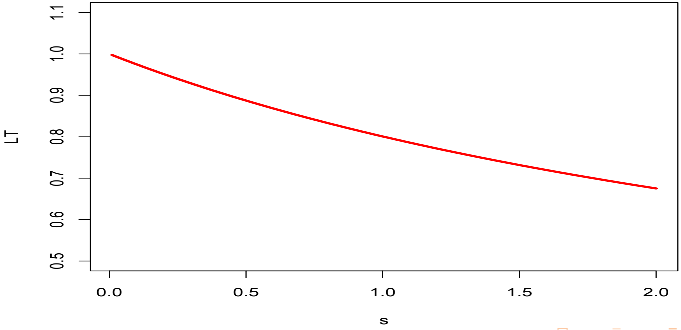
$$M_{c-1,c-2}(s) = \frac{(c-1)\mu}{\lambda + (c-1)\mu + s - \lambda M_{c,c-1}(s)}$$

$$\vdots$$

$$M_{21}(s) = \frac{2\mu}{\lambda + 2\mu + s - \lambda M_{3,2}(s)}$$

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}$$

LT Busy Period M/M/4/4, lambda=5, mu=6



The End.
Thank you!