Simulation of Queues with Arrivals Before Opening Time

By

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Abstract

In this study, we firstly derived the theoretical time-dependent distribution of the number of customers in the M/M/1 system with arrivals before opening time. Then, simulation methods were used to study the time-dependent distribution of the number of customers in three models, which are a regular M/M/1 system, an M/M/1 system with arrivals before opening time and an M/M/3 system with arrivals before opening time. We compared the simulation results with analytic results in the regular M/M/1 case. We compared the simulation results of an M/M/1 system with arrivals before opening time to those of an M/M/3 system with arrivals before opening time.

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Chapter 1 Introduction

Section 1.1 Simulation and numerical analysis

Simulation and numerical analysis are almost as old as human civilization. A major impetus to developing numerical procedures was the invention of the calculus by Newton and Leibnitz, as this led to accurate mathematical models for physical reality, engineering, medicine, and business. These mathematical models cannot usually be solved explicitly, and numerical methods to obtain approximate solutions are needed [1]. Furthermore, in the real world, many complex stochastic processes may not be modeled exactly or easily. Simulation may be an efficient method to study these sophisticated processes.

The invention of the computer gave rise to the development and wide application of simulation and numerical analysis. Applications include atmospheric modeling, operational research and investment strategy. In an atmospheric study, these techniques are used to simulate the behavior of the Earth's atmosphere, to understand the possible effect of human activities on our atmosphere. A large number of variables need to be introduced. Many types of numerical analysis procedures are used in atmospheric modeling, including computational fluid mechanics and the numerical solution of differential equations. The system is so complex that the explicit solution is not available. Based on high speed computing techniques, methods of simulation and numerical analysis will result in a real-time forecast of climate and weather[2]. In operational research, simulation is widely used in allocation of resources, supply chain management and quality control. It may help managers make better decisions[3].

In the financial market, stock and option prices depend heavily on macroeconomic and microeconomic factors, such as the interest rate, monetary policies, employment rate, financial statements, and so on. Since the financial market undergoes apparently stochastic fluctuations, simulation is a natural tool. Researchers try to forecast these possible changes and assess the possible effects of these changes on stock indices and stock prices. In these cases, computer simulation may be the unique choice. Monte Carlo methods are widely used. In addition, other numerical algorithm may be adopted. The method used is determined by accuracy and speed. Ultimately, the method must give quick and accurate answers to real-time problems in financial market. These answers can help managers and stock traders adjust their portfolios and trading strategy[4]. Efficiency and reliability are key demands when methods of simulation and numerical analysis are employed.

Fortran, C, C++ and Java are popular computer programming languages. Matlab is the most widely used software package for simulation and numerical analysis[5]. It includes many toolboxes, which may be directly used to solve problems in the financial market, engineering and communication. Maple is another useful software package.

Section 1.2 Application of simulation techniques in queue problems

Simply, if a phenomenon is concerned with “lines”, “servers” or “waiting”, then it may be classified as a queueing problem. In real life, gas station flow is a queueing problem; the cars are in a line, and gas pumps are servers. In a grocery store, cashiers are servers. Similar situations exist in banks or emergency departments of hospitals. A recent application of queueing theory is the study of world wide web. The retrieval of information is another example of a queueing problem[6].

Briefly speaking, queueing systems include two factors: one is commodity flow, such as customers in a bank and a retrieval request in a computer network; the other is the server (or servers). Based on the commodity flow, a queueing system is categorized into steady state flow and transient flow. In steady state flow, the system is in an equilibrium state. In this situation, time is not taken into account when the system is analyzed. In transient flow, the characteristics of the system are concerned with time. The randomness, unpredictability, or the unsteady nature of this flow gives considerable complexity to the solution and understanding of such problems. When time approaches infinity, transient flow may become steady state flow. For example, in this study, the queue is transient at the moment the server(s) open(s), but as time evolves, the system tends to steady state.

When queueing problems are studied, the distributions of interarrival and service times are often given via the arrival rates and service rates. Performance measures obtained include the expected waiting time, the expected number of customers in the system, and the distribution of the number of customers in the system. The relation between the expected values and the distributions of arrival and departure is also important. However, in most complex systems, we cannot build a clear and simple relation between the expected values and the parameters of the distributions. Therefore, methods of simulation have to be employed.

In this paper, some systems with arrivals before opening time are studied via simulation techniques. The situation of arrivals before opening time exists widely, for example, customers arrive before a bank opens. These types of systems involve transient flow. So the expected values and the distribution of the number of customers in the system depend on time. The distribution of the number of customers is a critical issue. This determines the expected values. Therefore, we will concentrate on the study of this distribution. For every model, the distribution of the number of customers in the system at different time points is reported. As time evolves, the system approximates the steady-state system. Comparing the distribution from simulation with the one from theoretical study at steady state, we may check whether our simulation is good, and determine when the system may be regarded to be in steady state. The evolution of the system from transient to steady state is another interesting issue, that is, from chaos to order.

Chapter 2 Models and theoretical distributions

In this chapter, three models will be studied. The chapter includes a description of models and the theoretical distribution of the number of customers in the system at some time points. The three models are (1) regular M/M/1 model; (2) M/M/1 model with arrivals before opening time; (3) M/M/3 model with arrivals before opening time.

Section 2.1 Regular M/M/1 model

For this model, the system has one server and an unlimited waiting room. The interarrival and service time are exponential with a constant arrival rate r_a and a constant service rate r_d. At the opening time (t=0), no arrival is in the system. We study the process from transient to steady state. Therefore, we do not take closing of the system into account[6]. The server follows the " first comes, first served" policy.

Let $P_k(t)$ be the probability of k customers in the system at time t. Then the differential-difference equations of the regular M/M/1 model are:

$$\frac{dP_k(t)}{dt} = -(r_a + r_d)P_k(t) + r_a P_{k-1}(t) + r_d P_{k+1}(t) \qquad k \geq 1$$

$$\frac{dP_0(t)}{dt} = -r_a P_0(t) + r_d P_1(t) \qquad k=0 \tag{2.1}$$

Initial condition: $P_0(0)=1$

These differential-difference equations may be solved analytically by many methods[6],[7] or the numerical solution may be obtained via computer numerical analysis. The explicit formula is given below.

$$P_k(t) = e^{-(r_a + r_d)t}[\rho^{\frac{(k)}{2}} I_k(\alpha t) + \rho^{\frac{(k-1)}{2}} I_{k+1}(\alpha t) + (1-\rho)\rho^k \sum_{j=k+2}^{\infty} \rho^{\frac{(-j)}{2}} I_j(\alpha t)]$$

(2.2)

where

$$\rho = \frac{r_a}{r_d} \quad (2.3)$$

$$\alpha = 2 r_d \rho^{\frac{1}{2}} \quad (2.4)$$

$$I_k(x) \equiv \sum_{m=0}^{\infty} \frac{(x/2)^{k+2m}}{(k+m)!m!} \qquad k \geq -1 \quad (2.5)$$

and here $I_k(x)$ is the modified Bessel function of the first kind of order k.

When the above equations are employed to calculate the distribution of the number of customers in the system at time t, no condition is imposed on ρ. However, if ρ is greater than 1, that is, the arrival rate is larger than the service rate. Then as time approaches infinity, the limiting probabilities will be zero for any finite k. In that case, the modified Bessel function will diverge. If ρ is less than 1, the Bessel function always converges. The steady state probabilities are non zero and sum to 1. Therefore, when time moves from a finite value to infinity, the system evolves to steady state flow from transient flow. In steady state, the distribution of the number of customers in the system and other expected values are independent of time. The theoretical distribution for the steady state probability of k customers in the system is:

$$P(k) = \rho^k \times (1-\rho) \quad . \quad (2.6)$$

where k=0, 1, 2…., and ρ is defined in equation 2.3.

For the regular M/M/1 model, the distribution of the number of customers in the system is simple when the system is in equilibrium. The time-dependent distribution is complex. For more complex queue models, the explicit formula for the time-dependent probability may be not available. This is why the simulation method is used.

Section 2.2 M/M/1 model with arrivals before opening time

In the M/M/1 model with arrivals before opening time, the time when the server starts to work is defined as the time origin (t=0). However, some customers arrive at the system and wait in the line before the server opens. Therefore, when the server begins to work, there may already be some customers in the system. Service follows the "first comes, first served" policy.

Assume that there are i customers in the system at t = 0. Then the differential-difference equation 2.1 is still correct, but the initial condition changes. The new initial condition is k = i at t = 0, i=0,1,2... By solving equation 2.1 again, the time-dependent distribution can be shown below[6]:

$$P_k(t|i) = e^{-(r_a+r_d)t}[\rho^{\frac{(k-i)}{2}} I_{k-i}(\alpha t) + \rho^{\frac{(k-i-1)}{2}} I_{k+i+1}(\alpha t) + (1-\rho)\rho^k \sum_{j=k+i+2}^{\infty} \rho^{\frac{(-j)}{2}} I_j(\alpha t)] \quad (2.7)$$

Where $P_k(t|i)$ is the conditional probability that there are k customers in the system at time t, given i customers in the system at t = 0.

Property: Let p(i) be the probability of i customers in the system at time 0. Then the unconditional probability of k customers in the system at t (t ≥ 0)[7] is :

$$P_k(t) = \sum_{i=0}^{\infty} p_k(t|i) \times p(i) \quad (2.8)$$

In our study, we assume arrivals before opening time follow a nonhomogenous Poisson process. Therefore, the distribution of the number of arrivals before opening time[7] is :

$$p(i) = \frac{(m)^i}{i!} e^{-m} \qquad i= 0, 1, 2\ldots \tag{2.9}$$

where

$$m = \int_{-\infty}^{0} \lambda(t)dt \tag{2.10}$$

We assume $\lambda(t)$ has an exponential form. It is common in many real life situations that many customers arrive just before opening time with a few arriving earlier. Thus we choose

$$\lambda(t) = e^{0.05\times t} \tag{2.11}$$

although other choices can be made. Therefore, in our case

$$m = \frac{1}{0.05} = 20 \tag{2.12}$$

For practical simulation, we choose the time limit prior to opening to be –200 to 0, so,

$$m(t) = \int_{-200}^{0} \lambda(t)dt = 20(1 - e^{-10t}) \tag{2.13}$$

As discussed in section 2.1, the above equations may be used to calculate the distribution of the number of customers in the system at any time. However, the after-opening arrival rate has to be less than the service rate. Then the Bessel function will always converge. Shortly after opening time, the effect of any arrivals before opening time is obvious. As time approaches infinity, $p_k(t|i)$ approximates $p(k)$. So for large t, an M/M/1 system with arrivals before opening time behaves the same as a regular M/M/1. Equation 2.6 will be applied to two M/M/1 models.

Section 2.3 M/M/3 model with arrivals before opening time

This model has three servers and one unlimited waiting room. The three service rates are constants; they may be same or different. Customers may arrive at the system before the servers start to work. Arrivals before opening time follow a nonhomogeneous Poisson process, exactly the same as described by equations 2.9 to 2.13. As in the M/M/1 process, the three servers follow a " first comes, first served" policy.

For the M/M/3 system, the time-dependent distribution of the number of customers in the system is not available in the literatures, even for the system beginning with 0 customers at t=0, as the differential-difference equations are complex. For the steady state system, the distribution is not available when three service rates are different and rules must be chosen as to which server to join. The steady state distribution with three same service rates may be derived easily[6].

Assume r_a is the arrival rate after the opening time, and r_d is the service rate of each server. So, the total service rate R_k is:

$R_k = \min[kr_d, 3r_d]$ where k = 0, 1, 2, 3… is the number of customers in the system

The theoretical distribution of the number of customers in the steady system is:

$$p_k = p_0 \frac{(3\rho)^k}{k!} \qquad k \le 3, \qquad (2.14)$$

$$p_k = p_0 \frac{(3)^3 \rho^k}{3!} \qquad k \geq 3, \tag{2.15}$$

where

$$\rho = \frac{r_a}{3r_d} < 1 \tag{2.16}$$

$$p_0 = [\sum_{k=0}^{2} \frac{(3\rho)^k}{k!} + \frac{(3\rho)^3}{3!}(\frac{1}{1-\rho})]^{-1} \tag{2.17}$$

In some special cases, although the three service rates are different, the theoretical steady state distribution of the number of customers in the system may be found. For example, if two of the service rates are 0, that is, two corresponding servers close and the system is actually an M/M/1 model. Therefore, the theoretical distribution is described by equation 2.6.

Section 2.4 Summary

In this chapter, three models were described. The time-dependent theoretical distributions of the number of customers are available for the two M/M/1 models. However, the explicit forms are complex. So, it is not easy to use them in practical situations. The time-dependent theoretical distribution is not available for the M/M/3 model with arrivals before opening time. Therefore, a simulation method and numerical analysis are employed. They will be studied in next chapter.

Chapter 3 Simulation algorithms

In this chapter, some basic simulation methods will be introduced. In addition, these techniques will be employed to simulate the two M/M/1 models and the M/M/3 model with arrivals before opening time processes.

Section 3.1 Simulation of an exponential distribution

Let the rate be r. Then the pdf of an exponential distribution is[7],[8]:

$$p(t) = re^{-rt} \qquad t \geq 0 \tag{3.1}$$

Therefore, the corresponding cdf is:

$$F(t) = 1 - e^{-rt} \qquad t \geq 0 \tag{3.2}$$

From the equation 3.2, we may derive :

$$t = -\frac{\ln(1-F)}{r} \tag{3.3}$$

Since the domain of F is [0, 1], F may be directly generated from random number U(0,1), and the corresponding t is generated based on equation 3.3. Since (1-F) is a random number in the range of [0, 1], therefore, equation 3.3 can be replaced by

$$t = -\frac{\ln(F)}{r} \tag{3.4}$$

The algorithm for generating an exponential distribution is:

Step 1: Generate the random number u in [0,1].

Step 2: set $t = -\frac{\ln(u)}{r}$.

Recall that the t in equation 3.4 and step 2 of the above algorithm is the interval between two consecutive events. If continuous time is required, we replace the step 2 above by the one below:

Step 2 : $t = t_{current} - \frac{\ln(u)}{r}$

The above algorithm is used to obtain the arrival times after opening time and departure times.

Section 3.2 The rejection method and simulation of nonhomogenous Poisson process

Often, if the random variable X has density function f(x), but there is not the simple relation between the CDF and x like equation 3.2. Therefore, the variable X with density function f(x) may not be generated directly. On the other hand, suppose the random value x has a density function g(x) that is easy to be obtain. We can create the random value x with density f(x) by generating y from function g and then accepting this generated value with a probability proportional to f(y)/g(y). Specially, let c be a constant such that

$$\frac{f(y)}{g(y)} \leq c \quad \text{for all y} \qquad (3.5)$$

The above technique is called the rejection method[8]. The algorithm is :

Step 1: Create Y having density g, using a random number u_1.

Step 2: Create a random number u_2.

Step 3: If $u_2 \le \dfrac{f(y)}{cg(y)}$, set x=y. Otherwise, return to step 1.

The rejection method and the above algorithm may be employed to generate a random value x from a nonhomogeneous Poisson process[8].

A homogeneous Poisson process assumes stationary increments at rate r, and the probability with k events in a unit time is:

$$p(k) = \frac{r^k e^{-r}}{k!} \tag{3.6}$$

The relation between the Poisson process and the corresponding exponential distribution is used to generate random value k by considering the corresponding exponential distribution. Suppose the events occurring in time range (0, T) are collected, where, T ≥ 0. We may successively generate the interarrival times, and stop when their sum exceeds T. For the case, -T ≤ t ≤ 0, the algorithm is described below:

Step 1: Set the initial condition: t =0 and k=0.

Step 2: Produce a random number u.

Step 3: $t = t + \dfrac{\ln(u)}{r}$. If | t |> T, stop. (Note: in this step, t ≤ 0, events occur at [-T,0]; when $t = t - \dfrac{\ln(u)}{r}$, t ≥ 0, events occur at [0,T]).

Step 4: k = k +1, S(k) = t.

Step 5: Go to Step 2.

In Chapter 2, the nonhomogeneous Poisson process was mentioned. The nonhomogeneous Poisson process is an extremely important counting process for modeling purposes, because it allows the rate to vary with time. In our study, it is employed to model the arrivals before opening time, for the arrival rate may increase as we approach the opening time. This kind of process is complex; therefore, it is not easy to derive explicit and analytical results. Simulation methods will be used to analyze such models.

Suppose the arrival rate for a nonhomogeneous Poisson process is defined by r(t), and we want to simulate the first T time units of this process. We may select a constant r such that

$$r(t) \le r \quad \text{for all } -T \le t \le 0 \tag{3.7}$$

So, we consider a homogeneous Poisson process with a constant rate r. We know how to simulate the homogeneous process. Therefore, the nonhomogeneous process may be simulated by the rejection method. The algorithm is:

Step 1: Set t = 0 and k = 0.
Step 2: Generate a random number u.
Step 3: $t = t + \frac{\ln(u)}{r}$, if | t | > T, stop. (Note: in this step, t ≤ 0, since events occur at [-T,0]).

Step 4: Generate a random number v.
Step 5: If $v \le \frac{r(t)}{r}$, set k = k + 1, S(k)=t.
Step 6: Go to step 2.

The final k represents the number of events in the period from -T to 0. The vector S = (S(1), S(2)….) records the time when the event occurs.

Section 3.3 The simulation of two M/M/1 processes

In this study, two M/M/1 processes are simulated, one is a regular M/M/1 process and the other is an M/M/1 process with arrivals before opening time. In fact, the general idea for the algorithm is the same, the difference is the selection of some arguments in the process of arrivals before opening time. If T = 0 (see the algorithm of nonhomogeneous process in section 3.2), the regular M/M/1 process is simulated. Otherwise, arrivals before opening time exist.

The algorithm above will be used to simulate complex systems like M/M/1 and M/M/3 processes. They are the important components. In addition, the simulation of a probabilistic model involves generating the stochastic mechanism of the model and recording the resultant flow of the model over time. Therefore, some variables have to be defined in order to keep track of the evolution of the process over time and determine relevant quantities[6].

Section 3.3.1 Definition of some variables

Time variable t: It represents the amount of time that has evolved.

Counter variables: These variables count the number of times that certain events have occurred by time t. We define some counter variables: (1) Nob: the number of customers in the system at opening time. If no arrivals at t=0, Nob = 0; (2) Noa: the number of customers who arrive after the server opens; (3) Na: the number of arrivals by time t; (4)Nd: the number of departures by time t;

System state variables: This defines the "state of system" at the time t. We define variable n as the number of customers in the system.

Event variables: (1)ta: arrival time; (2) td: departure time.

Output variables: In this simulation, the system state variable n is also an output variable. When waiting time is studied, the vectors recording the arrival time and departure time should be output variables. Ab: the vector recording the arrival time before opening time; Aa: the vector recording the arrival time after opening time. D: the vector recording the departure time. Two vectors recording the arrival time are required, since the method of calculating waiting time is different for the two types of arrivals.

Section 3.3.2 Algorithm

Generating arrivals before opening time

Set time T; If we want no arrival before opening time, set T = 0. In general, T defines the starting time to collect arrivals before opening time.

Simulate a nonhomogeneous process on the interval (-T,0), and record the arrival time Ab and the number of arrivals by opening time Nob.

Initialize

Set t = 0; Na = Nob; Noa=0; Nd = 0;

Set n = Nob; ta = t + random arrival time

If n = 0, td = ∞ (since server is empty);

Otherwise td = t + random departure time (one customer is assigned to this server).

EL = [ta, td]; (a vector which stores next arrival time and departure time)

Case 1 ta = min (EL); (Next event is an arrival)

Set t = ta (moving time to ta).

If t > tstop, stop. (tstop is criterion of stopping simulation)

Otherwise:

Reset Na = Na + 1 (one more customer arrives);

Reset Noa = Noa + 1 (one more customer arrives after opening time)

reset n = n + 1 (one more customer is in the system);

reset Aa(Noa) = t (recording the arrival time);

reset ta = t + random arrival time (next arrival time).

If n = 1 (the server is available, the current arrival is assigned to the server), reset: td = t + random service time (next departure time).

EL = [ta, td];

Case 2 td = min(EL); (next event is a departure).

Set t = td ; (move time to td)

If t > tstop, stop.

Otherwise:

Reset: Nd = Nd +1 (one more customer leaves)

Reset: n = n-1 (one more customer leaves)

Reset: D(Nd) = t (recording the departure time)

If n = 0, td = ∞ (no customer is in the system, the server is empty).

Otherwise:

td = t + random departure time.

EL = [ta, td];

Collect output data: repeat case1 and case 2 until t > tstop, then no reset occurs, stop simulating and collect n the number of customers in the system at t = tstop.

The above algorithm is a complete process to simulate an M/M/1 system. When the system is simulated N times, and N is very large, the probability of n customers in the system may be estimated with the frequency of occurrence of n customers in the system at t = tstop.

Section 3.4 The simulation of the M/M/3 process with arrivals before opening time

The simulation of an M/M/3 queue with arrivals before opening time is a sophisticated process. It involves three servers. Therefore, the relation among three servers should be defined; and the policy of assigning customers to servers should be clearly described. In addition, more variables are required to keep track of the evolution of the system, and to define system states and count the number of events.

In this study, three servers are regarded as be independent. When a customer is assigned to a specific server, this server works; otherwise, it has a rest and waits for a new assignment no matter whether other servers are busy or not. As described in chapter 2, the service policy is "first comes, first served". When more than one customer are in the queue, and more than one server are available, the customers will be assigned to an available server randomly. We assume a single line for the three servers.

Section 3.4.1 Definition of some variables

Time variable t: It represents the amount of time that has evolved.
Counter variables: These variables count the number of times that certain events have occurred by time t. We define some counter variables: (1) Nob: the number of customers in the system at opening time, if no arrivals at t=0, Nob = 0; (2) Noa: the number of arrivals after opening time; (3) Na: the number of arrivals by time t. It is sum of Nob and Noa; (3) Nd: the number of departures by time t; (4) C1, C2, and C3 are the number of departures by time t from server 1, server 2 and server 3, respectively.
System state variables: These define the "state of system" at the time t. We define a vector ss, where ss(1) : the number of customers in the system; ss(2), ss(3), ss(4): the indicator of the status of the server 1, server 2 and server 3. For example, ss(2) = n means that the nth customer is served by server 1, and ss(2)= 0 represents server 1 is available.
Event variables: (1) ta: arrival time; (2) t1, t2 and t3: departure times from server 1, server 2 and server 3, respectively.

Output variables: In this simulation, the system state variable ss(1) is also an output variables. When waiting time is studied, the vectors recording the arrival time and departure time should be output variables. Ab: the vector recording the arrival time before opening time; Aa: the vector recording arrival time after opening time. D: the matrix recording the departure time.

Section 3.4.2 Algorithm

Generating arrivals before opening time

Set time T; If we want no arrival before opening time, set T = 0. In general, T defines the starting time to collect arrivals before opening time.

Simulate a nonhomogeneous process on the interval (-T,0), and record the arrival time Ab and the number of arrivals by opening time Nob.

Initialize

Set t = 0; Na = Nob; Noa=0; C1 = 0; C2 = 0; C3 = 0;

Set n = Nob; ta = t + random interarrival time (next arrival time)

Set t1 = ∞; t2 = ∞; t3 = ∞; (since three servers are empty);

Set ss(1) = Nob (the number of customers in the system), ss(2)=ss(3)=ss(4)=0 (three servers are available)

At t = 0, randomly assigning customers to servers

Case 1 Na = 1 (one customer is in the system at t = 0)

Randomly determine which server will be used.

If server i is used by customer 1, where i = 1, 2, 3, then:

Reset: ss(i+1) = 1 (renew the system state variables)

Reset: ti = t + random service time of server i (renew departure time from Server i)

Case 2, Na =2 (two customers are in the system at t = 0)

Randomly determine which server will be used by customer 1

If server i is used by customer 1, where i = 1, 2, 3, then:

Reset: ss(i+1) = 1 (renew the system state variable).

Reset: ti = t + random service time of server i (renew departure time from Server i).

Randomly determine which empty server will be used by customer 2.

If server j is used by customer 2, where j = 1, 2, 3 with $j \neq i$, then:

Reset: ss(j+1) = 2 (assign customer 2 to server j)

Reset: tj = t + random service time of server j (renew the departure time from server j).

Case 3, Na ≥ 3 (more than two customers are in the system at t = 0)

Randomly determine which server will be used by customer 1

If server i is used by customer 1, where i = 1, 2, 3, then:

Reset: ss(i+1) = 1 (assign customer 1 to server i).

Rest: ti = t + random service time of server i (renew the departure time from server i).

Randomly determine which empty server will be used by customer 2.

If server j is used by customer 2, where j = 1, 2, 3 with $j \neq i$, then:

Reset: ss(j+1) = 2 (assign customer 2 to server j)

Reset: tj = t + random service time of server j (renew the departure time from server j).

Reset: ss(k+1) = 3 (assign customer 3 to server k, where k = 1, 2, 3 with $k \neq i \neq j$).

Reset: tk= t + random service time of server k (renew the departure time from server k).

EL = [ta, t1, t2, t3] (a matrix stores next arrival time and departure time).

Simulate the system as time evolves

Case 1 ta = min (EL); (Next event is an arrival)

Set: t = ta (moving time to ta).

If t > tstop, stop. (tstop is the simulation stopping criterion)

Otherwise:

Reset Na = Na + 1 (one more customer arrives);

Reset Noa = Noa + 1 (one more customer arrives after opening time)

Reset Aa(Noa) = t (recording the arrival time);

Reset: ta = t + random arrival time (next arrival time).

If ss = 0 (no customer is in the system), then:

Reset: ss(1) = ss(1) +1 (renew the number of customer in the system).

Randomly determine which server will be used.

If server i is used by customer Na, where i = 1, 2, 3, then:

Reset: ss(i+1) = Na (renew the system state variables)

Reste: ti = t + random service time of server i (renew departure time from Server i).

If ss(1)=1 & ss(i+1)≠0 (one customer is in the system, server i is used), then:

Reset: ss(1) = ss(1) +1 (renew the number of customer in the system).

Randomly determine which server will be used.

If server j is used by customer Na, where j = 1, 2, 3 with j ≠ i, then:

Reset: ss(j+1) = Na (renew the system state variables)

Reset: tj = t + random service time of server j (renew departure time from Server j).

If ss(1)=2 & ss(i+1)=0 (two customers are in the system, server i is available, i = 1, 2, 3), then:

Reset: ss(1) = ss(1) +1 (renew the number of customer in the system).

Reset: ss(i+1) = Na (renew the system state variables)

Reset: ti = t + random service time of server i (renew departure time from Server i).

If ss(1)>=3 (no server is free), then:

Reset: ss(1) = ss(1) +1 (renew the number of customer in the system).

EL = [ta, t1, t2, t3];

Case 2 ti = min(EL); (next event is a departure from server i).

Set t = ti ; (move time to ti)

If t > tstop, stop.

Otherwise:

Reset: Nd = Nd +1 (one more customer leaves from the system)

Reset: Ci = Ci + 1 (one more customer leaves from server i).

Set: ii = ss(i+1).

Reset: D(ii) = t (recording the departure time of customer ii from server i)

If ss(1) ≤ 3 (before customer ii leaves, less than three customers are in the system), then:

Reset: ss(1) = ss(1) -1 (one customer leaves from the system)

Reset: ss(i+1) = 0 (server I is free).

Rest: ti = ∞ (the server i is empty).

Otherwise:

Reset: ss(1) = ss(1) -1 (one customer leaves from the system).

Set: ssT=[ss(2), ss(3),ss(4)] (temporary matrix)

Set: m=max(ssT) (look for the max index be served).

Reset: ss(i+1)=m+1 (customer m+1 will be assigned to server i).

EL = [ta, t1, t2, t3]

Collect output data: repeat simulating the stochastic process along t until t > tstop, then no reset occurs, stop simulating and collect ss(1) the number of customers in the system at t = tstop.

The above algorithm is a complete process to simulate an M/M/3 with arrivals before opening time. When the system is simulated N times, and the N is very large, the probability of n customers in the system may be estimated with the frequency of occurrence of n customers in the system at t = tstop.

Section 3.5 Summary

In this chapter, first, we discussed the algorithm to simulate an exponential and a Poisson distribution. Second, these algorithms were employed to simulate a complex stochastic process, such as an M/M/1 system and an M/M/3 queue with arrivals before opening time. In order to keep track of the evolution, some counter variables and system state variables were defined. The various output variables were designed to satisfy different studies. The vectors recording arrival time and departure time may be outputted. However, since we focus on the distribution of the number of customers in the system, the number of customers in the system at t=tstop is a single output.

Chapter 4 Results and Analysis

Section 4.1 The theoretical distribution of the number of customers in a regular M/M/1 system

For an M/M/1 queue without arrivals at opening time, the theoretical time-dependent distribution of the number of customers in the system is described by equations 2.2 – 2.5. When the system is in steady state, the limiting probability is shown by equation 2.6. The modified Bessel function of the first kind of order k is directly calculated by the Matlab[5] function. Other parts are derived by a Matlab program, where the stop criterion of sum in equation 2.2 is set to be 10^{-8}. The arrival rate is 1.0, and the service rate is 1.5, but we do not define the unit. The service rate is greater than the arrival rate in order to converge as time approaches infinity.

At t =0: we assume the distribution for a regular M/M/1 process is

$P(k) = 1, \quad \text{where } k = 0$

$P(k) = 0, \quad \text{where } k = 1, 2, 3, \ldots$

The distribution at opening time (t = 0) is supposed.

At t = 2:

The exact distribution is shown in table 4.1.1

Table 4.1.1 The theoretical probability of the number of customers for a regular M/M/1 queue at t = 2

Number(k)	Prob
0	0.48483
1	0.28255
2	0.14094
3	0.06004
4	0.02201
5	0.00702
6	0.00197
7	0.00049
8	0.00011

The corresponding histogram is shown in Fig.4.1.1.

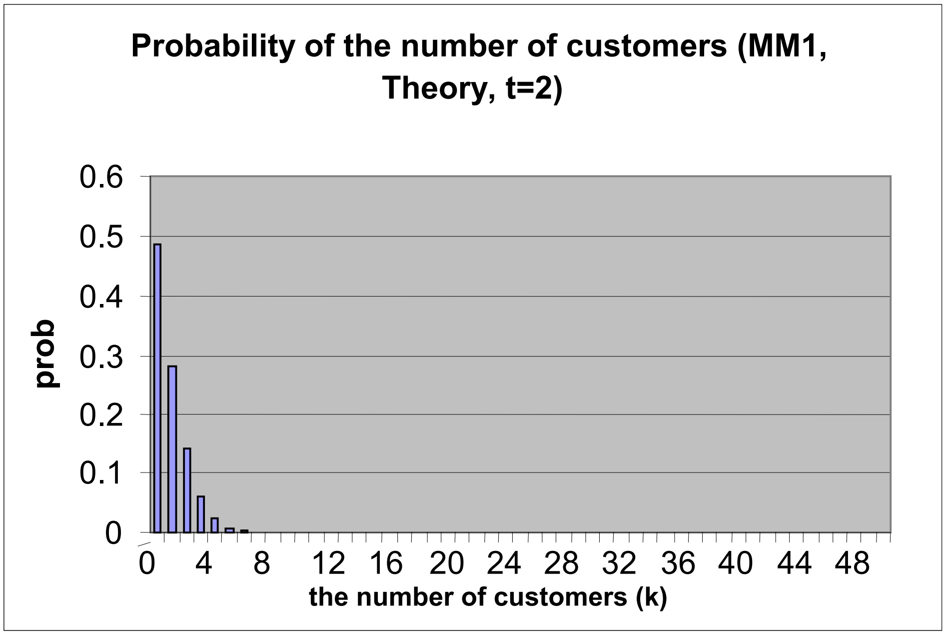

Fig 4.1.1 The theoretical distribution of the number of customers at t = 2 for a regular M/M/1 system

From table 4.1.1 and fig. 4.1.1, we know, for a regular M/M/1 queue at t = 2, the maximum probability is 0.4848 at k = 0. The probability decreases rapidly with k. When k is larger than 10, the probability is smaller than 10^{-5}. Based on the distribution above, we may derive the expected number of customers in the system at t = 2, which is 0.88.

At t = 20:

The distribution is shown in table 4.1.2 and Fig. 4.1.2

As shown in Table 4.1.2 and Fig. 4.1.2, the maximum probability is 0.3414 at k = 0. The expected number at t = 20 is 1.86.

Table 4.1.2 The theoretical probability of the number of customers for a regular M/M/1 queue at t = 20

Number(k)	Prob
0	0.341448
1	0.227072
2	0.150494
3	0.099297
4	0.065152
5	0.042462
6	0.027457
7	0.017595
8	0.011163
9	0.007004
10	0.004343
11	0.002658
12	0.001605
13	0.000956
14	0.000561
15	0.000324
16	0.000184
17	0.000103

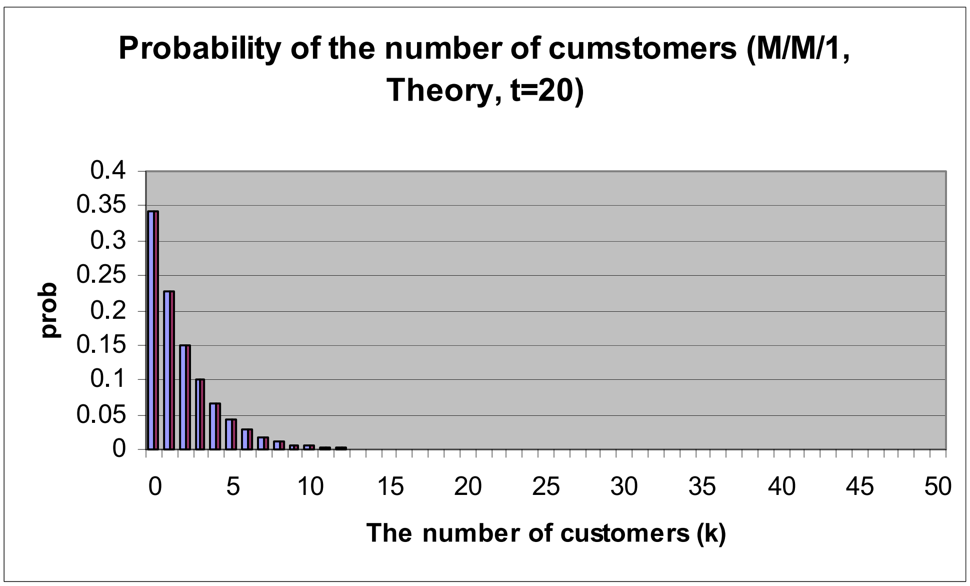

Fig 4.1.2 The theoretical distribution of the number of customers at t = 20 for a regular M/M/1 system

At t = 200:

The distribution is shown in table 4.1.3 and Fig. 4.1.3

Table 4.1.3 The theoretical probability of the number of customers for a regular M/M/1 queue at t = 200

Number(k)	Prob
0	0.333333
1	0.222222
2	0.148148
3	0.098765
4	0.065844
5	0.043896
6	0.029264
7	0.019509
8	0.013006
9	0.008671
10	0.005781

Number(k)	Prob
11	0.003854
12	0.002569
13	0.001713
14	0.001142
15	0.000761
16	0.000507
17	0.000338
18	0.000226
19	0.00015
20	0.0001
21	6.68E-05

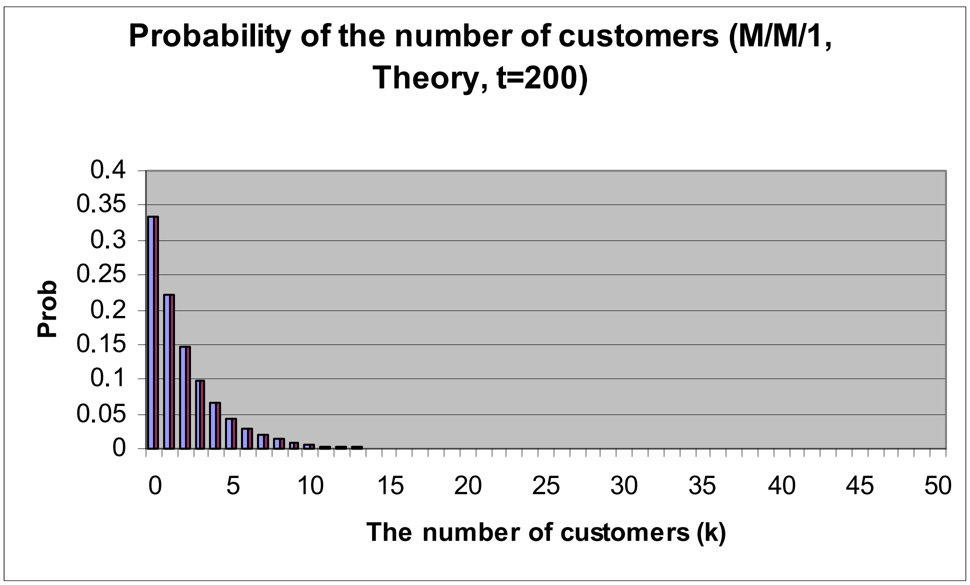

Fig 4.1.3 The theoretical distribution of the number of customers at t = 200 for a regular M/M/1 system

The data shown in Table 4.1.3 and Fig. 4.1.3 are calculated by using the equation 2.2. However, when the equation 2.6 is used, the results are exactly the same. As shown in Table 4.1.3 and Fig. 4.1.3, the maximum probability is 0.3333 at k = 0. The expected number at t = 200 is 2.0.

By comparison of tables and figures in this section, we know that: (1) The maximum probability occurs at k = 0 no matter how time shifts; (2) The distribution disperses as time evolves, (3) The expected number of customers in the queue increases as time increases. Therefore, a customer may have shorter waiting time when it arrives just after opening time.

Section 4.2 The distribution of the number of customers in a regular M/M/1 system via simulation

In this section, the distribution of the number of customers in a regular M/M/1 system via simulation will be shown. By comparison of results in this section and above, we will test if the program works well. The program is written via Matlab. At every time, the whole process will be simulated 10000 times. The probability of the number of customers is estimated by the frequency of occurrences.

At t = 0 :

P(k) = 1, where k = 0

P(k) = 0, where k = 1, 2, 3,……..

The distribution at opening time (t = 0) is supposed.

At t = 2:

The distribution is shown in table 4.2.1 and Fig. 4.2.1

Table 4.2.1 The probability of the number of customers for a regular M/M/1 queue via simulation at t = 2

number	times	prob
0	4882	0.4882
1	2783	0.2783
2	1397	0.1397
3	627	0.0627
4	221	0.0221
5	66	0.0066
6	18	0.0018
7	5	0.0005
8	1	0.0001

From Table 4.2.1 and Fig.4.2.1, the maximum probability is 0.4882 at k = 0. The distribution rapidly decreases as k increases. The expected number is 0.8833. Comparing Table 4.2.1 and Fig. 4.2.1 to Table 4.1.1 to Fig. 4.1.1, we may conclude that the simulation program works well.

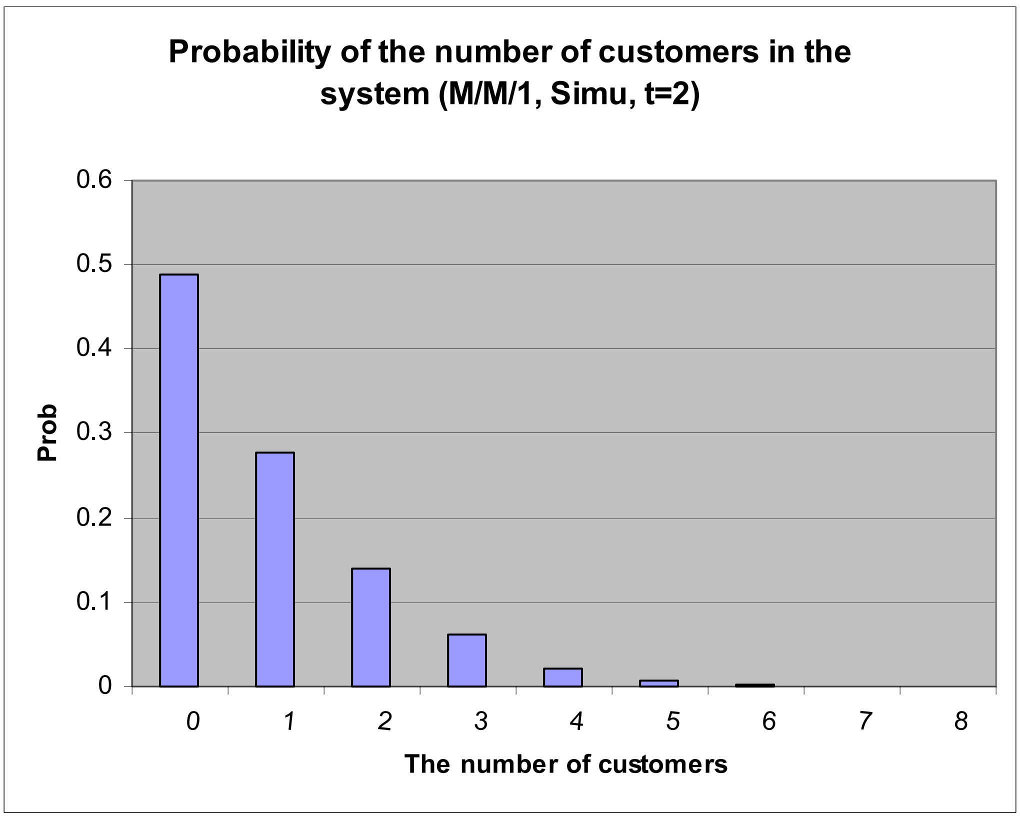

Fig. 4.2.1 The distribution of the number of customers at t = 2 for a regular M/M/1 system via simulation

At t = 20:

The distribution is shown in table 4.2.2 and Fig. 4.2.2

Table 4.2.2 The probability of the number of customers for a regular M/M/1 queue at t = 20 via simulation

Number(k)	Times	Prob
0	3439	0.3439
1	2225	0.2225
2	1529	0.1529
3	997	0.0997
4	664	0.0664
5	416	0.0416
6	271	0.0271
7	176	0.0176
8	110	0.011
9	85	0.0085
10	38	0.0038

Number(k)	Time	Prob
11	22	0.0022
12	10	0.001
13	7	0.0007
14	5	0.0005
15	5	0.0005
16	1	0.0001

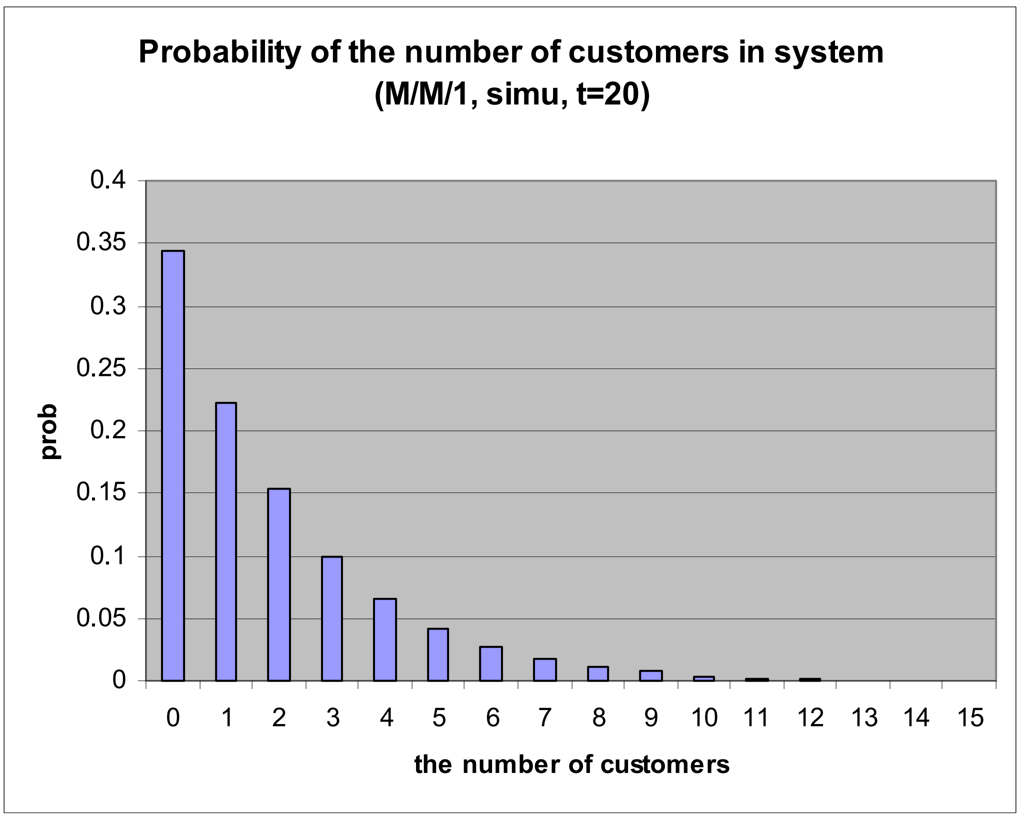

Fig. 4.2.2 The distribution of the number of customers at t = 20 for a regular M/M/1 system via simulation

From Table 4.2.2 and Fig.4.2.2, the maximum probability is 0.3439 at k = 0. The distribution rapidly decreases as k increases. The expected number is 1.8507.

At t = 200:

The distribution is shown in table 4.2.3 and Fig. 4.2.3

Table 4.2.3 The probability of the number of customers for a regular M/M/1 queue at t = 200 via simulation

Number	Times	Prob
0	3358	0.3358
1	2180	0.218
2	1465	0.1465
3	1021	0.1021
4	660	0.066
5	430	0.043
6	325	0.0325
7	207	0.0207
8	122	0.0122
9	85	0.0085

Number	Times	Prob.
10	54	0.0054
11	34	0.0034
12	21	0.0021
13	11	0.0011
14	10	0.001
15	6	0.0006
16	4	0.0004
17	5	0.0005
18	2	0.0002

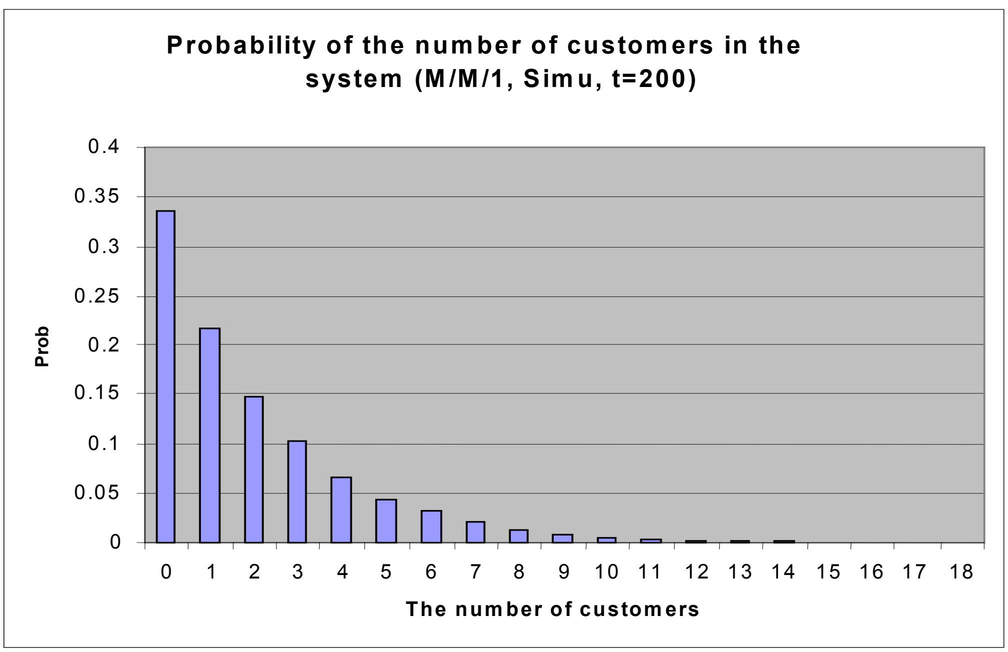

Fig. 4.2.3 The distribution of the number of customers at t = 200 for a regular M/M/1 system via simulation

The maximum probability is 0.3358 at k = 0. The expected number is 1.9872.

Comparing the tables and figures in this section to those in section 4.1, we conclude that the simulation works well.

Section 4.3 The distribution of the number of customers in an M/M/1 system with arrivals before opening time via simulation

In this section, the distribution of the number of customers in an M/M/1 queue with arrivals before opening time will be reported. The distribution results from simulating this process. The arrivals before opening time follow a nonhomogeneous Poisson process, which is described in equations 2.9 – 2.13. After opening time, the arrival rate and service rate are 1.0 and 1.5, respectively.

At t = 0:

The distribution is shown in Table 4.3.1 and Fig.4.3.1

Table 4.3.1 The distribution of the number of customers at t = 0 for an M/M/1 system with arrivals before opening time via simulation

Number	Times	Prob
0	0	0
1	0	0
2	0	0
3	0	0
4	0	0
5	1	0.0001
6	5	0.0005
7	4	0.0004
8	12	0.0012
9	29	0.0029
10	64	0.0064
11	93	0.0093
12	184	0.0184
13	274	0.0274
14	367	0.0367
15	455	0.0455
16	649	0.0649
17	761	0.0761
18	781	0.0781
19	869	0.0869
20	907	0.0907

Number	Times	Prob
21	888	0.0888
22	746	0.0746
23	694	0.0694
24	570	0.057
25	458	0.0458
26	358	0.0358
27	288	0.0288
28	207	0.0207
29	115	0.0115
30	79	0.0079
31	59	0.0059
32	34	0.0034
33	20	0.002
34	16	0.0016
35	6	0.0006
36	2	0.0002
37	0	0
38	2	0.0002
39	1	0.0001
40	1	0.0001
41	0	0
42	0	0
43	1	0.0001

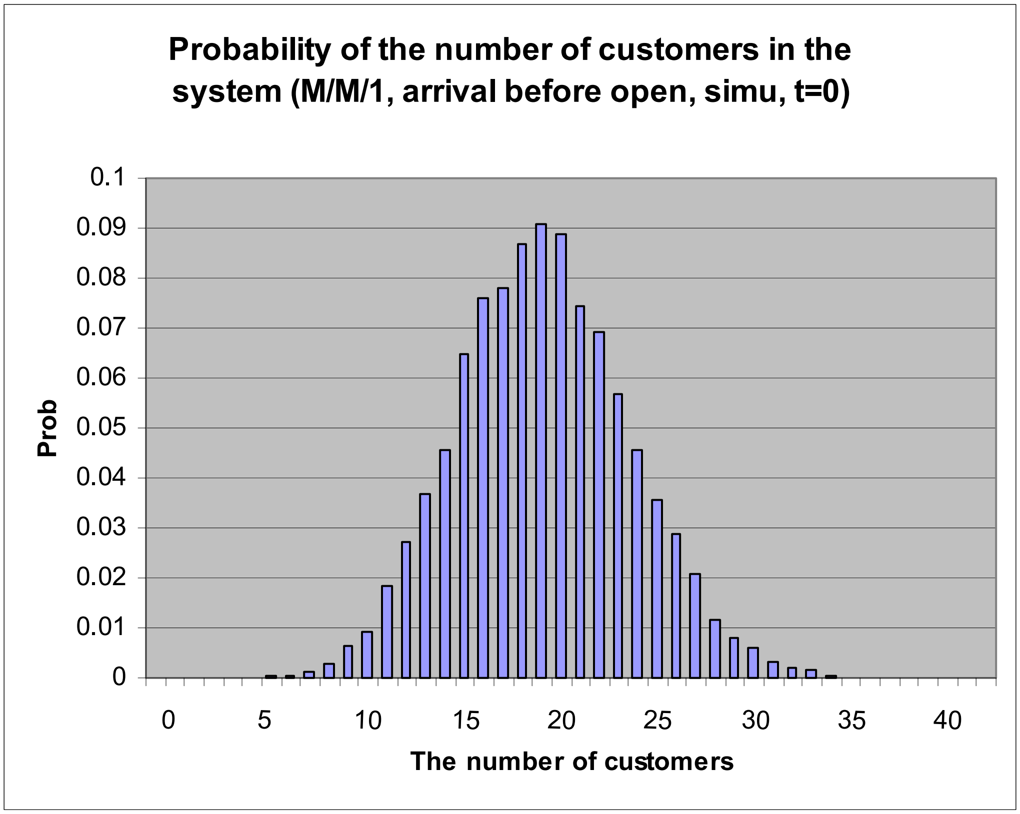

Fig. 4.3.1 The distribution of the number of customers at t = 0 for an M/M/1 system with arrivals before opening time via simulation

From Table 4.3.1 and Fig.4.3.1, the maximum probability is 0.0907 at k = 20. The expected number is 20.1184. The distribution is a Poisson distribution which looks approximately normal since the mean is fairly large.

At t = 2:

The distribution is shown in Table .4.3.2 and Fig.4.3.2

From Table 4.3.2 and Fig.4.3.2, the maximum probability is 0.083 at k = 19. The distribution looks symmetric. The expected number is 18.9362.

Table 4.3.2 The distribution of the number of customers at t = 2 for an M/M/1 system with arrivals before opening time via simulation

number	times	prob
0	1	0.0001
1	0	0
2	0	0
3	7	0.0007
4	5	0.0005
5	19	0.0019
6	25	0.0025
7	41	0.0041
8	52	0.0052
9	108	0.0108
10	151	0.0151
11	249	0.0249
12	318	0.0318
13	397	0.0397
14	479	0.0479
15	596	0.0596
16	716	0.0716
17	697	0.0697
18	828	0.0828
19	830	0.083
20	811	0.0811

Number	Times	Prob
21	699	0.0699
22	622	0.0622
23	562	0.0562
24	452	0.0452
25	373	0.0373
26	306	0.0306
27	202	0.0202
28	179	0.0179
29	88	0.0088
30	75	0.0075
31	48	0.0048
32	23	0.0023
33	17	0.0017
34	10	0.001
35	4	0.0004
36	2	0.0002
37	3	0.0003
38	1	0.0001
39	2	0.0002
40	2	0.0002

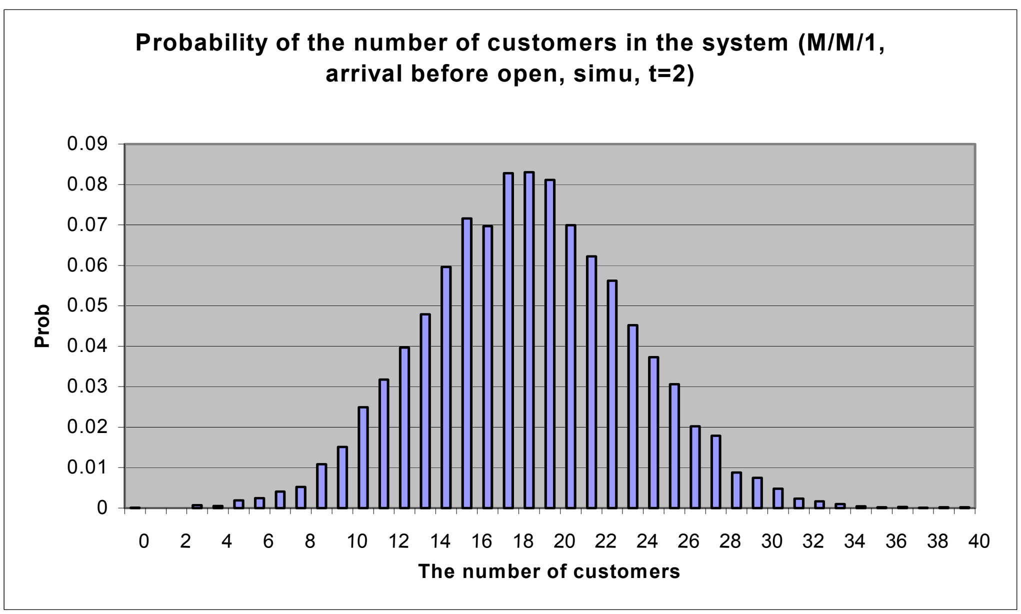

Fig. 4.3.2 The distribution of the number of customers at t = 2 for an M/M/1 system with arrivals before opening time via simulation

At t = 20:

The distribution is shown in Table 4.3.3 and Fig.4.3.3

Table 4.3.3 The distribution of the number of customers at t = 20 for an M/M/1 system with arrivals before opening time via simulation

Number	Times	Prob
0	748	0.0748
1	555	0.0555
2	438	0.0438
3	416	0.0416
4	390	0.039
5	403	0.0403
6	430	0.043
7	499	0.0499
8	513	0.0513
9	452	0.0452
10	464	0.0464
11	488	0.0488
12	488	0.0488
13	420	0.042
14	414	0.0414
15	418	0.0418
16	348	0.0348
17	343	0.0343
18	276	0.0276
19	279	0.0279
20	221	0.0221
21	184	0.0184
22	169	0.0169
23	142	0.0142

Number	Times	Prob
24	108	0.0108
25	77	0.0077
26	71	0.0071
27	63	0.0063
28	49	0.0049
29	40	0.004
30	23	0.0023
31	18	0.0018
32	17	0.0017
33	11	0.0011
34	11	0.0011
35	9	0.0009
36	1	0.0001
37	0	0
38	3	0.0003
39	0	0
40	0	0
41	0	0
42	0	0
43	0	0
44	0	0
45	0	0
46	0	0
47	0	0
48	1	0.0001

From Table 4.3.3 and Fig.4.3.3, there are at least two local maxima of probability. The first maximum is 0.0748 at k = 0; the second is 0.0513 at k = 8; the third is 0.0488 at k = 12. The third local maximum may be real or simply occurs due to our particular simulation. The expected number is 10.4463.

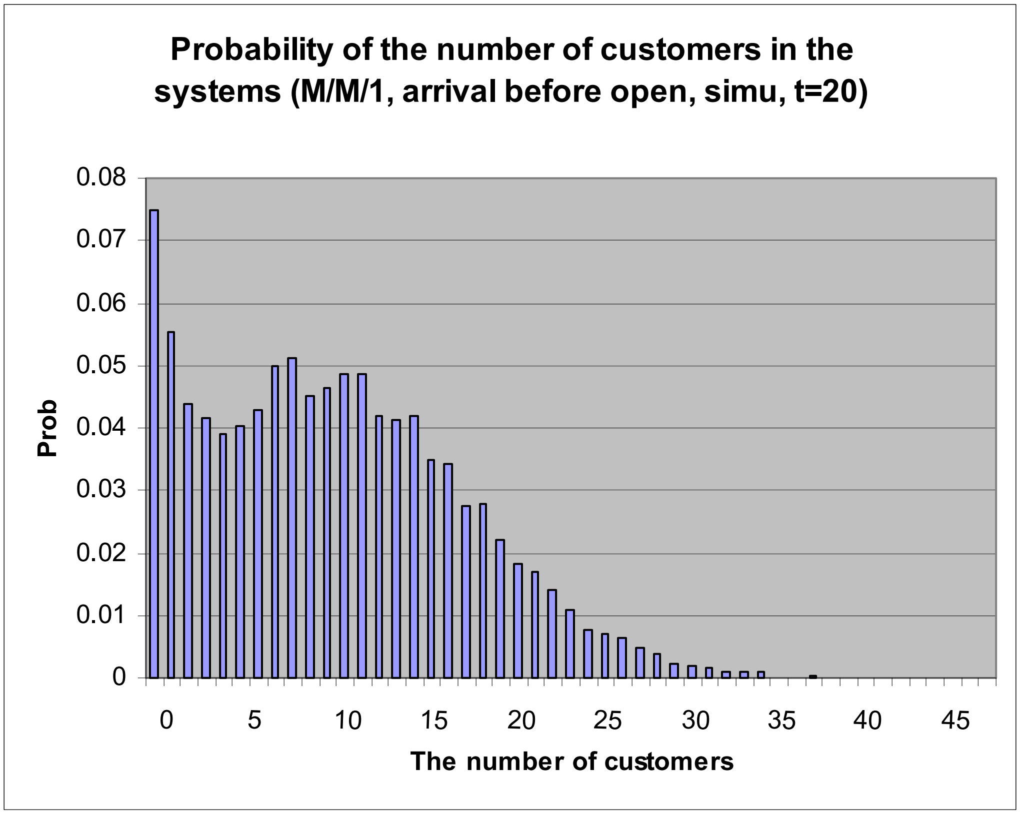

Fig. 4.3.3 The distribution of the number of customers at t = 20 for an M/M/1 system with arrivals before opening time via simulation

At t = 200:

The distribution is shown in Table 4.3.4 and Fig.4.3.4

Table 4.3.4 The distribution of the number of customers at t = 200 for an M/M/1 system with arrivals before opening time via simulation

Number	Times	Prob	Prob-st
0	3288	0.3288	0.333333
1	2214	0.2214	0.222222
2	1379	0.1379	0.148148
3	1033	0.1033	0.098765
4	723	0.0723	0.065844
5	444	0.0444	0.043896
6	300	0.03	0.029264
7	221	0.0221	0.019509
8	128	0.0128	0.013006
9	83	0.0083	0.008671
10	64	0.0064	0.005781

Number	Times	Prob.	Prob-st
11	36	0.0036	0.003854
12	30	0.003	0.002569
13	16	0.0016	0.001713
14	16	0.0016	0.001142
15	8	0.0008	0.000761
16	4	0.0004	0.000507
17	2	0.0002	0.000338
18	4	0.0004	0.000226
19	4	0.0004	0.00015
20	1	0.0001	0.0001
21	2	0.0002	6.68E-05

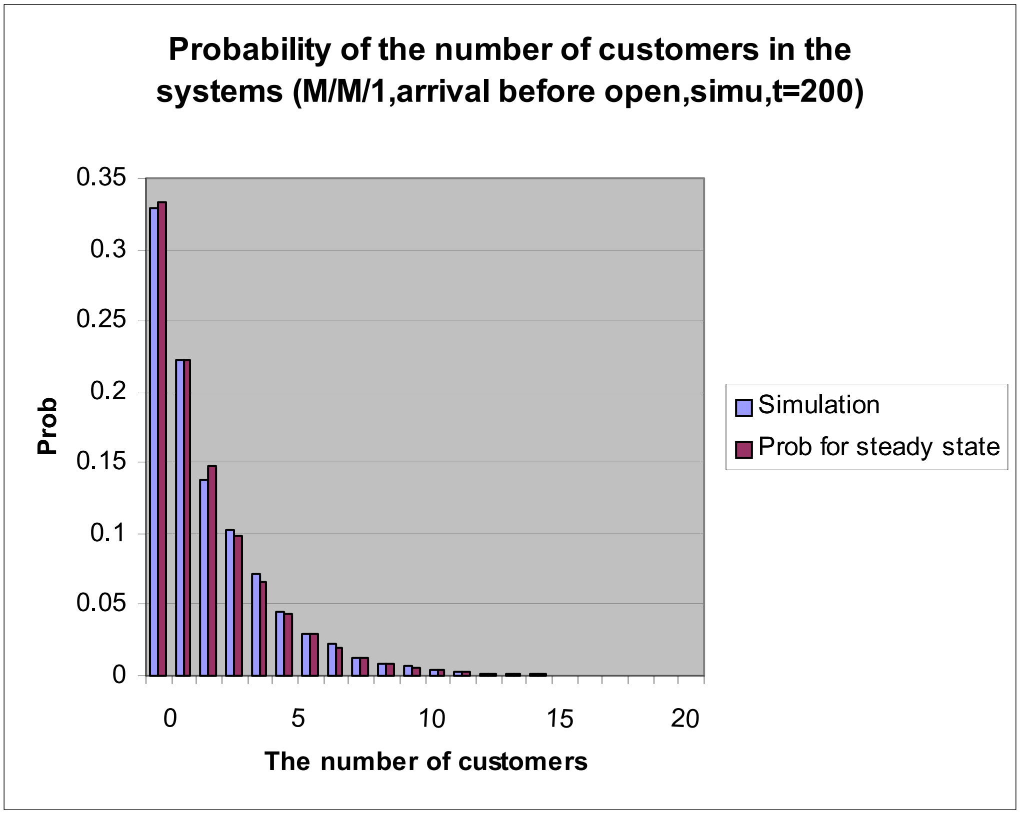

Fig. 4.3.4 The distribution of the number of customers at t = 200 for an M/M/1 system with arrivals before opening time via simulation

In table 4.3.4, Prob-st represents the probability with k customers in steady state. Prob-st is calculated from equation 2.6. By comparison of the two histograms in Fig.4.3.4, we conclude that, by t = 200, the system is in steady state.

Section 4.4 The distribution of the number of customers in an M/M/3 system with arrivals before opening time via simulation

In this section, we will report the distribution of the number of customers in an M/M/3 queue with arrivals before opening time. The distribution is derived by simulating this process. The arrivals before opening time are described in section 4.3. Therefore, the distribution at t = 0 is not discussed in this section. After opening time, the arrival rate is 1.0. The system includes three servers. The service rates of these servers are 0.4, 0.5 and

0.6, respectively. The sum of these service rates matches the service rate in the M/M/1 model .

At t = 2:

The distribution is shown in Table 4.4.1 and Fig.4.4.1

From Table 4.4.1 and Fig.4.4.1, the local maximums of the probability are 0.0801 at k = 18, or 20. The distribution looks symmetric. The expected number is 18.9986.

Table 4.4.1 The distribution of the number of customers at t = 2 for an M/M/3 system with arrivals before opening time via simulation

Number	Times	Prob
0	0	0
1	0	0
2	3	0.0003
3	1	0.0001
4	7	0.0007
5	7	0.0007
6	21	0.0021
7	35	0.0035
8	75	0.0075
9	102	0.0102
10	145	0.0145
11	244	0.0244
12	310	0.031
13	429	0.0429
14	487	0.0487
15	603	0.0603
16	653	0.0653
17	781	0.0781
18	801	0.0801
19	792	0.0792
20	801	0.0801

Number	Times	Prob
21	699	0.0699
22	624	0.0624
23	543	0.0543
24	466	0.0466
25	354	0.0354
26	300	0.03
27	229	0.0229
28	156	0.0156
29	104	0.0104
30	80	0.008
31	51	0.0051
32	39	0.0039
33	21	0.0021
34	16	0.0016
35	12	0.0012
36	3	0.0003
37	2	0.0002
38	0	0
39	3	0.0003
40	0	0
41	1	0.0001

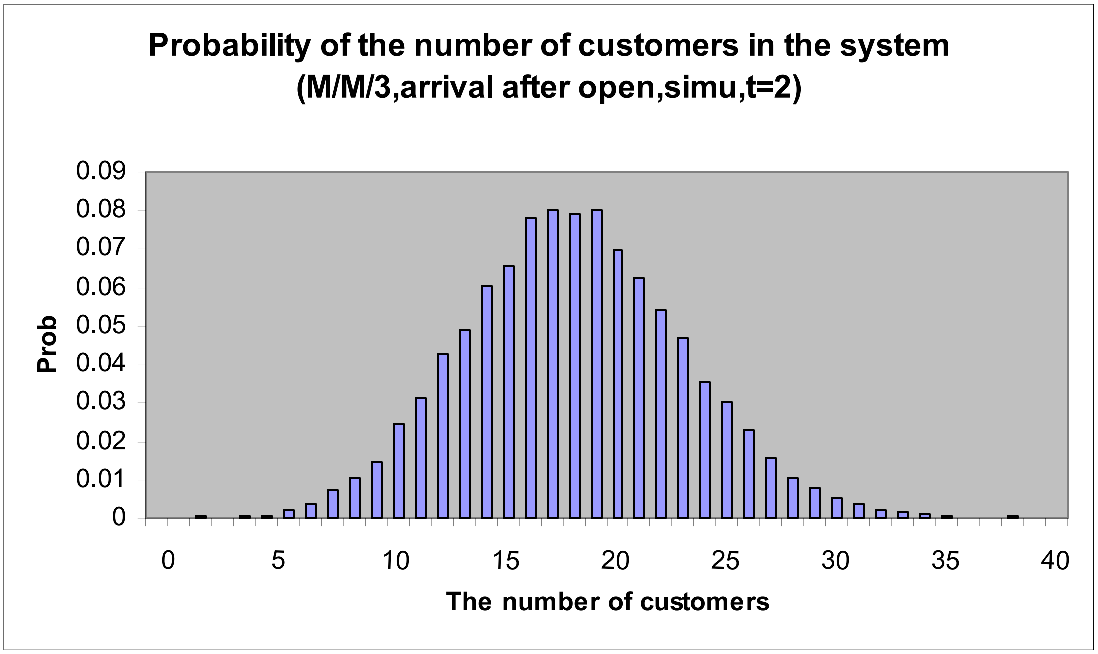

Fig. 4.4.1 The distribution of the number of customers at t = 2 for an M/M/3 system with arrivals before opening time via simulation

At t = 20:

The distribution is shown in Table 4.4.2 and Fig.4.4.2

Table 4.4.2 The distribution of the number of customers at t = 20 for an M/M/3 system with arrivals before opening time via simulation

Number	Times	Prob
0	239	0.0239
1	553	0.0553
2	610	0.061
3	505	0.0505
4	469	0.0469
5	500	0.05
6	478	0.0478
7	492	0.0492
8	495	0.0495
9	450	0.045
10	472	0.0472
11	463	0.0463
12	522	0.0522
13	426	0.0426
14	452	0.0452
15	387	0.0387
16	346	0.0346
17	319	0.0319
18	306	0.0306
19	278	0.0278
20	239	0.0239
21	202	0.0202
22	152	0.0152
23	130	0.013
24	112	0.0112
25	83	0.0083
26	76	0.0076
27	63	0.0063
28	44	0.0044
29	30	0.003
30	30	0.003
31	18	0.0018

Number	Times	Prob.
32	17	0.0017
33	12	0.0012
34	8	0.0008
35	6	0.0006
36	4	0.0004
37	3	0.0003

Number	Times	Prob
38	3	0.0003
39	1	0.0001
40	2	0.0002
41	2	0.0002
42	1	0.0001

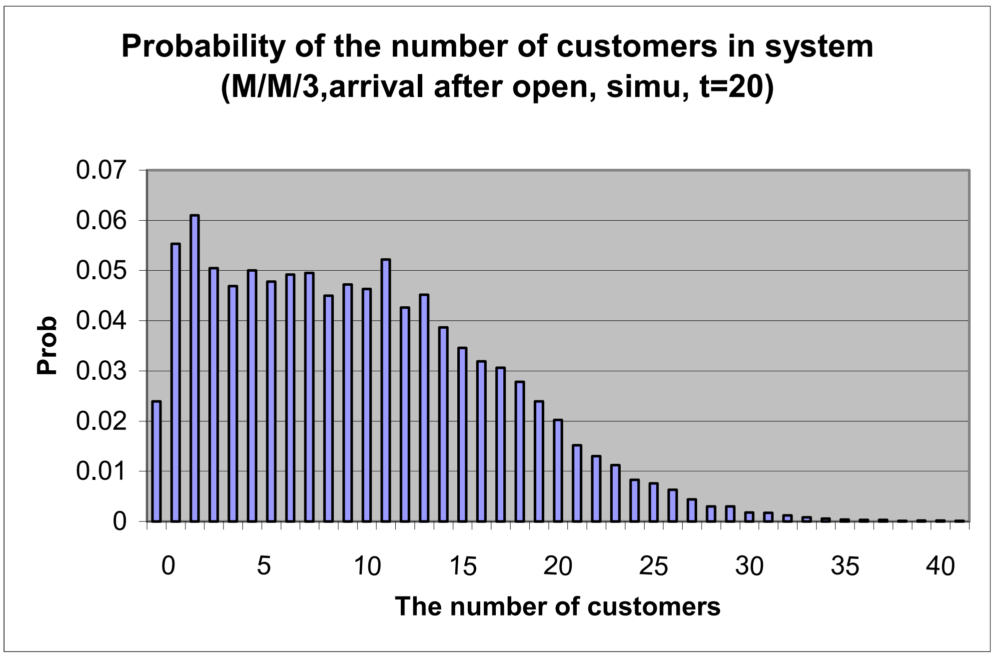

Fig. 4.4.2 The distribution of the number of customers at t = 20 for an M/M/3 system with arrivals before opening time via simulation

From Table 4.4.2 and Fig.4.4.2, it appears that there are two local maximums of probability. The first maximum is 0.061 at k = 2. The second is 0.0522 at k = 12. The expected number is 10.6873. Compared to the distribution shown in Fig.4.3.3, the probability distribution of the number of customers of an M/M/3 system with arrivals before opening time at t = 20 is different from the corresponding the M/M/1 system with arrivals before opening time. The distribution shown in Fig.4.3.3 has at least two local maximums, but here only two. The global maximum in Fig 4.3.3 occurs at k = 0, here at k = 2.

At t = 200:

The distribution is shown in Table 4.4.3 and Fig.4.4.3

From Table 4.4.3 and Fig.4.4.3, the maximum probability is 0.2296 at k = 2. The expected number is 2.8943. At t = 200, the distribution of the M/M/3 queue is different from that of the M/M/1 queue. First, the location of the maximum of probability is at k = 2 for this M/M/3 queue, but at k = 0 for the M/M/1 queue; second, the expected number of this M/M/3 queue is greater than that of the M/M/1 queue. This is reasonable because when there are one or two customers in the system, the M/M/3 model completes service slower than the M/M/1 model.

Table 4.4.3 The distribution of the number of customers at t = 200 for an M/M/3 system with arrivals before opening time via simulation (service rates are different)

Number	Times	Prob1
0	1109	0.1109
1	2166	0.2166
2	2296	0.2296
3	1503	0.1503
4	967	0.0967
5	645	0.0645
6	417	0.0417
7	289	0.0289
8	201	0.0201
9	134	0.0134
10	105	0.0105

Number	Times	Prob
11	52	0.0052
12	29	0.0029
13	34	0.0034
14	18	0.0018
15	12	0.0012
16	7	0.0007
17	8	0.0008
18	4	0.0004
19	1	0.0001
20	2	0.0002
21	1	0.0001

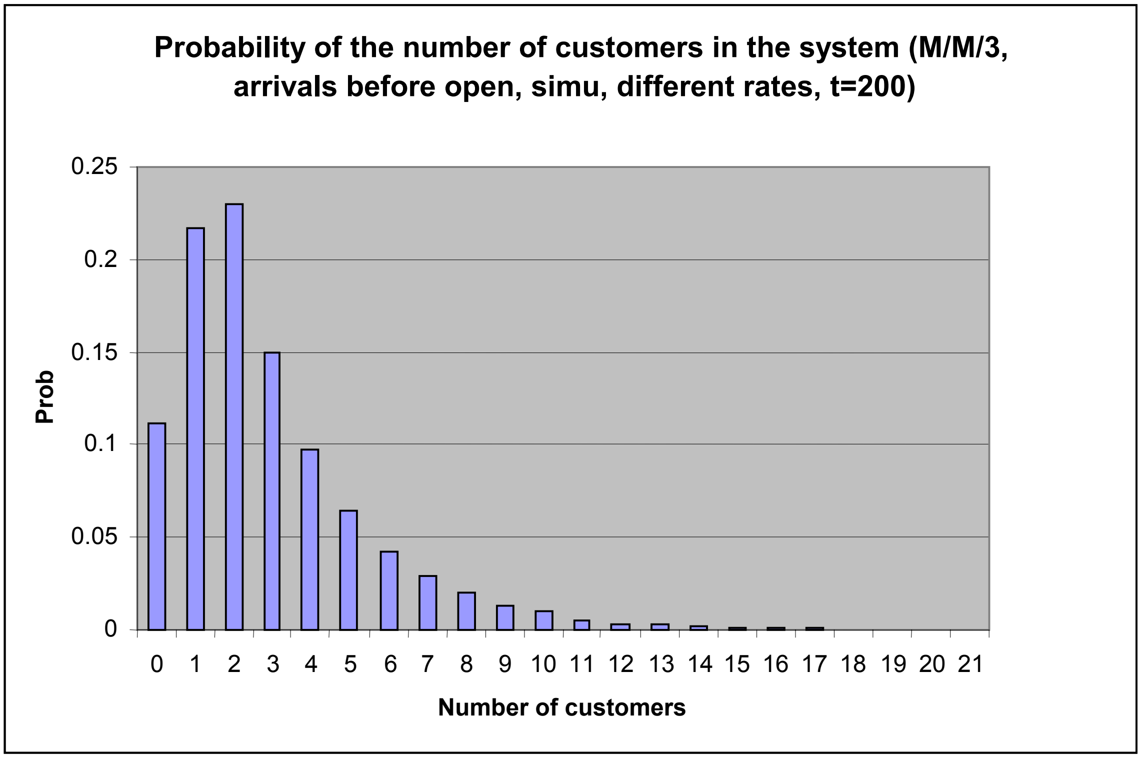

Fig. 4.4.3 The distribution of the number of customers at t = 200 for an M/M/3 system with arrivals before opening time via simulation (service rates are different)

At t = 200, we also simulate another M/M/3 queue with arrivals before opening time, which has three equal service rates of 0.5. In this case, we also compute the steady state probabilities.

Table 4.4.4 The distribution of the number of customers at t = 200 for an M/M/3 system with arrivals before opening time via simulation (service rates are same)

Number	Times	Prob	Prob-st
0	1088	0.1088	0.1111
1	2197	0.2197	0.2222
2	2251	0.2251	0.2222
3	1476	0.1476	0.14813
4	1010	0.101	0.098756
5	647	0.0647	0.065837
6	448	0.0448	0.043891
7	283	0.0283	0.029261
8	200	0.02	0.019507
9	141	0.0141	0.013005
10	89	0.0089	0.00867
11	53	0.0053	0.00578

Number	Times	Prob	Prob-st
12	41	0.0041	0.003853
13	28	0.0028	0.002569
14	22	0.0022	0.001713
15	6	0.0006	0.001142
16	8	0.0008	0.000761
17	2	0.0002	0.000507
18	3	0.0003	0.000338
19	1	0.0001	0.000226
20	0	0	0.00015
21	2	0.0002	0.0001
22	2	0.0002	6.68E-05
23	0	0	4.45E-05
24	2	0.0002	2.97E-05

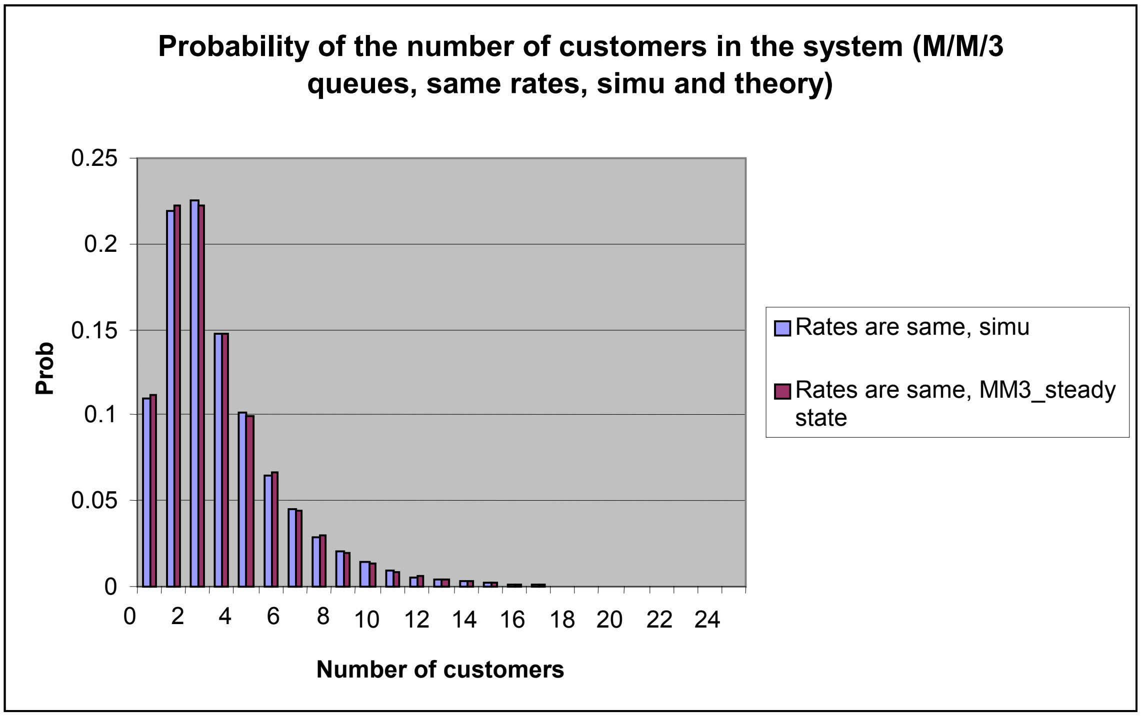

Fig. 4.4.4 The distribution of the number of customers at t = 200 for an M/M/3 system with arrivals before opening time via simulation (service rates are same)

The variable of prob-st represents the steady state probability with the number k in the system, calculated by using equations 2.14 – 2.17. By comparing the two distributions shown in Fig.4.4.4, we conclude that the program works well.

Chapter 5 Conclusion and Future Work

Conclusion

In this major paper, three queueing models were studied by using simulation. They were (1) regular M/M/1 queue; (2) M/M/1 queue with arrivals before opening time; (3) M/M/3 queue with arrivals before opening time. We may conclude that (1) the system will evolve from transient flow to steady state flow if the after-opening arrival rate is less than the total service rate; (2) for a regular M/M/1 queue, the expected number of customers in the system increases with time. Therefore, a customer arriving early will have shorter expected waiting time; (3) for queues with arrivals before opening time, if the expected number at $t=0$ exceeds the steady state expected number, then a customer arriving late will have shorter expected waiting time since the expected number decreases as time increases; (4) for the two queues with arrivals before opening time, the M/M/1 system has a lower expected number than the M/M/3 model especially when time increases.

Future work

In this study, the expected waiting time is an important topic for queueing models. In future, we may derive the expected waiting time by using simulation, then study relations between the expected waiting time and the parameters of arrival and service distributions. In the references, we may find theoretical results for the M/M/1 system in the steady state case. We may extend the study to a nonsteady state system, or a system with multiple servers, and then derive the empirical formula by using regression.

In this study, arrivals before opening time follow a nonhomogeneous Poisson process. The after-opening arrivals follow a homogeneous Poisson process, and the service time has an exponential distribution. In future, we may extend the study to other distributions. It may be difficult or impossible to obtain theoretical results, but we may easily simulate these processes and find numerical results.

Another interesting topic for future work is the evolution of a system from transient to steady state. We may study how to determine when a system is in a steady state, and study how long it takes for a system to evolve from transient to steady state. These systems may have arrivals before opening time, and different distributions.

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