Markov Chain Analysis For Vehicular Left-Turning Lanes

by

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Abstract

In this work we develop a queueing model of traffic at a signalized intersection. This model allows to predict the total queue length of left-turning and through vehicles accumulating at the intersection depending on traffic parameters and signalization scheme, which can be used to determine the optimal turning lane length.

The existing literature considers the queues of left-turning and through vehicles separately and does not study the queue which they form together. The recommendations about the turning lane length are based on the probabilities of turning lane blockage and overflow and do not consider the significance of the influence of such blockage or overflow on the overall traffic.

The new approach used in this work allows to consider the total queue of both left-turning and through vehicles and thus to study the problems at the intersection from the overall traffic standpoint.

Using the proposed model, the optimal turning lane length can be obtained based on user-specified criterion.

The model is implemented in a Matlab program using which the 95-th percentile queues for some combinations of parameters and different turning lane lengths are computed. The results of these computations are presented in a tabular form and followed by the author's comments.

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CHAPTER I

Introduction

1. Nature of the Problem

In the current work we consider the problem of managing the traffic at a signalized intersection. We assume that the vehicles arriving at this intersection are either left-turning or through. To prevent the disruption of through traffic by left-turning vehicles, the latter are stored in a turning lane. This turning lane should be able to store enough vehicles to avoid disruption. So, from the traffic standpoint, the length of the turning lane should be made as big as possible. However, we should take into account the fact that building of the longer turning lane costs more and requires additional space. That is why the optimal length should be a trade-off between these two standpoints.

We assume that a signalized intersection has the following phases: protected turn, permitted turn and red. During the protected phase left-turning vehicles are able to turn at any time as there is a red signal phase for opposing traffic at the same time. Through vehicles are stopped to allow the opposing turning vehicles to turn.

Permitted turn phase takes place when there is a green phase for opposing vehicles. During permitted phase left-turning vehicles are allowed to turn, but they should give right of way to opposing through vehicles. Through vehicles have a green phase in the meantime.

The third phase of the traffic light cycle is the red phase. During this phase no vehicles can turn or go through, allowing the vehicles from an intersecting road to move.

We consider that the red phase goes last in the traffic light cycle, and concerning permitted and protected phases, we consider two different cases: protected first or permitted first. This allows us to investigate which order is better.

Table 1 presented below shows the relation between phases for left-turning and through vehicles.

Table 1. Relation between phases for left-turning and through vehicles

Left-turning vehicles	Through vehicles
Protected Phase	Red Phase
Permitted Phase	Green Phase
Red Phase	Red Phase

In this work we develop a model allowing to study the queue of vehicles at the intersection waiting to turn left or go through depending on traffic parameters and signal phases order, and based on this to make a decision about turning lane length.

2. Literature Overview

There exist a number of research works on determining the optimal turning lane length.

One of the earliest and well-known papers is Harmelink (1967). He considers the length turning lanes at unsignalized intersections and presents the results in the form of graphs.

Chakroborty et al. (1995) proposes another model which gives more adequate results then the previous paper, and presents the recommended lane length in a tabular form. Also they compare the obtained results with the results of simulations.

Lertworawanich and Elefteriadou (2003) perform further development of the model for unsignalized intersections. They propose the new model based on the turning lane overflow probability and compare the obtained results with the existing guidelines.

All the above research works use the queueing theory approach to solving the problem.

Regarding the signalized intersection, Oppenlander and Oppenlander (1989) apply the rational procedure to determine the design length for left-turn or right-turn lanes with separate signal control. They use the arrival and service rate equations which as they mention are reasonably accurate for this type of intersections.

The research conducted by McCoy et al. (1994) uses the simulational approach for the right-turn lanes with the subsequent regression analysis.

Another research, Yekhshatyan and Schnell (2008), conducts the simulations with multivariate regression analysis as well. They consider right and left turning lanes at different types of intersections.

The queueing theory approach to determining the optimal left turning lane length at signalized intersections is considered in Kikuchi et al. (1993) and Kikuchi and Kronprasert (2010) which are discussed in detail later.

3. Lane Overflow and Blockage Approach

3.1. Approach description. The most natural approach to modelling traffic and turn signals at signalized intersections is through queueing theory. One of the papers that uses queueing theory for determining the left-turn lane length is Kikuchi et al. (1993). In this paper, the recommended length of the turning lane is calculated based on two types of problems at a signalized intersection.

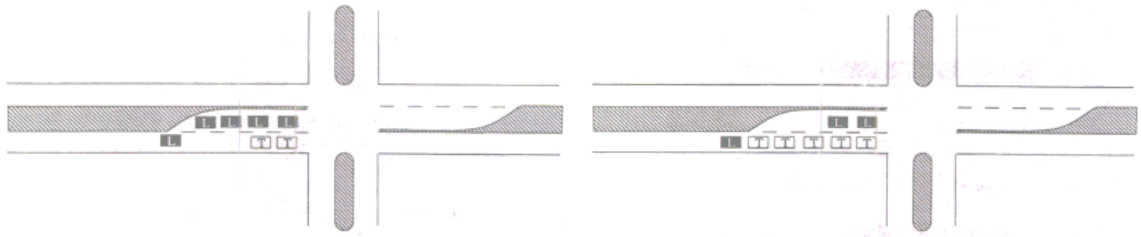
The first problem is the turning lane overflow. It consists in the following. When there is not enough space for all left-turning vehicles in a turning lane, those vehicles, which are not in the lane, will block the road for through traffic.

The second problem considered in this paper is the turning lane blockage. It can happen when number of through vehicles waiting is bigger than the number of vehicles that can be stored in the turning lane. In this case the vehicles which want to turn, but are not in the turning lane now, are not able to access the turning lane, even if there is a lot of space there.

Examples of situations in which these two problems arise are shown in the Figure 1 below (source: Kikuchi et al., 1993).

The desired turning lane length is such that prevents both of these problems. However, it is impossible to prevent these problems in 100% of cases, since there is always a chance that queue length will be bigger than the turning lane length. In the model proposed in Kikuchi et al. (1993) the recommended turning lane length allows the lane overflow

FIGURE 1. Lane Overflow and Blockage problems



(a) Lane Overflow (b) Lane Blockage

and blockage to happen, but with probabilities which are less than fixed values called threshold probabilities.

Firstly they find optimal lane length from the lane overflow standpoint (Figure 1a). This means that they find such turning lane length for which the probability for the turning lane not being able to store all arriving left-turning vehicles is less than the threshold probability. This probability is chosen to be 0.02. So this turning lane is able to store all left-turning vehicles in 98% of cases.

Similarly, they find optimal lane length from the lane blockage standpoint (Figure 1b). Such lane has the probability of blockage less than another threshold probability which is chosen to be 0.1. So the blockage does not occur in 90% of cases.

And finally, the optimal left-turning lane is found by simply choosing the maximum between the two lengths discussed above.

3.2. Limitations of Lane Overflow and Blockage Approach. The turning lane overflow and blockage problems are the main factors influencing the required turning lane length. However, the approach considering the probabilities of these two problems directly has some limitations.

Firstly, while considering these problems, it is important to take the phase of the traffic light circle into account. Assume that the situation on the road from the Figure 1a shown above occurs. The left-turning vehicles block the entrance to the through lane for an arriving through vehicle. However, if this occurs at the beginning of the protected phase, then some of the left-turning vehicles will leave the queue allowing the vehicles which block the entrance to the through lane to go forward and hence eliminating the blockage problem. So in this case the blockage is temporary and does not affect the through traffic since such vehicles cannot move anyway because of the red traffic light signal.

Similarly, if the situation from Figure 1b occurs and through vehicles are allowed to move, then they can clear the way for the left-turning vehicles, hence eliminating the turning lane blockage. So, again, the blockage can be temporary and not affect the traffic.

The second important thing that needs to be considered is that the significance of the turning lane blockage and overflow problems depends on the entire queue of all vehicles. The more a problem increases the overall queue, the more significant it is. If the blockage or overflow affects only few vehicles and no other vehicles arrive, then the overall traffic is not significantly disrupted. Then such cases can be considered as an insufficient grounds to increase the length of turning lane. However, if the problem occurrence leads to the vehicles not being able to complete their maneuver during the whole traffic light circle (whether it is a left turn or moving through), then this can create a traffic jam, in which case increasing of the turning lane is really required.

4. Assumptions to Relax

While considering the turning lane blockage, Kikuchi et al. (1993) assume that all vehicles arriving during the red phase are cleared during the following protected and permitted phases. This assumption often does not hold, especially on intersections with the intense traffic. Since the vehicles left from the previous phases increase the probabilities of lane blockage and overflow, the consequence of this assumption not being held is underestimation of the required turning lane length.

Furthermore, they assume that the number of vehicles able to turn during the permitted or protected left turn phase does not depend on the time of their arrival as long as they arrive by the end of the corresponding phase. However, since left turns require a particular amount of time and each vehicle must wait until all previous vehicles complete their turns before it can turn itself, if several vehicles arrive at the end of the phase, not all of them can complete their turns. This means that the later a vehicle arrives, the less likely it will have time to turn.

And finally, the number of vehicles able to turn during the permitted left turn phase, though depending on the opposing traffic volume, is assumed to be constant. However, since a permitted turn can be performed only during the gaps in the opposing traffic, this number is actually random.

In order to obtain more accurate estimation of the queue behaviour, all the discussed assumptions should be relaxed.

5. Accounting for Some of the Limitations

The effect of different signal phases mentioned before as a first limitation, is considered in Kikuchi and Kronprasert (2010). They study which states of the queues are acceptable and unacceptable for five different signalization schemes. These schemes include different sets and orders of phases. For each signalization scheme they use different time points of the cycle at which the states of the queues are studied. These time points are chosen as the points with the highest chance of through and left turning lane blockage or overflow.

The second problem that was considered in this paper is the assumption about clearing of all vehicles, which arrived during the red light phase, at the time of green light phase. In this paper they consider leftover vehicles, that is vehicles which are left from the previous phases.

The recommended length of turning lane is calculated as minimal length among those which provide that the total probability of an unacceptable state is less than a fixed threshold probability.

Thus, some of the open problems were considered by Kikuchi and Kronprasert (2010), but we still have some limitations left and some undesirable assumptions to relax.

The important thing to be determined here is criterion. We want to have an approach that allows us to find the optimal turning lane length, considering the system in general. So in this paper, the whole queue length at intersection is considered as a criterion for choosing the optimal turning lane length. The detailed explanation of this approach will be discussed in further chapters.

CHAPTER II

New Approach to the Problem

1. Approach Description

We consider our system as three queues at signalized intersection. The first one is the queue of turning vehicles and the second one is the queue of through vehicles. We call them "black queue" and "white queue" respectively, following the notation in Figure 1 where the left turning vehicles are shown as black rectangles and through vehicles as white ones. The last one is the queue of vehicles not being able to arrive in either of the two first queues due to lack of space in them. We call this a mixed queue because it consists of both left-turning (black) and through (white) vehicles. The vehicles from this queue take their place in one of the first mentioned two queues as soon as there appears a free space in them.

These three queues form one big queue, which is called total queue. Length of this queue can be found as summation of the mixed queue length and the maximum between two other queues lengths. This total queue length is considered as a criterion for choosing optimal turning lane length.

Our approach consists in determining the vectors of probabilities of having certain total queue lengths for different turning lane lengths. The decision about choosing an optimal turning lane length is made based on these probability vectors. However, in order to choose the optimal length, we should be able to compare these vectors.

One possible approach is to consider the probabilities of exceeding a particular total queue length. This is useful when one wants to ensure that the probability of total queue being longer than a specified value is less than some threshold level. The optimal turning lane length in this case is the minimal length which satisfies the condition.

Another approach is to consider the quantile queue length. Quantile length is the length for which probability of exceeding it is less than or equal to a specified threshold probability. This approach can be used the same way as the previous method if one wants to find the turning lane satisfying the same conditions. But if one wants to use another optimality criteria, the quantile length can be used as a measure of the total queue length since it is a fixed value whenever the total queue length is random. Then any criterion of optimality which is based on the fixed length, can be used.

In this work we use the latter approach and provide the algorithm of finding the quantile queue length for each set of traffic parameters and each turning lane length. The user of this algorithm then can choose the optimal turning lane length based on his/her own criterion. However if one wants to use some other characteristics of the queue rather than quantile length, that can be easily done based on the probability vectors which are also found in the algorithm.

Along with the algorithm, we also present the tables containing quantile lengths for some sets of parameters.

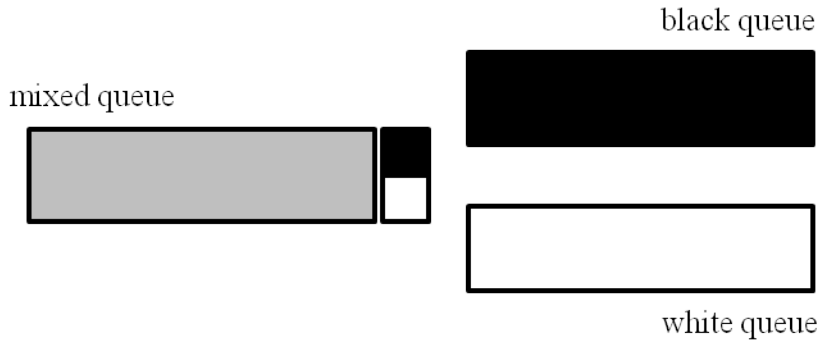
2. Mathematical Model

Now consider our approach from the mathematical point of view. As was written above, our system includes three queues: black queue, white queue and mixed queue. All those vehicles which are located in the turning lane, are, obviously turning vehicles, so they all are black. Those vehicles, which missed the turning lane entrance and went further, but have not crossed the road yet, and are still in our system, are white vehicles, because they are definitely don't want to turn.

Now notice, that until the vehicle from the mixed queue appears right before turning lane, it doesn't matter what colour it has. So, we do not consider which vehicles from the mixed queue are willing to turn or go through. But we consider colour of one vehicle, which stands right before turning lane. It can be a black vehicle, which means that as soon as space allows, it will move to the turning lane. Or it can be white. We consider this vehicle as part of either black or white queue, depending on its colour. See Figure 2.

Note that all queue lengths are expressed in numbers of vehicles. Turning lane length L is a fixed. The black and white queue length can

FIGURE 2. Black, white and mixed queues



take values from 0 to $L+1$, although they can't be both $L+1$ at the same time. The mixed queue takes values from 0 to ∞. The way to deal with the infinity will be discussed later.

We assume that the arrivals of black and white vehicles in any time interval of the length τ are independent and follow Poisson process with rate $\lambda_{black}\cdot\tau$ for black vehicles and $\lambda_{white}\cdot\tau$ for white vehicles. Also since white and black vehicles arrive independently, the arrivals of both black and white vehicles together follow Poisson distribution as well with the rate $\lambda\cdot\tau$ where the unit rate λ is the summation of the previous two: $\lambda = \lambda_{black} + \lambda_{white}$.

Note that this process can be equivalently described as the Poisson arrivals of vehicles with rate $\lambda\cdot\tau$ and they are white or black with the probabilities p and q respectively, where p and q have the following relation with the above mentioned λ_{black} and λ_{white}: $p = \lambda_{white}/\lambda$, $q = \lambda_{black}/\lambda$.

We consider the numbers of vehicles in black, white and mixed queue at the end of the red phase of traffic light. The end of the red phase is the point at which the maximum number of vehicles in the queue during the cycle is expected since they accumulate during the red phase (see Kikuchi et al., 1993). So it is the queue length at this point that determines the required turning lane length.

The numbers of vehicles in all the three queues at the end of the red phase at consequent cycles can be considered as the states of a Markov chain. That is, the probabilities of having certain numbers of vehicles in all three queues at the next cycle depend only on the numbers of vehicles in queues at the previous cycle. This assumption is reasonable if we assume that the traffic parameters do not change from cycle to cycle. And if one wants to account for non-constant traffic parameters during the day, he/she can consider the periods of time with different traffic parameters separately.

The states of the system are considered in discrete periods of time. Red light phase is considered as one time period, while the protected and permitted phases are divided into several small time intervals. These intervals have the same length and it is calculated as the greatest common divisor of protected phase length T_{pt}, permitted phase length T_{pm}, service time for black vehicles and service time for white ones.[1] Service time is time needed for either a black or white vehicle to be served, which are denoted as τ_{black} and τ_{white} respectively.

$$\tau_{int} = \gcd(T_{pt}, T_{pm}, \tau_{black}, \tau_{white}) \tag{1}$$

Such length allows us to construct both protected and permitted phases using these intervals. Moreover, now we can express service time as an integer multiple of these time intervals. When a vehicle arrives in one particular interval, it doesn't matter either it is at the beginning of this interval or at the end, it still can start being served in the same interval. However another vehicle which arrives after the one which started being served, should wait until the service of the previous vehicle ends before it can start being served. This partitioning allows us to take into account the fact that if several vehicles arrive at the end of the phase they can not all be served.

Note that the phase which follows the red has a delay at the beginning before the first vehicle starts its maneuver. So the service starts not from the very beginning. However, since no vehicles can be served during this delay, it can be simply considered as a part of the preceding red phase. Thus in future we assume that the delay is already included in the red phase and the service starts right from the beginning of the permitted or protected phase.

[1] If some of these numbers are not integers, we simply multiply them by some number making them all integer, and after finding the greatest common divisor we divide the answer by the same number. Due to computational reasons in practice it is better to firstly round off all values to integers or at least numbers with one decimal digit

CHAPTER III

Finding the Transition Matrices

As stated earlier, our approach is to find the quantile total queue length for each set of traffic parameters and each turning lane length. The first part of solving this problem is to determine the transition matrix. In this paper, the transition matrix is constructed from the separate matrices: a transition matrix for protected phase, a transition matrix for permitted phase, and a last one for the red signal phase. The transition matrix for the whole cycle then can be found by multiplying all these three matrices. Using it the quantile total queue length will be found.

In order to find each of these matrices, consider little intervals which the phases were divided into.

Introduce firstly the term "long" event.

Consider the event of serving a vehicle. This event takes some time, as the vehicle needs to quit the queue, and then either turn or go through. We take this time into account by supposing that at the same interval no more vehicles can be served except this one, and moreover, the next vehicle needs to wait some time until it can be served. Recall, that we introduced a concept "service time", the length of which is equal to integer multiple of intervals (see chapter II section 2). This is the time vehicle needs to wait.

But consider now another event. Suppose that there were $L+1$ vehicles in the queue, and one of them was served. As it takes some time for vehicle to be served, it takes some time also for $(L+1)$-st place to become free. Then the vehicles willing to come to the black or white queue need to wait this time until the queues stop being blocked, i.e. until the $(L+1)$-st vehicle will move and empty the $(L+1)$-st place. Since this depends on service time, and service time is so long that it takes one or more intervals for one vehicle to be served, the duration of this event should be taken into consideration as well. We call such events “long” events.

To take the duration of the “long” events into account, we suppose that if a vehicle needs to wait until this “long” event finishes, it won't be served in this interval.

For better understanding, consider an example. Suppose it is a permitted period, where both black and white vehicles are allowed to be served. Suppose also that there are $L+1$ vehicles in the black queue, there are no vehicles in the white queue, and there is one vehicle in the mixed queue. At the beginning of the interval the black vehicle will be served, all vehicles in the black queue will shift by one place, and the $(L+1)$-st place will be free. Suppose the vehicle from the mixed queue appears to be white. Now, when the queues are not blocked, it can easily go to the white queue. But it won't be served in this interval, because it waited until the $(L+1)$-st place became free.

As the length of this event is less than the whole serving time of vehicle, we suppose that $(L+1)$-st place will be already free by the beginning of the next interval, and so in the next interval this vehicle (white vehicle in the example above) can be served. So, our assumption is just that it can't be served in one particular interval in which this ”long” event begins.

Now introduce the term “instant” event.

We assume that it takes no time for a vehicle to go from the mixed queue to either black or white queue, when these queues are not blocked. We call such events “instant” events.

Now notice, that within one interval it doesn't matter when exactly the vehicle arrives. Despite of the exact arriving time within the particular interval, we suppose that the service of the vehicle starts at this interval if and only if the events needed to happen before this service can begin are “instant”. If there is need of “long” events before the service, the service can happen only in the next interval.

Hence, we can consider that all arrivals happen at the beginning of the interval. At the end of this arrival period, all arrivals are done, and all “instant” events are finished. The service will begin from this point, and as all “long” events include service, they will start here and continue until the end of the interval. So the service is considered only for those vehicles which are already in the queue at this point: the end of arrival period, and beginning of service. This ensures that only those vehicles which don't need to wait for "long" events will be able to be served.

So, we consider our process within this interval as if all the arrivals happen right at the beginning of the interval, and after that service of vehicles begins. In this case, we can find the transition matrix for each interval as multiplication of the two matrices: the arrival and the service matrix.

$$P^{int} = P^{arr} \cdot P^{serv} \tag{2}$$

Service of vehicles during different phases is different. But we will construct the general formulas for which the service at each phase will be special case.

1. Arrival Transition Matrix

Firstly, consider the arrival transition matrix. In order to simplify the explanation, firstly, we will construct an arrival matrix $\hat{P}^{arr}$ which is not exactly the arrival matrix from the equation (2), but the matrix P^{arr} will be easily calculated from $\hat{P}^{arr}$.

The states of the matrix $\hat{P}^{arr}$ are triples (b, w, m), where b represents the length of the black queue, w represents the length of the white queue, and m - length of the mixed queue. So, each entry of our matrix will represent transition from some state (b, w, m) to some state (b', w', m').

Note that although the matrix $\hat{P}^{arr}$ is indexed by triples, it is actually a usual square matrix where each pair of triples (b, w, m) and (b', w', m') represents a pair of integer indexes (say, i and i') of this matrix. This also concerns all the matrices indexed by several variables which we will use further. The correspondence between the multivariate indexes and the regular integer indexes that stand behind them will be given later, and now for simplicity we will use only multivariate indexes in all formulas.

Now we proceed to the construction of matrix $\hat{P}^{arr}$. Notice, that during the arrival period, no vehicles can be served, so number of vehicles can only increase. Here are three conditions on the transitions:

$$b' \geq b,\ w' \geq w,\ m' \geq m, \tag{3}$$

Those elements of transition matrix, for which these three conditions are not satisfied, are zeros, because such transitions are impossible.

From here on, for expressing the transition probabilities we will use the following notation:

(1) $\varphi(k, \lambda t) \equiv e^{-\lambda t} \cdot (\lambda t)^k / k!$ denotes a Poisson probability.
(2) p and q are the proportions of arriving white and black vehicles respectively which can be found using the following formulas:

$$p = \frac{\lambda_{white}}{\lambda_{black} + \lambda_{white}}, \qquad q = \frac{\lambda_{black}}{\lambda_{black} + \lambda_{white}}. \tag{4}$$

(3) C_k^n denotes a number of combinations from n elements taking k elements at a time, i.e.:

$$C_k^n = \frac{n!}{k!(n-k)!},\ 0 \leq k \leq n, \tag{5}$$

Consider three cases of the transitions. The first case represents the situation when both black and white queues are not overflowed after all arrivals and, therefore the mixed queue is empty. It can be expressed by the following conditions:

$$b' < L + 1,\ w' < L + 1, m' = 0. \tag{6}$$

In this case all the arriving vehicles go directly to the corresponding black or white queue. The conditions in (3) imply that no vehicles arrive in a mixed queue, i.e. $m' - m = 0$, and differences $b' - b \geq 0$, $w' - w \geq 0$ representing the number of arriving black and white vehicles respectively are non-negative. Such an arrival happens when exactly $b' - b + w' - w$ vehicles arrive, $b' - b$ of which are black, and $w' - w$ are white. Then the entries of transition matrix have the form given below:

$$\hat{P}^{arr}_{(b,w,m)\to(b',w',m')} = \underbrace{\varphi(b'-b+w'-w, \lambda t)}_{b'-b+w'-w \text{ arrive}} \cdot \underbrace{C^{b'-b+w'-w}_{b'-b} \cdot p^{w'-w} \cdot q^{b'-b}}_{w'-w \text{ of them are white and } b'-b \text{ are black}} \tag{7}$$

In the second case after all arrivals the black queue appears to be overflowed, i.e.

$$b' = L + 1, \; w' < L + 1. \tag{8}$$

The number of vehicles in mixed queue is arbitrary and depends on the number of arriving vehicles. This case can be split into two subcases: when this queue was already overflowed at the beginning of the interval, i.e. $b = L + 1$ and when it was not, i.e. $b < L + 1$.

In the first subcase the black queue is full and the white one is blocked by the last black vehicle, so all new arriving vehicles go to the mixed queue. So, $b' = b$ and $w' = w$. The probability of such a transition is equal simply to the probability that $m' - m$ vehicles arrive. See formula (9).

In the second subcase new arriving vehicles can go either to the black or to the white queue. They can go there only if both queues are not blocked yet. Since $w' < L + 1$ and $w < L + 1$, white vehicles won't block queues in this case, but since $b' = L + 1$ we know for sure that black vehicles do block the queues at the end. So for w' vehicles to be in the white queue at the end of this transition, they should come before the blockage, i.e. before number of vehicles in the black queue becomes $L + 1$. So we can not allow more than $L - b$ black vehicles to arrive before all necessary $w' - w$ white vehicles take their place in the white queue, so that total number of black vehicles in queue will not reach $L + 1$ and they won't block the white queue. Since none of the first $L - b$ black and $w' - w$ white vehicles block the entrance to the black and white queues, the order in which these first black and white vehicles arrive does not matter.

Hence, in order to have such transitions, $b' - b + w' - w + m' - m$ vehicles should come, and among the first $L - b + w' - w$ vehicles $L - b$ vehicles should be black and $w' - w$ should be white. Since $w' < L + 1$ and $L < L + 1$, black and white queues won't be blocked, and all $L - b$ black vehicles will get to the black queue as well as all $w' - w$ white vehicles will get to the white queue. The next vehicle arriving should be black, since we know that $b' = L+1$. After that no one can get to either the black queue, or to the white one. So, the remaining $m' - m$ vehicles will be stored in the mixed queue.

$$\hat{P}^{arr}_{(b,w,m)\to(b',w',m')} = \begin{cases} \varphi(m'-m, \lambda t), & \text{if } b = L+1,\ w' = w, \\ \varphi(b'-b+w'-w+m'-m, \lambda t) & \\ \quad \times C^{L-b+w'-w}_{w'-w} \cdot p^{w'-w} \cdot q^{L-b} \cdot q, & \text{if } b < L+1. \end{cases} \tag{9}$$

And finally, in the third case the white queue appears to be overflowed at the end of transition, so

$$b' < L+1, w' = L+1 \tag{10}$$

In this case we can apply all the same reasoning that we applied to the second case, except now white vehicle instead of black will block queues. So all mathematical formulas will be symmetric to those we had in the second case. Here is the formula for transition probabilities:

$$\hat{P}^{arr}_{(b,w,m)\to(b',w',m')} = \begin{cases} \varphi(m'-m, \lambda t), & \text{if } w = L+1,\ b' = b, \\ \varphi(b'-b+w'-w+m'-m, \lambda t) & \\ \quad \times C^{L-w+b'-b}_{b'-b} \cdot p^{L-w} \cdot q^{b'-b} \cdot p, & \text{if } w < L+1. \end{cases} \tag{11}$$

All the other transitions are impossible, so all entries of the arrival matrix, which were not discussed above are zero entries. Thus we have now Arrival Transition Matrix for one interval.

Here is the list of all cases with corresponding probabilities:

1. $b' < L+1, w' < L+1, m' = 0$

$$\Rightarrow \hat{P}^{arr}_{(b,w,m)\to(b',w',m')} = \varphi(b'-b+w'-w, \lambda t) \cdot C^{b'-b+w'-w}_{b'-b} \cdot p^{w'-w} \cdot q^{b'-b}$$

2. $b' = L+1, w' < L+1$
   1) $b = L+1, w' = w$

$$\Rightarrow \varphi(m'-m, \lambda t)$$

   2) $b < L+1$

$$\Rightarrow \varphi(b'-b+w'-w+m'-m, \lambda t) \cdot C^{L-b+w'-w}_{w'-w} \cdot p^{w'-w} \cdot q^{L-b} \cdot q$$

3. $b' < L+1, w' = L+1$

1) $w = L+1, b' = b$

$$\Rightarrow \varphi(m' - m, \lambda t)$$

2) $w < L+1$

$$\Rightarrow \varphi(b' - b + w' - w + m' - m, \lambda t) \cdot C_{b'-b}^{L-w+b'-b} \cdot p^{L-w} \cdot q^{b'-b} \cdot p$$

2. Service Transition Matrix

The second part of the interval transition matrix is service transition matrix.

Firstly note that by construction of the intervals, no more than one black and one white vehicle can be served in each interval. So while deriving the values of the transition matrix entries we need only to take into account whether one black, one white, both of them, or none of the vehicles are served. By serving here we mean starting to depart from the black or white queue, and by our assumptions, a vehicle which started its maneuver will finish it.

Recall, that interval length is equal to the greatest common divisor of several values, including service time of black vehicle and service time of white vehicle. This means that it takes integer number of little intervals for either black or white vehicle to be served. Hence, if one black (white) vehicle is served in a particular interval, no black (white) vehicles can be served in several subsequent intervals. We consider two variables r_b and r_w to take this fact into account. The r_b variable represents the number of intervals left until the next black vehicle can be served. If $r_b = 0$, it means that a vehicle can be served in this interval, if $r_b = 1$, it means that a vehicle can be served in the next interval, and so on. The same reasoning can be applied for the white queue and variable r_w.

Those variables should be included in the states of the system as the probabilities of transitions depend on them. Thus we have a service transition matrix where each state is expressed by five numbers (b, w, m, r_b, r_w), where we are already familiar with b, w, m, which represent numbers of vehicles in black, white and mixed queue respectively, and variables r_b, r_w have just been explained above. Each element of our transition matrix represents transition from state (b, w, m, r_b, r_w) to state (b', w', m', r_b', r_w').

Note, that for protected and permitted phases, service matrices should be different, because during permitted phase both black and white vehicles can be served, but during protected phase white vehicles are stopped, so they can't be served. In order not to write two different matrices, we will write only one matrix in general form, but we need to take this difference into consideration. So, we need to introduce two additional variables b_{serv} and w_{serv}. They both can take values 0 or 1, where 1 means that vehicles of corresponding type can be served during the particular phase and 0 means that they can not be served.

For simplicity of the following formulas consider two other variables. Notice, that whether or not the vehicle will be served depends on two factors. The first factor is the phase. The variables b_{serv} and w_{serv} show if this vehicle can be served depending on the phase. Variables r_b and r_w represents another factor. They show the ability for a vehicle to be served depending on service of the previous vehicles. If we combine those two factors, we will get the variables, that show if the vehicle actually can be served in this interval or not. Denote them by s_b and s_w. If $s_b = 1$ and there is vehicle in the black queue, it will be definitely served. If it is zero, the vehicle won't be served. The same can be said about white vehicles. This can be expressed mathematically by the formulas:

$$s_b = \begin{cases} 1, & \text{if } b_{serv} = 1 \text{ and } r_b = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

$$s_w = \begin{cases} 1, & \text{if } w_{serv} = 1 \text{ and } r_w = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Before we proceed to deriving the transition probabilities we need to introduce two more values.

As mentioned in chapter III, only those vehicles, which don't need to wait until the "long" event finishes, can be served in a particular interval. Such vehicles are in the system at the moment when the arrival period ends and the service period begins. As before, we consider transition from the state (b, w, m, r_b, r_w) to the state (b', w', m', r_b', r_w').

So, state (b, w, m, r_b, r_w) is the state at the beginning of the period we consider, i.e. service period. Then b and w are exactly those numbers of vehicles that we need to consider for service.

Now there are two possibilities for serving black vehicles: whether one black vehicle is served or none. One black vehicle can be served if number of vehicles in the black queue b is not equal to zero and if $s_b = 1$, which allows vehicle to be served. In this case the number of vehicles in the black queue will be reduced by one. Otherwise the vehicles won't be served and the number of vehicles in the black queue will remain the same.

So the number of vehicles left in the black queue just after the service b_{sb} can be expressed by the following formula:

$$b_{sb} = \begin{cases} b, & \text{if } b = 0, \\ b, & \text{if } b > 0 \text{ and } s_b = 0, \\ b - 1, & \text{if } b > 0 \text{ and } s_b = 1. \end{cases} \tag{14}$$

Similarly, for the white vehicles we introduce value w_{sw}:

$$w_{sw} = \begin{cases} w, & \text{if } w = 0, \\ w, & \text{if } w > 0 \text{ and } s_w = 0, \\ w - 1, & \text{if } w > 0 \text{ and } s_w = 1. \end{cases} \tag{15}$$

Consider these formulas in detail. The first case in each equation means that if there are no vehicles in the queue, the number of vehicles will stay the same as no vehicles can be served. If the queue is not empty, but vehicles are not allowed to be served, the number will stay the same. This is the second case of the equation above. And if the queue is not empty and, at the same time, vehicles are allowed to be served, the number of vehicles in the queue will be reduced by one.

The actual numbers of vehicles in the black and white queue b' and w' will be found later, and the formulas for them will include the introduced values b_{sb} and w_{sw}.

Formulas (14) and (15) can be rewritten in the following short form:

$$b_{sb} = \max\{b - s_b, 0\}, \tag{16}$$

$$w_{sw} = \max\{w - s_w, 0\}, \tag{17}$$

where $\max\{x, y\}$ denotes maximum between x and y.

Now we have all necessary notation and we are ready to write down formulas for elements of service transition matrix.

Firstly, we consider the change of r_b and r_w components of the state. Note that since r_b and r_w are the numbers of intervals to wait until the next vehicle can be served, they take the values from 0 up to a maximum values r_b^{max} and r_w^{max} respectively. The maximum number of intervals to wait appears in the interval following right after the service. By that time the vehicle was already being served during the previous interval, so the time to wait until its service completes is equal to the total number of intervals required for the service reduced by one. So r_b^{max} and r_w^{max} can be found as follows:

$$r_b^{max} = \frac{\tau_{black}}{\tau_{int}} - 1, \tag{18}$$

$$r_w^{max} = \frac{\tau_{white}}{\tau_{int}} - 1, \tag{19}$$

where τ_{black} is the time (expressed in time units rather than number of intervals) required for the vehicle to complete the turn, τ_{white} is the time required for the vehicle to go through, and τ_{int} is the duration of one interval.

The pattern of changing the r_b and r_w from the interval to interval is the following. If in the particular interval black vehicle is served, then r_b becomes equal to r_b^{max}, because that is the number of intervals to wait until the next service. For the next interval, there is one interval less to wait until service, so we need to reduce value of r_b by one. We continue doing this until r_b becomes equal to zero. It means that black vehicles don't need to wait for service and can be served in the current interval. Now whether the vehicle will be served in the current interval or not depends on other factors which will be discussed later. If it is not served, r_b remains 0 until the vehicle is served. And when the vehicle is served r_b becomes r_b^{max} again and the whole process repeats from the beginning. The same reasoning can be applied to the changes of r_w.

Mathematically this can be expressed as follows:

$$r_b' = \begin{cases} r_b^{max}, & \text{if } s_b = 1, b > 0 \\ \max\{0, r_b - 1\}, & \text{otherwise.} \end{cases} \tag{20}$$

$$r'_w = \begin{cases} r_w^{max}, & \text{if } s_w = 1, w > 0 \\ \max\{0, r_w - 1\}, & \text{otherwise.} \end{cases} \quad (21)$$

To derive the possible transitions from states (b, w, m, r_b, r_w) to (b', w', m', r'_b, r'_w) and their probabilities, we consider several different cases, listed below. The detailed explanation of these cases will be given later.

1. $m = 0$
    1) $b < L+1,\ w < L+1$
    2) $b = L+1,\ w < L+1$
    3) $b < L+1,\ w = L+1$
2. $m \neq 0$
    1) $b = L+1,\ w < L+1$
        a. $s_b = 0$
        b. $s_b = 1$
            (a) $b' = L+1$
            (b) $b' = L$
                i. $m \geq L+1-w_{sw}$
                ii. $m < L+1-w_{sw}$
    2) $b < L+1,\ w = L+1$
        a. ...

Here we do not present some cases similar to other ones in order to make the list shorter. We indicate them by the ellipsis (...) and discuss them later.

The first level of division into cases is based on the value of m: depending on whether it is zero or not. Consider firstly case 1. with $m = 0$. For this case there are three possibilities of having different numbers of vehicles in black and white queues listed in 1), 2) and 3).

Since there are no vehicles in the mixed queue in all these cases, no one can come either to the black or to the white queue. So, while finding the number of vehicles in a particular queue, we need to take into consideration only the number of vehicles in this queue and the variables s_b and s_w, that show us if the vehicles are allowed to be served in a given interval. The mixed queue remains empty because all the arrivals were already taken into account before and now we consider that no more arrivals happen. Whether the black vehicle will be served or not depends on the factors discussed before, and the b_{sb} variable took them into account. The same can be said about white vehicle. Hence, we have the following expressions for b', w' and m' in these cases:

$$b' = b_{sb},\ w' = w_{sw},\ m' = 0. \tag{22}$$

The expressions for r_b' and r_w' are given by the formulas (20) and (21) respectively.

Since in this case the number of vehicles in each queue at the end of the interval is determined, the corresponding transition probabilities are equal to 1:

$$P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r_b',r_w')} = 1. \tag{23}$$

Now consider the case 2. where $m \neq 0$. We can't have $b < L+1, w < L+1$ here, because mixed queue is not empty if and only if black or white queue is full. So, we have only two cases for $m \neq 0$: 2.1) and 2.2).

These two cases are symmetric, so we will discuss only the first one, and we will just give formulas for the second one.

Consider the case 2.1) with $b = L+1,\ w < L+1$. If $s_b = 0$ (case a.), which means that black vehicle cannot be served now, then we have:

$$b' = L + 1, w' = w_{sw}, m' = m. \tag{24}$$

Again, all three queue lengths are determined in this case, so the transition probability is equal to 1:

$$P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r_b',r_w')} = 1. \tag{25}$$

Recall that r_b' and r_w' are defined in (20) and (21).

Consider now situation where $s_b = 1$ (case b.), i.e. one black vehicle can be served. Then since $b = L + 1$ and $s_b = 1$, one black vehicle will be immediately served, the number of vehicles in the black queue will become L, and so they won't block the white queue any more, giving the vehicles from the mixed queue ability to take their places in the black and white queues. Now at the end of the interval there are two possible cases for b': one with $b' = L + 1$, case (a), and another one with $b' = L$, case (b).

For the case (a), firstly we describe the situations when b' can become $L+1$. Since $b' = L+1$, we already know that a black vehicle will block the black and white queues, and the question is when, i.e. how many white vehicles will be able to join the white queue before that. As written above, at the beginning of this transition, the black vehicle will be served, and there will be L vehicles in the black queue. Now consider the first vehicle in the mixed queue. If this vehicle is white, and moreover several first vehicles are white, than they all will go the white queue as long as there is enough space in there. When the vehicle in the mixed queue is black, then this process ends. This black vehicle will block the white queue. Of course, this can happen at the beginning, i.e. the first vehicle can be black, and no white vehicles will go to the white queue.

Now, let i be a number of white vehicles which come to the white queue, i.e. all white vehicles in the mixed queue before the first black vehicle. Note, that the value of i should be such that, firstly, it is less than or equal to $m-1$, the number of vehicles in the mixed queue deducted by 1, because all these vehicles come from the mixed queue and this queue also contains at least one black vehicle, which will block the queue. Secondly, the white queue should be able to store all i vehicles without overflow, because the subsequent black vehicle should be able to take its $(L+1)$-th place in the black queue. The number of vehicles able to be stored in a white queue is equal to $L-w_{sw}$ where we took into consideration the possibility of a white vehicle being served. Thus, the range I for i is the following:

$$I = \{i \mid 0 \leq i \leq \min\{L - w_{sw}, m-1\}\}\,, \tag{26}$$

where $\min\{x, y\}$ is minimum between x and y.

Then, there are the following possibilities for (b', w', m'):

$$b' = L+1,\ w' = w_{sw} + i,\ m' = m - i - 1,\ i \in I, \tag{27}$$

where the range I for i is given by (26).

From these formulas, it can be seen that the number of vehicles in the mixed queue is reduced by the same number by which the number of vehicles in the white queue was increased, because these vehicles came from the mixed to the white queue. Also, the mixed queue is reduced by one black vehicle which follows all i white vehicles. The case of $i = 0$ corresponds to the situation when the first vehicle in the mixed queue was black, and no white vehicles could go to the white queue.

The probability of such transition for fixed i that satisfies (26) is equal to the probability of the first i vehicles in the mixed queue being white, and the next vehicle being black. This black vehicle will block the queue and no one will leave the mixed queue since then. Since the probability that a particular vehicle in the mixed queue is white or black is p and q respectively, then the probability of such transition is:

$$P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = p^i q. \tag{28}$$

Now consider (b), i.e. such cases where $b' = L$. As stated, a black vehicle will be served immediately, and there will be L vehicles in the black queue. Now if a vehicle comes to the black queue, the length of it cannot be reduced, because black vehicles are not allowed to be served in this interval any more. But since we know that $b' = L$, as the number of black queue vehicles became equal to L in the beginning of this transition, we conclude that it will stay the same until the next interval, i.e. no more vehicles arrive and go away from the black queue. It means that we don't consider such situations, where the black vehicle from the mixed queue appeared in the $L+1$ place of the black queue. Hence, several first vehicles from the mixed queue are white, and the number of them is such that the process ends until the black vehicle appears in it.

This can happen in two situations. Either the white queue becomes full and it blocks the way for black vehicles, or all vehicles in the mixed queue are white and there is enough space in the white queue to store all of them. In both cases number of vehicles in the black queue doesn't change. For the first situation number of vehicles in the mixed queue is more than or equal to the number of vehicles needed to fill the white queue (case i.), and for the second case it is less (case ii.).

Consider the case i. with $m \geq L+1-w_{sw}$. The number of vehicles in the mixed queue is enough to fill the whole white queue, so if all vehicles coming from the mixed queue are white then the white queue length becomes $L+1$, and so white vehicle will block black queue ensuring that no black vehicles arrive to the black queue and its length remains L. The expressions for b', w' and m' are the following:

$$b' = L,\ w' = L+1,\ m' = m - (L+1-w_{sw}). \tag{29}$$

Probability of the transition from state (b, w, m, r_b, r_w) to (b', w', m', r'_b, r'_w) can be found as the probability that the first $L + 1 - w_{sw}$ vehicles in the mixed queue are white:

$$P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = p^{L+1-w_{sw}} \tag{30}$$

For the case ii., i.e. when $m < L + 1 - w_{sw}$, the number of vehicles in the mixed queue is not enough to make the white queue full. So, in this case, for black queue not to change its length, all vehicles in the mixed queue should be white. And all of them will go the white queue. The formulas for b', w' and m' become the following:

$$b' = L, \ w' = w_{sw} + m, \ m' = 0. \tag{31}$$

The transition probability is equal to the probability of all vehicles in the mixed queue being white:

$$P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = p^m. \tag{32}$$

This finishes the consideration of the case 2.1), i.e. when $b = L + 1, \ w < L+1$ and $m \neq 0$. The case 2.2) with $b < L+1, \ w = L+1$ and $m \neq 0$ is symmetric to 2.1) and differs only in exchanging b with w and q with p (and also corresponding b', r_b, r'_b, s_b and b_{sb} with w', r_w, r'_w, s_w and w_{sw} respectively). So for this case we just give the transition probabilities without any explanation.

Here is a list of all cases and corresponding transition probabilities, including those which were not explained.

1. $m = 0$

$$\left.\begin{array}{l} 1)\ b < L+1, \ w < L+1 \\ 2)\ b = L+1, \ w < L+1 \\ 3)\ b < L+1, \ w = L+1 \end{array}\right\} \Rightarrow \begin{cases} b' = b_{sb}, \ w' = w_{sw}, \ m' = 0 \\ P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = 1 \end{cases}$$

2. $m \neq 0$

1) $b = L+1, \ w < L+1$

a. $s_b = 0$

$$\Rightarrow \begin{cases} b' = L+1, w' = w_{sw}, m' = m \\ P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = 1 \end{cases}$$

b. $s_b = 1$

(a) $b' = L + 1$

$$\Rightarrow \begin{cases} b' = L+1,\ w' = w_{sw} + i,\ m' = m - i - 1,\ i \in I \\ P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = p^i q \end{cases}$$

(b) $b' = L$

i. $m \geq L + 1 - w_{sw}$

$$\Rightarrow \begin{cases} b' = L,\ w' = L+1,\ m' = m - (L + 1 - w_{sw}) \\ P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = p^{L+1-w_{sw}} \end{cases}$$

ii. $m < L + 1 - w_{sw}$

$$\Rightarrow \begin{cases} b' = L,\ w' = w_{sw} + m,\ m' = 0 \\ P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = p^m \end{cases}$$

2) $b < L + 1,\ w = L + 1$

a. $s_w = 0$

$$\Rightarrow \begin{cases} w' = L+1, b' = b_{sb}, m' = m \\ P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = 1 \end{cases}$$

b. $s_w = 1$

(a) $w' = L + 1$

$$\Rightarrow \begin{cases} w' = L+1,\ b' = b_{sb} + i,\ m' = m - i - 1,\ i \in I \\ P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = q^i p \end{cases}$$

(b) $w' = L$

i. $m \geq L + 1 - b_{sb}$

$$\Rightarrow \begin{cases} w' = L,\ b' = L+1,\ m' = m - (L + 1 - b_{sb}) \\ P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = q^{L+1-b_{sb}} \end{cases}$$

ii. $m < L + 1 - b_{sb}$

$$\Rightarrow \begin{cases} w' = L,\ b' = b_{sb} + m,\ m' = 0 \\ P^{serv}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = q^m \end{cases}$$

3. The Full Transition Matrices of the Phases

As mentioned before, the arrival matrix of section 1, is not exactly the one we need. Our actual states are characterized by five values (b, w, m, r_b, r_w). So we need to adjust the arrival transition matrix to this. The r_b and r_w are changed based on serving, and so we considered these changes while finding service matrix. Thus in the arrival matrix we assume that $r'_b = r_b$ and $r'_w = r_w$.

So the augmented arrival matrix P^{arr} can be found based on the arrival matrix $\hat{P}^{arr}$ as follows:

$$P^{arr}_{(b,w,m,r_b,r_w)\to(b',w',m',r'_b,r'_w)} = \begin{cases} \hat{P}^{arr}_{(b,w,m)\to(b',w',m')}, & \text{if } r'_b = r_b \text{ and } r'_w = r_w, \\ 0, & \text{otherwise,} \end{cases} \tag{33}$$

where r_b and r'_b take the values from 0 to r_b^{max}, and r_w with r'_w take the values from 0 to r_w^{max}. Note that if r_b^{max} and r_w^{max} are both equal to 0, then the augmented arrival matrix P^{arr} is equivalent to the arrival matrix $\hat{P}^{arr}$.

The transition matrix for one interval is the product of the augmented arrival matrix P^{arr} and the service matrix P^{serv} discussed before. Since the states of those two matrices are the same, the size is the same too, and therefore they can be multiplied. The mathematical formula for the size of both arrival and service matrix is the following:

$$\begin{aligned} N = (L+2)(L+2)(\hat{m}_{max}+1)(r_b^{max}+1)(r_w^{max}+1) \\ - \hat{m}_{max}(L+1)(L+1)(r_b^{max}+1)(r_w^{max}+1) \\ - (\hat{m}_{max}+1)(r_b^{max}+1)(r_w^{max}+1), \end{aligned} \tag{34}$$

where $\hat{m}_{max}$ is the adjusted maximum number of vehicles in the mixed queue, which is fixed here and its choice will be discussed in detail later (see section 1 of chapter IV). The explanation of this formula is simple. Firstly, number of all sets of numbers (b, w, m, r_b, r_w) was found, and then all impossible cases were deducted. The first term corresponds to the case when $b < L+1, w < L+1, m > 0$, and the second term corresponds to cases when b and w were at the same time $L+1$. These cases are impossible in the model considered, and so we don't include them into our matrix.

Now, in order to find the transition matrix for the whole phase we need firstly to find the number of intervals in the phase. This can be done using the following formulas:

$$n_{pt} = \frac{T_{pt}}{\tau^{int}}, \tag{35}$$

$$n_{pm} = \frac{T_{pm}}{\tau^{int}}, \tag{36}$$

where pt stands for protected phase, pm for permitted phase, and n_x, T_x and τ^{int} denote the number of intervals in the phase x, duration of the phase x and duration of one interval, respectively, with x being pt or pm. Regarding the red phase, there is no need to divide it into intervals, so the whole phase is considered as one interval.

The transition matrix for the whole phase can be found as a product of the transition matrices for each interval. Since all intervals are the same, this product reduces to raising the interval transition matrix to the power equal to the number of intervals in the phase:

$$P_{pt} = \left(P_{pt}^{arr} \cdot P_{pt}^{serv}\right)^{n_{pt}}, \tag{37}$$

$$P_{pm} = \left(P_{pm}^{arr} \cdot P_{pm}^{serv}\right)^{n_{pm}}, \tag{38}$$

$$P_{red} = P_{red}^{arr}. \tag{39}$$

Note that the red phase does not have a service component since all vehicles are stopped during it.

The matrices which appear on the right hand side of the expressions in (37)–(39) are slightly different from phase to phase.

Consider firstly the red phase. As was already mentioned before, the transition matrix does not have a service component. The r_b^{max} and r_w^{max} are both equal to 0 in this case. Since we do not partition the red phase into intervals, the length of the interval is set to be the length of the whole red phase. Then we get the following expression for P_{red}^{arr}:

$$P_{red}^{arr} = P^{arr}\Big|_{t=T_{red}, r_b^{max}=0, r_w^{max}=0}. \tag{40}$$

where the matrix P^{arr} is defined in (33). The conditions on the r_b^{max} and r_w^{max} lead to the transition matrix of the red phase being equivalent to $\hat{P}^{arr}$ for which the states are represented only by three values (b, w, m). This is reasonable because during the red phase, vehicles can only arrive at the system and so r_b and r_w do not make any sense here. This condition forces both of them to be equal to 0 so one can consider that they are not included into the states.

For the protected phase we have both arrival and service components. Also the phase is partitioned into intervals, so unlike the red phase the length of the interval here is in general not equal to the length of the whole phase. The white vehicles are stopped during this phase so we set $r_w^{max} = 0$ to exclude r_w from the consideration. The value of r_b^{max} is found by usual formula (18). Since the black vehicles are allowed to be served and white vehicles are not, the values for b_{serv} and w_{serv} are 1 and 0 respectively. Then we have the following formulas for P_{pt}^{arr} and P_{pt}^{serv} used in the equation(38):

$$P_{pt}^{arr} = P^{arr}\Big|_{t=\tau^{int}, r_w^{max}=0}, \tag{41}$$

$$P_{pt}^{serv} = P^{serv}\Big|_{b_{serv}=1, w_{serv}=0, r_w^{max}=0}. \tag{42}$$

Since r_w was excluded, the states here are represented by (b, w, m, r_b).

The permitted phase is slightly more complex. Firstly, both black and white vehicles are able to be served during this phase. Moreover, the service of the black vehicles is random since it depends on the opposing through traffic. It means that even if the black vehicle is at the beginning of the black queue in a particular interval,and it is allowed to be served (there are no other vehicles which are being served at this moment), it does not mean that it will be served in this interval. We assume that this randomness can be described by introducing the probabilities p_{pm} and q_{pm} which sum up to 1. The former represents the probability that during the interval there is a gap in the opposing traffic allowing vehicle to turn, and the latter represent the probability that opposing traffic does not allow vehicle to turn. Mathematically this can be expressed in the following formulas:

$$P_{pm}^{arr} = P^{arr}\big|_{t=\tau^{int}} \,, \tag{43}$$

$$P_{pm}^{serv} = p_{pm} \cdot P^{serv}\big|_{b_{serv}=1, w_{serv}=1} + q_{pm} \cdot P^{serv}\big|_{b_{serv}=0, w_{serv}=1} \,. \tag{44}$$

The probabilities p_{pm} and q_{pm} can depend on the interval number during the permitted phase, but they should be the same for the permitted phases of different circles. This means that these probabilities can be different for the first, second and any other interval in the permitted phase, but they are equal for all the first intervals of the permitted phases, they are equal for all the second intervals, and so on. This ensures that the final transition matrix of the whole cycle consisting of the protected, permitted and red phase is the same for all cycles. Otherwise the system becomes non-Markovian and requires different methods of analysis than used in current work. For simplicity but without loss of generality we assume that p_{pm} and q_{pm} are constant.

4. The Whole Cycle Transition Matrix

Now we have the transition matrices for all three phases of the cycle and we are close to obtaining the whole cycle transition matrix. The idea is just to multiply these three matrices in the order in which the corresponding phases follow each other during the cycle. However one can notice that these three matrices have different indexing since the states in the phases are represented by different number of variables. For the red phase we have triple (b, w, m), for protected phase – 4-tuple (b, w, m, r_b), and for permitted phase – 5-tuple (b, w, m, r_b, r_w). Since the variables r_b and r_w do not actually describe the queues, but are introduced as temporary variables, the final transition matrix for the whole cycle will represents the transitions between states expressed in terms of (b, w, m) only.

Firstly, consider variable r_w. It is used only in permitted phase. Note that at the beginning of the permitted phase r_w is always equal to 0 because regardless of the phase being protected or red light phase before the permitted, no white vehicles can be served in it, and so the first white vehicle can be served right in the first interval of the permitted phase and does not need to wait until the previous white vehicle is served. Also after the permitted phase is finished, there is no need to store the state r_w any more because white vehicles are not allowed to be served during the next two phases. Thus from the point of view of the whole permitted phase, we are interested in transitions from the states with $r_w = 0$ to the states with any r'_w. So we can just sum up the transition probabilities from the states $r_w = 0$ with regard to r'_w. Then we obtain a reduced transition matrix $\hat{P}_{pm}$ for the permitted phase which is indexed by the 4-tuples (b, w, m, r_b) entries of which can be found as follows:

$$\hat{P}_{pm\ (b,w,m,r_b)\to(b',w',m',r'_b)} = \sum_{r'_w=0}^{r_w^{max}} P_{pm\ (b,w,m,r_b,0)\to(b',w',m',r'_b,r'_w)}\ . \tag{45}$$

Now the matrices $\hat{P}_{pm}$ and P_{pt} have both the same indexing and can be multiplied following which we obtain the combined permitted-protected transition matrix. Consider two cases: where the protected phase happens after permitted, and other way around. We will consider both cases, and then they will be compared to each other.

$$P_{pmpt} = \begin{cases} \hat{P}_{pm} \cdot P_{pt}, & \text{if protected phase follows the permitted,} \\ P_{pt} \cdot \hat{P}_{pm}, & \text{if permitted phase follows the protected.} \end{cases} \tag{46}$$

From this point we do not need r_b any more, because we have already taken all the black vehicles service into consideration, and the only phase left is red. So similarly to excluding r_w from permitted transition matrix, we can exclude r_b from P_{pmpt} obtaining the reduced transition matrix $\hat{P}_{pmpt}$ indexed only by the triple (b, w, m) the entries of which are calculated as follows:

$$\hat{P}_{pmpt\ (b,w,m)\to(b',w',m')} = \sum_{r'_b=0}^{r_b^{max}} P_{pmpt\ (b,w,m,0)\to(b',w',m',r'_b)}\ . \tag{47}$$

Now we are able to multiply this matrix and the transition matrix of the red phase. Note that the order of multiplication does not matter here because the red phase and the combined permitted-protected phase follow each other cyclically. For certainty we put the transition matrix of the red phase last and obtain the following transition matrix of the whole cycle:

$$P \equiv P_{cycle} = \hat{P}_{pmpt} \cdot P_{red}. \tag{48}$$

CHAPTER IV

Finding Quantile Total Queue Length

1. Handling Infinity

In reality number of vehicles in the mixed queue can be any positive integer number, as was told in chapter II section 2. Since we won't consider infinite matrix, we need to find a suitable approach to this problem. Suppose we are interested in those states, where value m varies from 0 to the fixed number m_{max}. The way to determine this fixed number will be discussed later (see section 3 of this chapter)

Now let's consider transition matrix for all those states. If we construct transition matrix the way we did before in this paper, considering only states where $0 \leq m \leq m_{max}$, we won't take into account some possible transitions. For instance, in reality, our system can move to state with $m = m_{max}$ through the state with $m = m_{max} + 1$ (say, inside the cycle transitions were: $m_{max} \to (m_{max} + 1) \to m_{max}$), but we didn't take this into consideration, because we didn't consider the value $m_{max} + 1$ for m. It means that probability of moving to state $m = m_{max}$ will include only transitions through states where $0 \leq m \leq m_{max}$, and won't include all the rest probabilities. Hence, the computed transition probabilities will not be exact in this case. Thus we can conclude that another approach should be chosen.

Consider all states, from which our system can move to the states with $m = m_{max}$. Let's calculate firstly the maximum number of vehicles in the mixed queue, which can be gone from it after one cycle. Maximum number of black and white vehicles that can be served during the whole cycle can be found as the length of time period where they can be served dividing by time needed for vehicle to be served. These numbers should be rounded up, because if a vehicle starts being served, it will be served eventually due to our assumption (see chapter III section 2).

$$b_{max} = \left\lceil \frac{T_{pt} + T_{pm}}{\tau_{black}} \right\rceil, \qquad w_{max} = \left\lceil \frac{T_{pm}}{\tau_{white}} \right\rceil. \tag{49}$$

If during the cycle all black and white vehicles that can be served are served, then their space in the black and white queue will be occupied by vehicles from the mixed queue. It means that $b_{max} + w_{max}$ vehicles can leave the mixed queue during the cycle due to service.

We took into account number of vehicles that can be served, now consider all the other vehicles that can leave the mixed queue. Note that in order to have non-zero mixed queue there should be $L+1$ vehicles in either black or white queue. So, minimum number of vehicles in black and white queue together is $L+1$ vehicles: $L+1$ vehicles in one queue and 0 vehicles in another one. Thus, maximum number of available places in these queues is $l = 2L + 1 - (L + 1) = L$. These places can be taken by the vehicles from the mixed queue, and so L vehicles more can leave the mixed queue.

Hence, we get that the overall maximum number of vehicles that can leave the mixed queue during the cycle is $L + b_{max} + w_{max}$. It means that the state with the biggest m from which the system can move to the state with $m = m_{max}$ vehicles in the mixed queue is the state with $m = m_{max} + L + b_{max} + w_{max}$. If $m > m_{max} + L + b_{max} + w_{max}$, then there is no way that after one cycle there will be m_{max} vehicles in the mixed queue. So, to know probabilities of transitions to the states with $m = m_{max}$ vehicles we need to know the probabilities of transitions from all states up to $m = m_{max} + L + b_{max} + w_{max}$. If the system goes to any other states it can not move to the state with $m = m_{max}$ during the cycle. In order to find the probabilities of transition to the states with $m < m_{max}$, the values for m less than $m_{max} + L + b_{max} + w_{max}$ should be considered. Thus we obtain the adjusted maximum number of vehicles in the mixed queue $\hat{m}_{max}$:

$$\hat{m}_{max} = m_{max} + L + b_{max} + w_{max}. \tag{50}$$

So, in order to obtain all transition probabilities between the states with the number of vehicles in mixed queue varying from 0 to m_{max}, we need firstly to construct a bigger transition matrix: with all m such that $0 \leq m \leq \hat{m}_{max}$, and then reduce it in such way that it has only states with m from 0 to m_{max}.

This ensures that all computed transition probabilities among the states with $m \leq m_{max}$ are exact.

2. Steady-state Vector

Since we reduce the real transition matrix, which has infinite numbers of states, the matrix we have now is no longer a transition matrix, because its rows do not sum to 1. Following Kikuchi et al. (1993), we simply change the last column of the matrix according to the formula:

$$P_{in} = 1 - \sum_{j=0}^{n-1} P_{ij}, \quad i = 0, 1, \ldots, n, \tag{51}$$

where indexes i and j are the integer indexes corresponding to the triples (b, w, m) and (b', w', m'), and n is the maximum index of matrix P, i.e. the size of the matrix P is $(n+1) \times (n+1)$:

$$n = (L+2)(L+2)(m_{max}+1) - (L+1)(L+1)m_{max} - (m_{max}+1). \tag{52}$$

This formula is a special case of formula (34) with r_b^{max}, and r_w^{max} set to 0 and $\hat{m}_{max}$ replaced with m_{max}.

Now when the obtained matrix P satisfies the conditions of a transition matrix, i.e. all its entries belong to the interval $[0, 1]$ and each row sums to 1, we can obtain a steady state vector π by solving the following system of equations:

$$\begin{cases} \pi &= \pi P, \\ \sum_i \pi_i &= 1. \end{cases} \tag{53}$$

The solution of this system exists and it is unique if the Markov chain represented by the matrix P is irreducible and positive recurrent (see Ross, 2009). If, in addition, this Markov chain is aperiodic, the obtained steady-state vector is the limiting probability vector.

We are searching for the limiting probability vector, so we need to show that our Markov chain is irreducible, aperiodic and positive recurrent.

The Markov chain is called irreducible if the transition from any state to any state is possible in a finite number of steps. Consider the state $(0, 0, 0)$ (the corresponding integer index is 1). We can show that the transition from this state to any other state is possible in one step. Consider the transition to the particular state (b, w, m). We can assume that no vehicles arrive during the protected and permitted phase. Then in order for the system to move to the desired state (b, w, m) it is sufficient that $b + w + m$ vehicles arrive during the red phase, and if $b \leq w$ then the first b vehicles of them are black and the next w vehicles are white, and if $b > w$ then we have the opposite situation: first w vehicles are white and the next b vehicles are black. Clearly, this case is an example of transition to the state (b, w, m) which has non-zero probability. This proves that the probability of moving to any state from the state $(0, 0, 0)$ is positive.

The probabilities of the opposite transitions, from any state (b, w, m) to $(0, 0, 0)$, are positive as well, since there is non-zero probability that there will be no arrivals during a finite number of cycles, and we can always find the sufficiently large number of cycles after which all the vehicles in the queues will be served no matter how long the queues are. This proves that probability of transition from any state (b, w, m) to any state (b', w', m') is positive since there exists a transition $(b, w, m) \to (0, 0, 0) \to (b', w', m')$ with non-zero probability.

The aperiodicity can be proved as follows. The irreducible Markov chain is aperiodic if and only if any arbitrary state of it is aperiodic (see Ross, 2009). Consider the state $(0, 0, 0)$. In this state all three queues are empty. So if no vehicles arrive during the whole cycle, no vehicles will be served either. So the system will transit into the same state $(0, 0, 0)$. Since this transition occurs in one step, the state $(0, 0, 0)$ is aperiodic. Thus, the whole Markov chain is aperiodic.

Now consider the positive recurrence property of our Markov chain. Regarding the finite version of the matrix P, the corresponding finite-state Markov chain is positive recurrent. Indeed, in a finite-state Markov chain all recurrent states are positive recurrent (see Ross, 2009). Now, if any arbitrary state of the irreducible Markov chain is recurrent then all states are recurrent. Hence, since any finite-state Markov chain has at least one recurrent state, we conclude that the Markov chain represented by the finite matrix P is positive recurrent.

However, as regards the infinite Markov chain, i.e. the original Markov chain with $m = 0, \dots, \infty$, it is not always positive recurrent. The necessary condition for the positive recurrence here is that the average numbers of black and white vehicles arriving during the whole cycle is less than the effective numbers of black and white vehicles which can be served during the cycle:

$$\bar{N}_b^{arr} < \bar{N}_b^{serv}, \quad \bar{N}_w^{arr} < \bar{N}_w^{serv}, \tag{54}$$

where $\bar{N}_b^{arr}$ is the average number of black vehicles which arrive during the whole cycle and $\bar{N}_b^{serv}$ is the effective number of black vehicles which can be served, and $\bar{N}_w^{arr}$ and $\bar{N}_w^{serv}$ are the same numbers for white vehicles. All these numbers are computed using the following formulas:

$$\bar{N}_b^{arr} = \lambda_{black}(T_{pm} + T_{pt} + T_r), \qquad \bar{N}_b^{serv} = \left\lceil \frac{T_{pt}}{\tau_{black}} \right\rceil + \left\lceil \frac{T_{pm}}{\tau_{black}} p_{pm} \right\rceil, \tag{55}$$

$$\bar{N}_w^{arr} = \lambda_{white}(T_{pm} + T_{pt} + T_r), \qquad \bar{N}_w^{serv} = \left\lceil \frac{T_{pm}}{\tau_{white}} \right\rceil. \tag{56}$$

If these conditions do not hold, the queue in the system will infinitely grow. Note that we a little overestimate the service to ensure that the conditions are necessary.

Thus, from all discussed above we see that since the finite Markov chain is irreducible, aperiodic and positive recurrent, there exists the unique solution π of the system of equations (53) regardless of the fact if the actual infinite-state Markov chain is positive recurrent or not.

For the positive recurrence of the infinite-state Markov chain, we firstly check the necessary condition discussed above, and if it does not hold we know for sure that the system does not have limiting vector. And if this condition holds, we use the heuristic method to determine whether the limiting vector exists or not. We increase the value of m_{max} and study how the results change. If the Markov chain is positive recurrent, the results should stabilize after some m_{max}. So if they do not stabilize for large values of m_{max} we consider that the Markov chain is not positive recurrent and hence does not have the limiting probability vector. It means that the queue in the system will grow infinitely for the considered set of parameters.

The entries $\pi_0, \ldots, \pi_n$ of the obtained vector π represent the probabilities that the Markov chain is in the states $0, 1 \ldots, n$ when the Markov chain works in steady-state mode. Every index $0, 1, \ldots, n$ corresponds to some triple (b, w, m), so we obtained probabilities of the system being in states with all possible sets of values (b, w, m).

3. Finding the Turning Lane Length Based on Quantile Total Queue

Denote the index corresponding to (b, w, m) as $\mathrm{I}(b, w, m)$. It can be found as follows:

$$\mathrm{I}(b,w,m) = \begin{cases} b(L+2)+w, & \text{if } m = 0, \\ 2(m-1)(L+1)+(L+1)(L+3)+\hat{w}+\hat{b}, & \text{if } m \neq 0 \end{cases} \tag{57}$$

where

$$\hat{w} = \begin{cases} w, & \text{if } b = L+1 \\ L+1, & \text{otherwise,} \end{cases} \qquad \hat{b} = \begin{cases} 0, & \text{if } b = L+1 \\ b, & \text{otherwise} \end{cases} \tag{58}$$

Now we are ready to find the probabilities of the system having different total queue length.

We are going to get the vector π^L, where the first component of it π_0^L is the probability of having the total queue length equal to 0, the second component π_1^L is the probability of total queue length being equal to 1, etc.

Notice that given values (b, w, m) the total length can be calculated using the following formula:

$$L_T(b,w,m) = m + \max\{b, w\}, \tag{59}$$

where $L_T(b, w, m)$ is the total queue length, and $\max\{x, y\}$ is maximum number between x and y.

Note, that if we find total length for each index of vector π (for each set of (b, w, m), which is the same), in general we will have several indices corresponding to the same total queue length. So in order to find probability of total length being exactly $0, 1, \ldots,$ or n, we need to sum all such entries of vector π, corresponding lengths of which are all equal to the length we need.

$$\pi_i^L = \sum_{\substack{(b,w,m): \\ L_T(b,w,m)=i}} \pi_{\mathrm{I}(b,w,m)}, \tag{60}$$

where $\mathrm{I}(b, w, m)$ is defined in (57).

Now based on the vector π_L one can find the optimal total queue length. It was already mentioned that optimality of queue length can be considered from different points of view (see chapter II section 1) and so different values of interest can be found based on π_L. In this paper we consider the quantile queue length, that is such length for which probability of exceeding it is equal to or less than a specified threshold probability, as the main value of interest:

$$N_\alpha^*(L) = \min \left\{ n \left| \left(1 - \sum_{i=0}^{n} \pi_i^L \right) \leq \alpha \right. \right\}, \tag{61}$$

where α is a fixed threshold probability, and $N_\alpha^*(L)$ is the α-quantile queue length for the fixed turning lane length L (see Kikuchi et al., 1993).

The interest in finding the quantile total queue length can be explained by the fact that it is a single value which characterizes the random queue length very good, giving the understanding of how good the intersection works.

One can compute $N_\alpha^*(L)$ for different values of L and choose the optimal turning lane length based on some criterion which uses these values. For example, one can minimize the cost of building the turning lane of the length L but at the same time include the penalty function which increases with $N_\alpha^*(L)$.

Recall, that in chapter IV section 1 the maximum number of vehicles m_{max} in the mixed queue was fixed. Having this number fixed, one can apply all techniques discussed in this paper and find the total queue length using formula (61).

However the obtained value $N_\alpha^*(L)$ is not the actual one because we used not the actual infinite vector of probabilities. Instead of it we used special techniques to construct a reduced transition matrix, and found the reduced stationary vector.

In order to get the quantile queue length close to the real one, we need to study the behaviour of computed total queue length for different values of m_{max}. The bigger value of m_{max} is, the closer to the actual value of $N^*_\alpha(L)$ we are. So, when we make this value bigger, we get better results. Since we can't make it infinitely large, we need to find the minimum value of m_{max}, beginning from which the values of quantile total queue length does not change. That is how we choose the value for m_{max}.

4. Quantile Total Queue Length for Different Parameters

We implement the model proposed in this work in a Matlab program. This program computes the limiting probability vector for the total queue length for the specified input parameters. Based on this vector it computes the α-quantile queue length $N^*_\alpha(L)$.

We run this program for different sets of parameters and present the results in a tabular form.

Among the parameters we have service time for both white and black vehicles, which we consider to be fixed. In our computations we assume $\tau_{white} = 1$ and $\tau_{black} = 3$. As threshold level the 5% probability is chosen.

Also we have other parameters such as durations of the protected, permitted and red phases, the volumes of both through and left-turning vehicles, and the probability for left-turning vehicle to be able to turn during the permitted phase p_{pm}. We consider different values for these parameters to compare the results.

For p_{pm} the values 0.3 and 0.7 were chosen. This way we can see how the opposing through vehicles influence the traffic. The values of the other parameters are taken from Kikuchi and Kronprasert (2010). We consider the following values for volumes of through and left-turning vehicles: 200, 400, 600, and 800 vehicles per hour. The percentage of the left-turns among all arriving vehicles takes values 30%, 50% and 70%. For different percentage different durations of the phases are considered. For 30% the duration of the protected, permitted, and red phases is 15 sec, 30 sec, and 45 sec respectively. For 50% we have 19 sec, 26 sec, and 45 sec; for 70%: 25 sec, 20 sec, and 45 sec.

The results are presented in the Tables 2–7 below.

TABLE 2. 95-th percentile total queue length for duration of protected/permitted/red phase equal to 15/30/45 seconds and $p_{pm} = 0.3$

Through	Left-Turn	Turning lane length														
Volume	Volume	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
140	60	6	5	5	5	5										
		7	7	6	6	6										
280	120	9	9	9	8	8	8	8	8							
		11	11	11	10	10	10	9	9							
420	180	13	13	12	12	11	11	11	10	10	10	10	10			
		16	15	15	15	14	14	14	13	13	13	13	13			
560	240	21	18	16	15	15	14	14	13	13	13	13	13	13	13	13
		23	20	19	19	18	18	18	17	17	17	16	16	16	16	16

TABLE 3. 95-th percentile total queue length for duration of protected/permitted/red phase equal to 15/30/45 seconds and $p_{pm} = 0.7$

Through	Left-Turn	Turning lane length														
Volume	Volume	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
140	60	6	5	5	5	5										
		7	7	6	6	6										
280	120	9	9	9	8	8	8	8	8							
		11	11	11	10	10	10	9	9							
420	180	13	12	12	11	11	11	11	10	10	10	10	10			
		16	15	15	15	14	14	14	13	13	13	13	13			
560	240	16	15	15	15	14	14	14	13	13	13	13	13	13	13	13
		20	19	19	19	18	18	18	17	17	17	16	16	16	16	16

TABLE 4. 95-th percentile total queue length for duration of protected/permitted/red phase equal to 19/26/45 seconds and $p_{pm} = 0.3$

Through Volume	Left-Turn Volume	Turning lane length														
		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
100	100	6 6	5 6	5 5	5 5											
200	200	10 11	9 10	8 10	8 9	7 8	7 8	7 8								
300	300	19 20	15 16	14 15	12 14	11 13	10 12	10 12	10 11	10 11	10 11					
400	400	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞

TABLE 5. 95-th percentile total queue length for duration of protected/permitted/red phase equal to 19/26/45 seconds and $p_{pm} = 0.7$

Through Volume	Left-Turn Volume	Turning lane length														
		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
100	100	5 6	5 6	5 5	5 5											
200	200	9 11	9 10	8 10	7 9	7 8	7 8	7 8								
300	300	13 16	12 15	11 14	11 13	10 13	9 12	9 11	9 11	9 11	9 11					
400	400	24 25	19 21	17 19	15 18	14 17	13 16	13 16	12 15	11 14	11 13	11 13	11 13			

For each set of the parameters discussed above these tables contain the determined 95-th percentile queue length for several values of turning lane length. The smallest possible turning lane length we consider is two vehicles. This length is recommended by Yekhshatyan and

TABLE 6. 95-th percentile total queue length for duration of protected/permitted/red phase equal to 25/20/45 seconds and $p_{pm} = 0.3$

Through	Left-Turn	Turning lane length														
Volume	Volume	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
60	140	6 6	6 6	5 5	5 5											
120	280	11 11	10 10	9 9	9 9	9 8	8 8	8 8								
180	420	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞
240	560	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞

TABLE 7. 95-th percentile total queue length for duration of protected/permitted/red phase equal to 25/20/45 seconds and $p_{pm} = 0.7$

Through	Left-Turn	Turning lane length														
Volume	Volume	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
60	140	6 6	5 6	5 5	5 5											
120	280	10 11	9 10	9 9	8 9	8 8	8 8	8 8								
180	420	18 18	15 15	14 14	13 13	12 13	12 12	12 12	11 11	11 11	11 11					
240	560	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞	∞ ∞

Schnell (2008) as a minimum turning lane length. The 95-th percentile queues were found for all turning lane lengths up to the case when the lengths of the turning lane and the total queue are equal. We assume that there is no need in turning lane which is bigger than the 95-th percentile queue.

In each table, one can see two rows of numbers for each fixed through and left-turn volume. Recall, that while finding the transition matrix, we considered two cases: the first one is when the protected phase goes first and is followed by permitted, and the second case is the opposite: the permitted phase goes first and is followed by protected (see formula (46)). So the numbers in the first of two rows represent the total queue length in the first case and numbers in another row correspond to the second case.

5. Analysis of the Results

5.1. Optimal order of phases. Firstly notice that the total queue length in the upper rows is usually smaller than in the lower rows. It means that the intersection works better when protected phase goes before permitted. This can be explained by the following argument.

For the case when the percentage of left-turning vehicles is 30% there are not many of these vehicles, but they still can disrupt through traffic. So if protected phase goes first, some of the left-turning vehicles get served thus decreasing the probability of blocking the road for white vehicles.

As regards the case with 50% percents of left-turning vehicles, the previous argument still applies here. The main reason for it is that left-turning vehicles service time used in computations is three times bigger than service time of through vehicles ($\tau_{black} = 3$ vs $\tau_{white} = 1$). So it is better to start serving left-turning vehicles first so that they do not make through vehicles wait.

Now consider the last two tables (Table 6 and Table 7). Here the percentage of left-turning vehicles is big (70%). We see that almost all lengths for different orders of protected and permitted phases are the same. And moreover, for the volume of 400 vehicles per hour, $p_{pm} = 0.3$, and turning lane length equal to 6, the total queue length for the case with permitted first is bigger. It could be caused by very big left-turning volume. Since the majority of arriving vehicles are left-turning, it is crucial not to disrupt their service. So in this case it is better to start with permitted phase which can clear the queue of through vehicles thus decreasing the probability of turning lane blockage and allowing to serve more left turns during the following protected phase.

Note that actually the cases with 30%, 50% and 70% of left turns can not be compared directly since they have different durations of protected and permitted phases. These durations are taken from Kikuchi and Kronprasert (2010) where they are chosen so to make traffic better. Thus, for instance, the duration of protected period for larger percentage of the turning vehicles is chosen to be longer than for the smaller.

However, even with different durations of phases, one can see the influence of the left-turn percentage on the optimal order of protected and permitted phases.

5.2. Infinite queues. Notice that for some sets of parameters, the 95-th percentile queues are marked as infinite. For these cases the average number of vehicles arriving during the cycle is greater than the effective number of vehicles that can be served, which means that the necessary condition of positive recurrence given in (54) does not hold. This implies that the equation (53) does not have solutions in infinite-state case (see chapter IV section 2).

If one tries to run the program for these sets of parameters for different m_{max}, each time the output for the total queue length will be the maximum total length possible for chosen m_{max}. So if one does not check the necessary condition (54) and runs the program right away, the obtained output should be considered as an indicator of possible not positive recurrence.

In such cases the system does not have a steady-state vector, and the queue tends to increase from cycle to cycle. So the limiting 95-th percentile (as well as any other quantile) total queue length is infinite.

This means that increasing the turning lane will not have any effect for such traffic, and hence, one should consider increasing the proportion of protected and permitted phases compared to red or building additional through and turning lanes.

5.3. Dependence on the opposing traffic. As mentioned, we consider the lengths for different values of the probability p_{pm} with which turning vehicles are able to turn during the permitted period.

This probability is determined by the intensity of the opposing traffic: the more intense this traffic is, the less gaps are available for turning vehicles to make their turn. So while we do not consider the opposing volume explicitly, it is actually represented by the value of p_{pm}.

The results for probabilities 0.3 and 0.7 are very similar, but still we can notice that the queues for $p_{pm} = 0.7$ are shorter, which makes sense, because in this case more left-turning vehicles can be served. For some cases increasing this probability from 0.3 to 0.7 even leads to the queue becoming finite while it is infinite for $p_{pm} = 0.3$. So in this case the opposing traffic affects the queue dramatically.

So, we can conclude that, as expected, the intense opposing traffic increases the total queue length and even can lead to unstable system with infinite queue.

6. Finding Optimal Turning Lane

Note that the Tables 2–7 do not contain the recommended values for turning lane length, as it was, for example, in Kikuchi et al. (1993) or Kikuchi and Kronprasert (2010). They choose some criterion and obtain the turning lane length which optimizes this criterion. We, on the other hand, leave the choice of the optimality criterion to user, and provide him or her with the necessary information about the queue in the system allowing to make an informed decision about the optimal turning lane length.

In order to obtain the optimal turning lane length using our model, one should compare the resulting total queue length for different sizes of turning lane and choose the one which gives the sufficiently short total queue while being not very long itself.

Here are some facts which should be taken into account while making this decision.

Firstly, if the total queue length is the same for different turning lane lengths, it is reasonable to choose the smallest turning lane among them. It will save both money and space while providing the same 95-th percentile queue. You can see a lot of examples of such situation in the Tables 2–7.

Secondly, if we can make total queue much smaller by changing the length of the turning lane only by one vehicle, it is probably reasonable to do so. For example, see Table 5. For the through and left-turn volumes both equal 400 vph, the increase of turning lane length from $L = 2$ to $L = 3$ in the protected-phase-first case decreases the total queue length by 5 (from 24 to 19). So such increase gives a very high effect.

The decision about the further increase of the turning lane depends on the available funds and space for turning lane construction, and also on the requirements to the intersection. If the available resources allow to increase the turning lane length to 10, the corresponding total queue length can be decreased almost twice (from 19 to 11). The further increase of turning lane is unreasonable since it does not affect the total queue.

If there is a restriction on total queue length that somebody wants not to exceed, he or she can easily make their decision about turning lane length based on the presented tables (or similar tables obtained for other sets of parameters). It suffices to choose the minimal turning lane for which the total queue does not exceed the specified length. This restriction can be, for example, based on the distance to the previous intersection on the road.

In general, the decision about the optimal turning lane is made by comparing the money needed to be spent on the certain turning lane and the result that can be obtained.

7. Conclusions

In this work we proposed the model of determining the total queue length of through and left-turning vehicles at signalized intersection. This model allows to find the length of the queue for various sets of traffic volumes, percentages of left turns, probabilities of finding the gap in the opposing traffic (for permitted turn), as well as different durations of the signalization phases, orders of the phases and turning lane lengths.

The queue length can be obtained for protected first, permitted first and permitted only signalization schemes, although the results for the latter scheme were not presented in this work.

The proposed model does not provide user with recommendations about the turning lane length. Instead, it gives the user all necessary information about the queue in the system for different values of turning lane length, which is required in making an informed decision.

The model can be used not only for choosing the optimal turning lane length but also for choosing the optimal signalization scheme as well as the optimal duration of signal phases.

All procedure of applying the model was implemented in a Matlab program using which the total queue length was computed for some sets of parameters. The obtained results were properly analysed and interpreted by author.

The ability of the model to compute the probabilities of actual total queue length, compared to the probabilities of the certain events considered in existing literature, makes the proposed model a powerful tool in making decisions about managing traffic at signalized intersections.

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