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THE SIGNATURE OF THE SHAPOVALOV FORM ON IRREDUCIBLE VERMA MODULES

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Abstract. A Verma module may admit an invariant Hermitian form, which is unique up to a real scalar when it exists. Suitably normalized, it is known as the Shapovalov form. The collection of highest weights decomposes under the affine Weyl group action into alcoves. The signature of the Shapovalov form for an irreducible Verma module depends only on the alcove in which the highest weight lies. We develop a formula for this signature, depending on the combinatorial structure of the affine Weyl group.

Classifying the irreducible unitary representations of a real reductive group is equivalent to the algebraic problem of classifying the Harish-Chandra modules admitting a positive definite invariant Hermitian form. Finding a formula for the signature of the Shapovalov form is a related problem which may be a necessary first step in such a classification.

1. Introduction

1.1. **Unitarizability and invariant Hermitian forms.** Classically, the fundamental concept of Fourier analysis was that an essentially arbitrary function could be expanded as a linear combination of exponentials. The more recent development of ideas in group theory has illuminated the dependence of results in Fourier analysis on group-theoretic concepts, resulting in the movement from Euclidean spaces to the more general setting of locally compact groups. Results such as the Peter-Weyl Theorem give us a means of decomposing function spaces of a compact group G into an orthogonal direct sum of subspaces expressed in terms of characters of irreducible unitary representations of G. Equipped with this decomposition and knowledge of these simpler subspaces, one can reformulate problems in analysis in more tractable settings. Quantum mechanics is another source of problems connected to unitary representations. Because of its implications for many different areas of mathematics and physics, the study of unitary representations has been an active area of research.

The irreducible unitary representations of an abelian group are one dimensional (characters). In the case of a locally compact abelian group, Pontrjagin showed that the unitary dual (the set of equivalence classes of irreducible unitary representations) $\widehat{G}_u$, furnished with pointwise multiplication of characters as the product, has the structure of a locally compact abelian group. In this situation, the unitary dual has the additional property that its unitary dual is G.

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Investigation of the non-abelian case began with the study of compact groups. In the 1920s, Weyl described the irreducible unitary representations of a compact, connected Lie group. For a locally compact group, (for example, a real or complex reductive group), the problem of describing the unitary dual remains unsolved, with the exception of some special cases.

In the interests of classifying the irreducible unitary representations, we wish to study a broader family of representations: those which admit an invariant Hermitian form. Unitarity of a representation amounts to the existence of a positive definite invariant Hermitian form on the underlying vector space, hence our objective will be, in particular, to investigate the signatures of invariant Hermitian forms and to understand how positivity can fail.

1.2. **Historical background.** Let G be a real reductive Lie group and let K be a maximal compact subgroup of G. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the corresponding Lie algebras, and let $\mathfrak{g}$ and $\mathfrak{k}$ be their complexifications. A Harish-Chandra module M is a complex vector space which is:

a) a $(\mathfrak{g}, K)$-module:
M has compatible actions by $\mathfrak{g}$ and K, and every $m \in M$ lies in a finite-dimensional K-invariant subspace
b) admissible:
the i-isotypic subspace of M is finite-dimensional for every irreducible unitary representation i of K
c) finitely generated over $U(\mathfrak{g})$.

To an admissible representation (π, V) of G, we associate in a natural way a Harish-Chandra module $V_{K-\text{finite}}$, known as *the Harish-Chandra module of* V. We define $V_{K-\text{finite}}$, the set of K-finite vectors, to be the set of vectors which lie in a finite dimensional K-invariant subspace of V.

For irreducible unitary representations, infinitesimal equivalence (the Harish-Chandra modules are isomorphic) implies unitary equivalence. Furthermore, for any irreducible Harish-Chandra module M with a positive definite invariant Hermitian form, one can construct an irreducible unitary representation (π, V) so that M is the Harish-Chandra module of V. (See [10].) It follows that classifying the irreducible unitary representations of G is equivalent to the algebraic problem of classifying the Harish-Chandra modules admitting a positive definite invariant Hermitian form.

Verma modules may admit an invariant Hermitian form, which is unique up to multiplication by a real scalar when it exists. Suitably normalized, this Hermitian form is called the Shapovalov form. Finding a formula for the signature of the Shapovalov form is a related problem which may be a necessary first step in classifying the Harish-Chandra modules admitting a positive definite invariant Hermitian form. The Shapovalov form on $M(\lambda)$ exists for λ in a subspace of $\mathfrak{h}^*$, where $\mathfrak{h}$ is a maximally compact Cartan subalgebra. This will be discussed further in Section 2. Previously, Nolan Wallach computed the signature of the Shapovalov form for a region corresponding roughly to the intersection of that subspace with the negative Weyl chamber (a computation which we extend in this paper). In the following, we will describe its implications for the unitarizability of $(\mathfrak{g}, K)$-modules.

In lectures at the Institute for Advanced Studies in 1978, Zuckerman introduced an algebraic method of constructing all admissible $(\mathfrak{g}, K)$-modules using homological algebra machinery known as cohomological induction. (See [8].)

Let L be a θ-stable Levi subgroup of G with corresponding complexified Lie algebra $\mathfrak{l}$, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$. Observe that representations of $\mathfrak{l}$ can be extended to representations of $\mathfrak{q}$ by allowing $\mathfrak{u}$ to act trivially.

Let $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ be the category of $(\mathfrak{g}, \mathfrak{k})$-modules. Consider the induction functor

$$\operatorname{ind}_{\mathfrak{q},\mathfrak{l}\cap\mathfrak{k}}^{\mathfrak{g},\mathfrak{l}\cap\mathfrak{k}}(Z) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} Z$$

from $\mathcal{C}(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$ to $\mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$. This functor, when applied to $Z = \mathbb{C}_\lambda \otimes V$ where $\lambda \in z(\mathfrak{l})^*$ and V is an $(\mathfrak{l}, L \cap K)$-module, produces what are known as generalized Verma modules. When applied to $Z = \mathbb{C}_\lambda$ in the special case where our parabolic subalgebra is a Borel subalgebra, it produces the Verma module of highest weight λ.

Let the functor $\Gamma : \mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}) \to \mathcal{C}(\mathfrak{g}, \mathfrak{k})$ be such that $\Gamma(V)$ is the set of $\mathfrak{k}$-finite vectors of V. The functor Γ is covariant and left exact. As $\mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ has enough injectives, we can form the Zuckerman functors: $\Gamma^j : \mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}) \to \mathcal{C}(\mathfrak{g}, \mathfrak{k})$, where Γ^j is the j^{th} derived functor of Γ.

By composing the induction functor with the Zuckerman functors Γ^j, we obtain cohomological induction functors which take $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})$-modules to $(\mathfrak{g}, \mathfrak{k})$-modules.

In [2], Enright and Wallach show for admissible $V \in \mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ and m equal to the dimension of the compact part of $\mathfrak{u}$ that $\Gamma^j(V^h) \simeq \left(\Gamma^{2m-j}(V)\right)^h$, where the superscript h denotes Hermitian dual . In particular, if V admits a non-degenerate invariant Hermitian form so that $V^h \simeq V$, then $\Gamma^m(V) \simeq (\Gamma^m(V))^h$. Thus $\Gamma^m(V)$ also admits a non-degenerate invariant Hermitian form.

Subsequently in [12], Wallach lifts information concerning the signature of the invariant Hermitian form on $V \in \mathcal{C}(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})$ to the invariant Hermitian form on the generalized Verma module $\operatorname{ind}_{\mathfrak{q},\mathfrak{l}\cap\mathfrak{k}}^{\mathfrak{g},\mathfrak{l}\cap\mathfrak{k}}(\mathbb{C}_\lambda \otimes V)$ (known as the *Shapovalov form*, which we will describe further in the following subsection). Finally, he lifts that information, using knowledge of the isomorphism $\Gamma^m(X) \simeq (\Gamma^m(X))^h$, to the form on the cohomologically induced $(\mathfrak{g}, \mathfrak{k})$-module $\Gamma^m\left(\operatorname{ind}_{\mathfrak{q},\mathfrak{l}\cap\mathfrak{k}}^{\mathfrak{g},\mathfrak{l}\cap\mathfrak{k}}(\mathbb{C}_\lambda \otimes V)\right)$. He concludes that if the form on V is positive definite and λ lies in a particular region bounded by hyperplanes, which we shall call the Wallach region, then the $(\mathfrak{g}, \mathfrak{k})$-module produced is also unitarizable.

In this paper, we extend the formula for the signature of the Shapovalov form beyond the Wallach region. We compute the signature of the Shapovalov form for all irreducible Verma modules which admit an invariant Hermitian form. The formula is stated in Theorem 6.12.

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2. An introduction to the Shapovalov form

We will use the following notation:

- $\mathfrak{g}_0$ denotes a real semisimple Lie algebra
- θ is a Cartan involution of $\mathfrak{g}_0$

- $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the Cartan decomposition corresponding to θ
- $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a θ-stable Cartan subalgebra and corresponding Cartan decomposition

We drop the subscript 0 to denote complexification. We let $B(\cdot,\cdot)$ denote the Killing form, which is a symmetric, invariant, non-degenerate bilinear form on $\mathfrak{g}$. We let $(\cdot,\cdot)$ denote the symmetric bilinear form on $\mathfrak{h}^*$ induced by B.

Definition 2.1. A Hermitian form $\langle\cdot,\cdot\rangle$ on a $\mathfrak{g}$-module V is **invariant** if it satisfies

$$\langle Xv, w\rangle + \langle v, \bar{X}w\rangle = 0$$

for every $X \in \mathfrak{g}$ and every $v, w \in V$, where $\bar{X}$ denotes the complex conjugate of X with respect to the real form $\mathfrak{g}_0$.

We wish to define the Hermitian dual of a representation of $\mathfrak{g}$. In order to do so, we first define the conjugate representation:

Definition 2.2. Given a representation (π, V), we define the **conjugate representation** $(\bar{\pi}, \bar{V})$ as follows: the vector space $\bar{V}$ is the same vector space as V, but with the following definition of multiplication by a complex scalar: $z\bar{\cdot}v = \bar{z}\cdot v$ where by $\cdot$ and $\bar{\cdot}$ we mean scalar multiplication in V and $\bar{V}$, respectively. We define $\bar{\pi}(X) = \pi(\bar{X})$ for all $X \in \mathfrak{g}$, where conjugation is with respect to the real form $\mathfrak{g}_0$.

Observe that every weight μ of V under π gives rise to a weight $\bar{\mu}$ of $\bar{V}$ under $\bar{\pi}$, where $\bar{\mu}(H) = \overline{\mu(\bar{H})}$ for every $H \in \mathfrak{h}$.

Definition 2.3. The **Hermitian dual** of the representation (π, V) is (π^h, V^h), the conjugate representation of the contragredient representation of (π, V).

If V is the direct sum of weight spaces V_μ for $\mu \in I$, then V^h is the direct product of weight spaces $V^h_{-\bar{\mu}}$ for $\mu \in I$.

Theorem 2.4. *An irreducible representation (π, V) admits a non-degenerate invariant Hermitian form if and only if (π, V) is isomorphic to a subrepresentation of its Hermitian dual.*

Definition 2.5. In the case where (π, V) is the Verma module $M(\lambda)$ with generator v_λ, the **Shapovalov form**, which we will denote by $\langle\cdot,\cdot\rangle_\lambda$, is the invariant Hermitian form for which $\langle v_\lambda, v_\lambda\rangle_\lambda = 1$.

According to the previous theorem, in order to determine when the Shapovalov form exists, we wish to determine when a Verma module embeds in its Hermitian dual.

Pick some positive system of roots $\Delta^+(\mathfrak{g},\mathfrak{h})$ and let $\mathfrak{b}$ be the corresponding Borel subalgebra and $\mathfrak{n}$ its nilradical. The production functor is defined by

$$\mathrm{pro}_{\mathfrak{b}}^{\mathfrak{g}}(V) = \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), V),$$

where V is a $\mathfrak{b}$-module. We have $\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(V)^h \simeq \mathrm{pro}_{\bar{\mathfrak{b}}}^{\mathfrak{g}}(V^h)$ (Lemma 5.13, [11]). We conclude that the Hermitian dual of the Verma module $M(\lambda) = \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ is $\mathrm{pro}_{\bar{\mathfrak{b}}}^{\mathfrak{g}}(\mathbb{C}_\lambda^h) = \mathrm{Hom}_{\bar{\mathfrak{b}}}(U(\mathfrak{g}), \mathbb{C}_{-\bar{\lambda}})$. Now $\mathrm{Hom}_{\bar{\mathfrak{b}}}(U(\mathfrak{g}), \mathbb{C}_{-\bar{\lambda}})$ has the same weights as $U(\mathfrak{g}) \bigotimes_{\bar{\mathfrak{b}}^{op}} \mathbb{C}_{-\bar{\lambda}}$. We conclude from universality properties of Verma modules that the Verma module $U(\mathfrak{g}) \bigotimes_{\bar{\mathfrak{b}}^{op}} \mathbb{C}_{-\bar{\lambda}}$ embeds into the Hermitian dual of $M(\lambda)$. From this, we conclude that $M(\lambda)$ admits an invariant Hermitian form if $-\bar{\lambda} = \lambda$ and

$\overline{\Delta^+(\mathfrak{g},\mathfrak{h})} = \Delta^-(\mathfrak{g},\mathfrak{h})$. Observe that we must have $\mathfrak{b}\cap\bar{\mathfrak{b}} = \mathfrak{h}$. In the following, we will determine for which $\mathfrak{h}$ and λ these conditions are satisfied.

Assume that $\mathfrak{h}_0$ is θ-stable. For a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ with Cartan decomposition $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, a root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ is imaginary valued on $\mathfrak{t}_0$ and real valued on $\mathfrak{a}_0$. A root α is **imaginary** if it vanishes on $\mathfrak{a}_0$ and **real** if it vanishes on $\mathfrak{t}_0$. If α has support on both $\mathfrak{t}_0$ and $\mathfrak{a}_0$, then it is **complex**.

We define $\theta\alpha$ by $(\theta\alpha)(H) = \alpha(\theta^{-1}H)$ for every $H \in \mathfrak{h}$. If $X_\alpha \in \mathfrak{g}_\alpha$, then

$$[H,\theta X_\alpha] = \theta([\theta^{-1}H, X_\alpha]) = \alpha(\theta^{-1}H)\theta X_\alpha = (\theta\alpha)(H)\theta X_\alpha.$$

Therefore if α is a root, then $\theta\alpha$ is a root. We have $\theta\mathfrak{g}_\alpha = \mathfrak{g}_{\theta\alpha}$.

We define $\bar\alpha$ by $\bar\alpha(H) = \overline{\alpha(\bar H)}$ for every $H \in \mathfrak{h}$. As $\bar{\cdot}$ is involutive and since $[\bar X, \bar Y] = \overline{[X,Y]}$, arguing as for θ, we conclude that $\bar\alpha$ is a root if α is a root. Also, $\bar{\mathfrak{g}}_\alpha = \mathfrak{g}_{\bar\alpha}$. Note that $\bar\alpha = \alpha$ if and only if α is real, and $\bar\alpha = -\alpha$ if and only if α is imaginary.

In fact, $\theta\alpha$ and $\bar\alpha$ are related by $\theta\alpha = -\bar\alpha$ as α is imaginary valued on $\mathfrak{t}_0$ and real valued on $\mathfrak{a}_0$.

Since $\theta\alpha = \alpha$ for imaginary α, therefore $\theta\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$, whence $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha\cap\mathfrak{k}\oplus\mathfrak{g}_\alpha\cap\mathfrak{p}$. As $\mathfrak{g}_\alpha$ is one-dimensional, either $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha\cap\mathfrak{k}$ or $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha\cap\mathfrak{p}$. We call an *imaginary* root α **compact** if $\mathfrak{g}_\alpha \subset \mathfrak{k}$ and **noncompact** if $\mathfrak{g}_\alpha \subset \mathfrak{p}$.

We define $B_\theta(\cdot,\cdot) = -B(\cdot,\theta\cdot)$. As B is symmetric and invariant and as θ is an involutive automorphism of $\mathfrak{g}$, B_θ is symmetric. Since $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces corresponding to the eigenvalues 1 and -1 of θ, respectively, we conclude that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to B. The decomposition $\mathfrak{h} = \mathfrak{t}\oplus\mathfrak{a}$ is both direct and orthogonal, hence $\mathfrak{h}^* = \mathfrak{t}^*\oplus\mathfrak{a}^*$ is an orthogonal decomposition of $\mathfrak{h}^*$ with respect to the non-degenerate symmetric bilinear form induced by B. For every $\alpha \in \mathfrak{h}^*$, we let $\alpha = \alpha_{\mathfrak{t}} + \alpha_{\mathfrak{a}}$ be the decomposition of α under this direct sum. Note that $\alpha|_{\mathfrak{t}} = \alpha_{\mathfrak{t}}|_{\mathfrak{t}}$, $\alpha|_{\mathfrak{a}} = \alpha_{\mathfrak{a}}|_{\mathfrak{a}}$, and $\alpha_{\mathfrak{t}}$ and $\alpha_{\mathfrak{a}}$ are orthogonal.

A Cartan subalgebra $\mathfrak{h}$ is **maximally compact** or **fundamental** if the compact part has largest possible dimension. In this case, there are no real roots, whence every root has non-trivial restriction to $\mathfrak{t}$ (see Proposition 6.70 of [7]). Suppose $\mathfrak{h}$ is maximally compact. If $X_\alpha \in \mathfrak{g}_\alpha$ where α is complex, then $\theta\alpha = \alpha_{\mathfrak{t}} - \alpha_{\mathfrak{a}}$ and α have the same restriction to $\mathfrak{t}$. The vectors $X_\alpha + \theta X_\alpha \in \mathfrak{k}$ and $X_\alpha - \theta X_\alpha \in \mathfrak{p}$ both have $\mathfrak{t}$-weight $\alpha_{\mathfrak{t}}$. If $\alpha \in \Delta(\mathfrak{k},\mathfrak{t})$ arises from the imaginary root $\beta \in \Delta(\mathfrak{g},\mathfrak{h})$, then β is the only root restricting to α. If α arises from a complex root β, then β and $\theta\beta$ are the only roots restricting to α. We may think of $\Delta(\mathfrak{g},\mathfrak{h})$ as $\Delta(\mathfrak{k},\mathfrak{t})\sqcup\Delta(\mathfrak{p},\mathfrak{t})$, where $\Delta(\mathfrak{k},\mathfrak{t})$ and $\Delta(\mathfrak{p},\mathfrak{t})$ overlap in the part coming from complex roots. Therefore we may think of the compact roots as $\Delta(\mathfrak{k},\mathfrak{t})$ and the noncompact roots as $\Delta(\mathfrak{p},\mathfrak{t})$.

Lemma 2.6. *We have* $\overline{\Delta^+(\mathfrak{g},\mathfrak{h})} = \Delta^-(\mathfrak{g},\mathfrak{h})$ *for some appropriate choice of* $\Delta^+(\mathfrak{g},\mathfrak{h})$ *if and only if every* $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ *has non-trivial restriction to* $\mathfrak{t}$ *(i.e.* $\mathfrak{h}$ *is maximally compact).*

Proof. $\Rightarrow$: This direction is clear as we cannot have $\bar\alpha = \alpha$, and so none of the roots are real.

$\Leftarrow$: Conversely, if $\mathfrak{h}$ is maximally compact, then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. We know that $\mathfrak{k}$ is a reductive Lie subalgebra and every $\alpha \in \Delta(\mathfrak{k},\mathfrak{t})$ is the restriction of some $\beta \in \Delta(\mathfrak{g},\mathfrak{h})$ to $\mathfrak{t}$. Choose a positive system $\Delta^+(\mathfrak{k},\mathfrak{t})$ for $\Delta(\mathfrak{k},\mathfrak{t})$ defined by some regular element $r_{\mathfrak{k}} \in \mathfrak{t}^*$. We can arrange for $r_{\mathfrak{k}}$ to be regular with respect to the root system $\Delta(\mathfrak{g},\mathfrak{h})$ also as every $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ has non-zero restriction to $\mathfrak{t}$. We define $\Delta^+(\mathfrak{g},\mathfrak{h})$ to be the positive system of $\Delta(\mathfrak{g},\mathfrak{h})$ corresponding to $r_{\mathfrak{k}}$.

Since $(\alpha, r_{\mathfrak{k}}) = (\alpha|_{\mathfrak{t}}, r_{\mathfrak{k}})$, we conclude that $\Delta^+(\mathfrak{g},\mathfrak{h})$ is compatible with $\Delta^+(\mathfrak{k},\mathfrak{t})$: if $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})$ and $\alpha|_{\mathfrak{t}} \in \Delta(\mathfrak{k},\mathfrak{t})$, then $\alpha|_{\mathfrak{t}} \in \Delta^+(\mathfrak{k},\mathfrak{t})$. Furthermore, as $\bar{\alpha} = -\alpha_{\mathfrak{t}} + \alpha_{\mathfrak{a}}$, we see that we have $\overline{\Delta^+(\mathfrak{g},\mathfrak{h})} = \Delta^-(\mathfrak{g},\mathfrak{h})$. □

Remark 2.7. We may also write $\overline{\Delta^+(\mathfrak{g},\mathfrak{h})} = \Delta^-(\mathfrak{g},\mathfrak{h})$ as $\theta\Delta^+(\mathfrak{g},\mathfrak{h}) = \Delta^+(\mathfrak{g},\mathfrak{h})$.

We may satisfy the condition $-\bar{\lambda} = \lambda$ by selecting λ to be imaginary–that is, it takes imaginary values on $\mathfrak{t}_0 \oplus \mathfrak{a}_0$. In conclusion,

Proposition 2.8. *Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a Borel subalgebra of $\mathfrak{g}$. If $\mathfrak{h} = \mathfrak{b} \cap \bar{\mathfrak{b}}$, $\mathfrak{h}$ is maximally compact, λ is imaginary, and the positive system $\Delta^+(\mathfrak{g},\mathfrak{h})$ corresponding to $\mathfrak{b}$ is θ-stable, then the Verma module $M(\lambda) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ admits a non-degenerate invariant Hermitian form.*

Henceforth, we work in the setting where a non-degenerate invariant Hermitian form on a given Verma module exists. In this case, how does one construct the Shapovalov form?

For $X \in \mathfrak{g}$, let $X^* = -\bar{X}$ and extend the map $X \mapsto X^*$ to an involutive antiautomorphism of $U(\mathfrak{g})$ by $1^* = 1$ and $(xy)^* = y^*x^*$ for every $x, y \in U(\mathfrak{g})$. We have $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n} + \mathfrak{n}^{op}U(\mathfrak{g}))$ from the triangular decomposition of $U(\mathfrak{g})$. Let p be the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ under this direct sum.

For $x, y \in U(\mathfrak{g})$, by invariance, $\langle xv_\lambda, yv_\lambda\rangle_\lambda = \langle y^*xv_\lambda, v_\lambda\rangle_\lambda$. Since $\mathfrak{n}$ acts on v_λ by zero, therefore $\langle U(\mathfrak{g})\mathfrak{n}v_\lambda, v_\lambda\rangle_\lambda = 0$. As any element of $U(\mathfrak{g})v_\lambda$ is a sum of vectors of weight no more than λ, it follows that any element of $\mathfrak{n}^{op}U(\mathfrak{g})v_\lambda$ is a sum of vectors of weight strictly less than λ. By invariance, $\langle \mathfrak{n}^{op}U(\mathfrak{g})v_\lambda, v_\lambda\rangle_\lambda = 0$. We conclude that

$$\langle xv_\lambda, yv_\lambda\rangle_\lambda = \langle p(y^*x)v_\lambda, v_\lambda\rangle_\lambda = \lambda(p(y^*x))\,\langle v_\lambda, v_\lambda\rangle_\lambda = \lambda(p(y^*x)).$$

We see from this construction that an invariant Hermitian form on a Verma module is unique up to multiplication by a real scalar.

Let v and w be vectors of weight $\lambda - \mu$ and $\lambda - \nu$, respectively. Since

$$\begin{array}{ccccc} \langle Hv, w\rangle_\lambda & = & -\langle v, \bar{H}w\rangle_\lambda & & \\ \| & & \| & & \\ (\lambda-\mu)(H)\,\langle v, w\rangle_\lambda & & -(\bar{\lambda}-\bar{\nu})(H)\,\langle v, w\rangle_\lambda & = & (\lambda+\bar{\nu})(H)\,\langle v, w\rangle_\lambda, \end{array}$$

we conclude that $\langle v, w\rangle_\lambda = 0$ if $\mu \neq -\bar{\nu} = \theta\nu$. The Shapovalov form pairs the $\lambda - \mu$ weight space with the $\lambda - \theta\mu$ weight space. Since the dimension of each weight space of $M(\lambda)$ is finite, therefore by restricting our attention to each weight space and the weight space to which it is paired individually, we may discuss the signature and the determinant of the Shapovalov form. For the purpose of such a discussion, we study the classical Shapovalov form.

There is a unique involutive automorphism σ of $\mathfrak{g}$ such that

$$\sigma(X_i) = Y_i, \quad \sigma(Y_i) = X_i, \quad \sigma(H_i) = H_i$$

where the X_i, Y_i, H_i are the canonical generators of $\mathfrak{g}$. It induces an involutive automorphism of $U(\mathfrak{g})$, which we will also denote by σ. We know that

$$p(\sigma(x)) = p(x) \qquad \forall\, x \in U(\mathfrak{g})$$

(see [6]). The classical Shapovalov form, which we denote by $(\cdot,\cdot)_S$, is defined by

$$(xv_\lambda, yv_\lambda)_S = \lambda(p(\sigma(x)y)) \qquad \forall\, x, y \in U(\mathfrak{g}).$$

It is symmetric, bilinear, and $(M(\lambda)_{\lambda-\mu}, M(\lambda)_{\lambda-\nu})_S = 0$ if $\mu \neq \nu$.

A theorem of Shapovalov states that the determinant of the classical Shapovalov form on the $\lambda - \mu$ weight space is

$$\prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})} \prod_{n=1}^{\infty} \left((\lambda+\rho, \alpha^\vee) - n\right)^{P(\mu - n\alpha)}$$

up to multiplication by a scalar. Here, P denotes Kostant's partition function.

Comparing the formula $\langle xv_\lambda, yv_\lambda \rangle_\lambda = \lambda(p(y^*x))$ to the formula for the classical Shapovalov form $(xv_\lambda, yv_\lambda)_S = \lambda(p(\sigma(x)y)) = \lambda(p(\sigma(y)x))$, we see that when μ is imaginary, the determinant of a matrix representing $\langle \cdot, \cdot \rangle_\lambda$ on the $\lambda - \mu$ weight space differs from the classical formula above by the determinant of a change of basis matrix. When μ is complex so that the $\lambda - \mu$ and $\lambda - \theta\mu$ weight spaces are paired, we see that the form $\langle \cdot, \cdot \rangle_\lambda$ on $M(\lambda)_{\lambda-\mu} + M(\lambda)_{\lambda-\theta\mu}$ can be represented by a matrix of the form

$$\begin{matrix} \lambda - \mu \\ \lambda - \theta\mu \end{matrix} \begin{pmatrix} 0 & A \\ \bar{A}^t & 0 \end{pmatrix}$$

where A and $\bar{A}^t$ differ from matrices representing the classical Shapovalov form on the $\lambda - \theta\mu$ and $\lambda - \mu$ weight spaces, respectively, by multiplication by change of basis matrices. Thus the determinant of this matrix, up to multiplication by a scalar, is

$$\prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})} \prod_{n=1}^{\infty} \left((\lambda+\rho, \alpha^\vee) - n\right)^{P(\mu - n\alpha)} \left((\lambda+\rho, \alpha^\vee) - n\right)^{P(\theta\mu - n\alpha)}.$$

Unfortunately, when the subspace under consideration has dimension greater than one, a formula for the determinant is insufficient for the purposes of computing the signature.

The radical of the Shapovalov form is the unique maximal submodule of $M(\lambda)$, hence the form is non-degenerate precisely for the irreducible Verma modules. The Shapovalov determinant formula indicates precisely where the Shapovalov form is degenerate, and consequently where $M(\lambda)$ is reducible: on the affine hyperplanes $H_{\alpha,n} := \{\lambda + \rho \,|\, (\lambda+\rho, \alpha^\vee) = n\}$ where α is a positive root and n is a positive integer. We conclude that in any connected set of purely imaginary λ avoiding these reducibility hyperplanes, as the Shapovalov form never becomes degenerate, the signature corresponding to some fixed μ remains constant.

The largest of such regions, which we name the **Wallach region**, is the intersection of the negative open half spaces

$$\left(\bigcap_{\alpha \in \Pi} H^-_{\alpha,1} \right) \bigcap H^-_{\widetilde{\alpha},1}$$

with $i\mathfrak{h}_0^*$, where $\widetilde{\alpha}^\vee$ is the highest coroot, Π the set of simple roots corresponding to our choice of Δ^+, and $H^-_{\beta,n} = \{\lambda + \rho \,|\, (\lambda+\rho, \beta^\vee) < n\}$.

In [12], Wallach shows for fixed imaginary μ that the diagonal entries in a matrix associated to the Shapovalov form $\langle \cdot, \cdot \rangle_{\lambda+t\xi}$ and the $\lambda + t\xi - \mu$ weight space have higher degree in t than the off-diagonal entries. Thus, choosing λ and ξ appropriately so that $\lambda + t\xi$ lies in the Wallach region for all $t \geq 0$, an asymptotic argument which examines the signs of the diagonal entries for large t yields a formula for the signature of the Shapovalov form within the entire Wallach region.

Definition 2.9. Denote the signature of the Shapovalov form on the $\lambda - \mu$ and $\lambda - (-\bar{\mu})$ weight space(s) of $M(\lambda)$ by $(p(\{\mu, -\bar{\mu}\}), q(\{\mu, -\bar{\mu}\}))$. The **signature character** of $\langle \cdot, \cdot \rangle_\lambda$ is

$$ch_s M(\lambda) = \sum_{\{\mu, -\bar{\mu}\} \subset \Lambda_r^+} (p(\{\mu, -\bar{\mu}\}) - q(\{\mu, -\bar{\mu}\}))\, e^{\lambda - \frac{\mu - \bar{\mu}}{2}}$$

where Λ_r^+ denotes the positive root lattice.

Here we make the observation that if $\mu \in \Lambda_r^+$ is complex, then the Shapovalov form pairs the two distinct weight spaces $M(\lambda)_{\lambda-\mu}$ and $M(\lambda)_{\lambda-(-\bar{\mu})}$ so that $p(\{\mu, -\bar{\mu}\})$ and $q(\{\mu, -\bar{\mu}\})$ are equal. In other words, $p(\{\mu, -\bar{\mu}\}) - q(\{\mu, -\bar{\mu}\}) = 0$. Thus we may write the signature character as

$$ch_s M(\lambda) = \sum_{\substack{\mu \in \Lambda_r^+ \\ \mu \text{ imaginary}}} (p(\mu) - q(\mu))\, e^{\lambda - \mu}.$$

Theorem 2.10. *(Wallach,*[12]*) Suppose $\lambda|_{\mathfrak{a}_0} \equiv 0$. The signature character of $M(\lambda)$ for $\lambda + \rho$ in the Wallach region is*

$$ch_s M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+(\mathfrak{p}, \mathfrak{t})} \left(1 - e^{-\alpha}\right) \prod_{\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t})} \left(1 + e^{-\alpha}\right)}.$$

This is a rewording of a special case of Lemma 2.3 of [12]. Here, in translating from the language of Section 2 of [12] to our language, we choose H to correspond to $ir_\mathfrak{k}$. Then $\mathfrak{q} = \mathfrak{b}$, $\mathfrak{l} = \mathfrak{h}$, $\mathfrak{u} = \mathfrak{n}$, $\mathfrak{u}_n = \bigoplus_{\alpha \in \Delta^+(\mathfrak{p}, \mathfrak{t})} \mathfrak{g}_\alpha$, $\mathfrak{u}_k = \bigoplus_{\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t})} \mathfrak{g}_\alpha$, $\Delta(\mathfrak{u}_n) = \Delta(\mathfrak{p}, \mathfrak{t})$, and $\Delta(\mathfrak{u}_k) = \Delta(\mathfrak{k}, \mathfrak{t})$. The system of positive roots Φ^+ for $(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})$ is empty, and therefore the Weyl group $W_{\mathfrak{l} \cap \mathfrak{k}}$ is trivial, $\rho_{\mathfrak{l} \cap \mathfrak{k}} = 0$, and $D_{\mathfrak{l} \cap \mathfrak{k}} = 1$. We choose V to be the trivial representation. Therefore $D_{\mathfrak{l} \cap \mathfrak{k}} ch_s(V) = 1$.

Observe that the formula for the signature character makes sense due to our results concerning pairings of non-imaginary weight spaces and our characterization of the roots corresponding to a maximally compact Cartan subalgebra.

Our goal is to extend Wallach's result (Theorem 2.10) to all irreducible Verma modules which carry an invariant Hermitian form. The strategy is as follows:

Suppose $\lambda + \rho$ lies in the hyperplane $H_{\alpha,n}$, where α is a positive root and n is a positive integer, but for all other positive roots β, $(\lambda + \rho, \beta^\vee)$ is not an integer. Then for non-zero ξ and for non-zero t in a neighbourhood of 0, $\langle \cdot, \cdot \rangle_{\lambda + t\xi}$ has radical $\{0\}$. Since $\langle \cdot, \cdot \rangle_\lambda$ has radical isomorphic to the irreducible Verma module $M(\lambda - n\alpha)$, the signature character must change by plus or minus twice the signature character of $\langle \cdot, \cdot \rangle_{\lambda - n\alpha}$ across $H_{\alpha,n}$. (This will be discussed more rigorously in Section 3.)

Roughly, by taking a suitable path from λ to the Wallach region and keeping track of changes as we cross reducibility hyperplanes, we arrive at an expression for the signature of $\langle \cdot, \cdot \rangle_\lambda$ in terms of the signature in the Wallach region. We shall describe this more concretely in Section 4.

3. The Jantzen filtration

Given a finite-dimensional complex vector space E and an analytic family $\langle \cdot, \cdot \rangle_t$ of Hermitian forms defined on E for $t \in (-\delta, \delta)$ so that $\langle \cdot, \cdot \rangle_t$ is non-degenerate for

$t \neq 0$ and degenerate for $t = 0$, we define the Jantzen filtration of E as follows:

$$E = E_0 \supset E_1 \supset \cdots \supset E_N = \{0\}$$

where $e \in E_n$ for $n \geq 0$ if there exists an analytic function $f_e : (-\varepsilon, \varepsilon) \to E$ for some $\varepsilon > 0$ such that

(1) $f_e(0) = e$
(2) $\langle f_e(t), e' \rangle_t$ vanishes to order at least n at $t = 0$ for any $e' \in E$.

For $e, e' \in E_n$, define

$$\langle e, e' \rangle^n = \lim_{t \to 0} \frac{1}{t^n} \langle f_e(t), f_{e'}(t) \rangle_t$$

which is independent of choice of f_e and $f_{e'}$. We have the following results (see Section 3 of [11]):

Theorem 3.1. *(Vogan, [11]) The form $\langle \cdot, \cdot \rangle^n$ on E_n is Hermitian with radical E_{n+1}, and therefore it induces a non-degenerate Hermitian form on E_n/E_{n+1}, which we also denote $\langle \cdot, \cdot \rangle^n$. Let (p_n, q_n) be the signature of $\langle \cdot, \cdot \rangle^n$, (p, q) be the signature of $\langle \cdot, \cdot \rangle_t$ for $t \in (0, \delta)$, and (p', q') be the signature of $\langle \cdot, \cdot \rangle_t$ for $t \in (-\delta, 0)$. Then*

$$\begin{aligned} p &= p' + \sum_{n \ odd} p_n - \sum_{n \ odd} q_n \\ q &= q' + \sum_{n \ odd} q_n - \sum_{n \ odd} p_n. \end{aligned}$$

For the remainder of this section, let $\lambda_t : (-\varepsilon, \varepsilon) \to i\mathfrak{h}_0^*$ be an analytic map satisfying the following conditions:

(1) For some positive root α and positive integer n, $\lambda_0 \in H_{\alpha,n}$.
(2) $\lambda_0 \notin H_{\beta,m}$ for $\beta \neq \alpha, \theta\alpha, \alpha + \theta\alpha$ and m an integer.
(3) For $t \neq 0$, λ_t is imaginary (so the Shapovalov form exists) but does not lie in any hyperplanes of the form $H_{\beta,m}$ where β is a root and m is an integer.

We may view $M(\lambda_t)$ as realized on a fixed vector space V for every t in $(-\varepsilon, \varepsilon)$ via $M(\lambda_t) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda_t} = U(\mathfrak{n}^{op}) \otimes \mathbb{C}_{\lambda_t}$. From now on, we will identify V with $U(\mathfrak{n}^{op})$ and the $-\mu$ weight space of $U(\mathfrak{n}^{op})$ with the $\lambda_t - \mu$ weight space of $M(\lambda_t)$ without further comment. Since $\langle xv_{\lambda_t}, yv_{\lambda_t} \rangle_{\lambda_t} = \lambda_t(p(y^* x))$ for $x, y \in U(\mathfrak{g})$, therefore $\langle \cdot, \cdot \rangle_{\lambda_t}$ is an analytic family of Hermitian forms on V. The Jantzen filtration of V is

$$V = V_0 \supset V_1 \supset \cdots \supset V_N = \{0\}$$

where V_j is defined as E_j was, with the additional stipulation that f_e take values in a fixed finite-dimensional subspace of V. As before, we define a Hermitian form $\langle \cdot, \cdot \rangle^j$ on V_j with radical V_{j+1}. We remark that the chain of subspaces is indeed finite as each V_j is invariant under $\mathfrak{g}$ and $M(\lambda_0)$ has finite length.

As we have an $\mathfrak{h}$-invariant orthogonal decomposition of V into finite dimensional subspaces with respect to the Shapovalov form, we may view $\langle \cdot, \cdot \rangle_{\lambda_t}$ as a collection of analytic families of Hermitian forms on each finite dimensional weight space (or pair of weight spaces) of V. From orthogonality, we may further conclude that for $e \in M(\lambda_t)_{\lambda_t - \mu}$, we may take f_e to have values in $M(\lambda_t)_{\lambda_t - \mu}$. Therefore the Jantzen filtration of V gives us Jantzen filtrations of each finite dimensional subspace in our orthogonal decomposition of V, and Theorem 3.1 holds for each of these subspaces. For μ imaginary, let $(p(\mu), q(\mu))$ be the signature of $\langle \cdot, \cdot \rangle_{\lambda_t}$ on

$M(\lambda_t)_{\lambda_t-\mu}$ for $t \in (0,\varepsilon)$ and $(p'(\mu), q'(\mu))$ be the signature for $t \in (-\varepsilon, 0)$. Let $(p_j(\mu), q_j(\mu)$ be the signature of $\langle\cdot,\cdot\rangle^j$ on the $-\mu$ weight space of V_j/V_{j+1}. Then

$$\begin{aligned} p &= p' + \sum_{j \text{ odd}} p_j - \sum_{j \text{ odd}} q_j \\ q &= q' + \sum_{j \text{ odd}} q_j - \sum_{j \text{ odd}} p_j \end{aligned}$$

as before.

In determining the Jantzen filtration of V corresponding to $\langle\cdot,\cdot\rangle_{\lambda_t}$, $\mathfrak{g}$-invariance of the different levels of the filtration establishes strong restrictions on the possible values of the V_j. (We note that for what follows, we may also use the Kazhdan-Lusztig Conjecture to obtain the Jantzen filtration, but that gives us more information than we need.) We have two cases:

Case 1: α is imaginary and $H_{\alpha,n}$ the only reducibility hyperplane containing λ_0.

By our choice of λ_0, $M(\lambda_0)$ has only one non-trivial submodule: $M(\lambda_0 - n\alpha)$. Its multiplicity must be one as $M(\lambda - n\alpha)$ is a free $U(\mathfrak{n}^{op})$-module by choice of λ (see Theorem 7.6.6 of [1]). Therefore our Jantzen filtration must be

$$M(\lambda_0) \supset M(\lambda_0 - n\alpha) \supset \cdots \supset M(\lambda_0 - n\alpha) = V_N \supset \{0\}.$$

According to the Shapovalov determinant formula, up to multiplication by a scalar, the determinant of the form $\langle\cdot,\cdot\rangle_{\lambda_t}$ on the $\lambda_t - n\alpha$ weight space is

$$\prod_{m=1}^{\infty} \prod_{\beta \in \Delta^+(\mathfrak{g},\mathfrak{h})} ((\lambda_t + \rho, \beta^\vee) - m)^{P(n\alpha - m\beta)}.$$

The only factor which is zero when $t = 0$ is the factor corresponding to $\beta = \alpha$ and $m = n$. Since $P(0) = 1$, as we go from $t > 0$ to $t < 0$, the determinant changes sign. Therefore N must be odd and (p_N, q_N) or (q_N, p_N) must be the signature of the Shapovalov form on $M(\lambda_0 - n\alpha)$. Thus:

Proposition 3.2. *In the setup of this section, suppose α is imaginary. If $t_1 \in (0,\varepsilon)$ and $t_2 \in (-\varepsilon, 0)$, then*

$$\begin{aligned} ch_s M(\lambda_{t_1}) &= e^{\lambda_{t_1} - \lambda_{t_2}} \cdot ch_s M(\lambda_{t_2}) \pm 2e^{\lambda_{t_1} - \lambda_0} ch_s M(\lambda_0 - n\alpha) \\ &= e^{\lambda_{t_1} - \lambda_{t_2}} \cdot ch_s M(\lambda_{t_2}) \pm 2 ch_s M(\lambda_{t_1} - n\alpha). \end{aligned}$$

Case 2: α is complex (so λ_0 is contained in both $H_{\alpha,n}, H_{\theta\alpha,n}$, and also in $H_{\alpha+\theta\alpha,2n}$ if $\alpha + \theta\alpha$ is a root).

We know that $M(\lambda_0 - n\alpha)$ is a submodule of $M(\lambda_0)$ as $(\lambda_0 + \rho, \alpha^\vee) = n$. As λ_0 and ρ are imaginary, therefore $(\lambda_0 + \rho, -\bar{\alpha}^\vee) = -\overline{(\bar{\lambda}_0 + \bar{\rho}, \alpha^\vee)} = \overline{(\lambda_0 + \rho, \alpha^\vee)} = \bar{n} = n$, whence $M(\lambda_0 - n(-\bar{\alpha}))$ must also be a submodule of $M(\lambda_0)$.

Key to describing the Jantzen filtration in this case is the usage of results of Bernstein, Gelfand, and Gelfand, described in [1]. Let $J(\lambda)$ denote the unique largest submodule of $M(\lambda)$ and $L(\lambda) = M(\lambda)/J(\lambda)$ the corresponding simple quotient.

Proposition 3.3. *(Proposition 7.6.1, [1]) The Verma module $M(\lambda)$ has a Jordan-Hölder series and every simple subquotient of $M(\lambda)$ is isomorphic to $L(\mu)$ for some μ belonging to $W \cdot (\lambda + \rho) \cap (\lambda + \rho - \Lambda_r^+) - \rho$.*

Beware that the notation in [1] includes a shift by ρ.

Theorem 3.4. *(Theorem 7.6.6,* [1]*) For $\mu, \lambda \in \mathfrak{h}^*$,*

$$\dim \operatorname{Hom}(M(\mu), M(\lambda)) \leq 1.$$

Theorem 3.5. *(Bernstein-Gelfand-Gelfand, Theorem 7.6.23 of* [1]*) For $\lambda, \mu \in \mathfrak{h}^*$,*

$$M(\mu) \subset M(\lambda) \iff \begin{array}{c} \exists\, \alpha_1, \cdots, \alpha_m \in \Delta^+(\mathfrak{g}, \mathfrak{h}) \text{ such that} \\ \lambda + \rho \geq s_{\alpha_1}(\lambda + \rho) \geq \cdots \geq s_{\alpha_m} \cdots s_{\alpha_1}(\lambda + \rho) = \mu + \rho. \end{array}$$

Remark 3.6. The above conditions are equivalent to $\mu + \rho \in W(\lambda + \rho)$ and $\mu \leq \lambda$ in the case where $\mathfrak{g}$ is type A_2 (see Remark 7.8.10, [1]).

If α and $-\bar{\alpha} = \theta\alpha$ are orthogonal: We have $(\lambda_0 - n\alpha + \rho, (\theta\alpha)^\vee) = (\lambda_0 + \rho, (\theta\alpha)^\vee) = n$. By symmetry and our discussion above, we have the following containment of Verma modules:

$$\begin{array}{ccccc} & & M(\lambda_0) & & \\ & \diagup & & \diagdown & \\ M(\lambda_0 - n\alpha) & & & & M(\lambda_0 - n\theta\alpha) \\ & \diagdown & & \diagup & \\ & & M(\lambda_0 - n(\alpha + \theta\alpha)) & & \end{array}.$$

Note that $L(\lambda_0)$ is in the 0^{th} level of the Jantzen filtration since the radical of $\langle \cdot, \cdot \rangle_{\lambda_0}$ is the unique largest submodule of $M(\lambda_0)$. As each copy of $L(\lambda_0 - n\alpha)$ is paired with a copy of $L(\lambda_0 - n\theta\alpha)$, the pair makes no contribution to the change in signature character as t changes sign, whether or not it is contained in an even or odd level of the filtration. Thus only $L(\lambda_0 - n(\alpha + \theta\alpha)) = M(\lambda_0 - n(\alpha + \theta\alpha))$ may make a contribution to the change in signature character. By Theorems 3.4 and 3.5, the multiplicity of $L(\lambda_0 - n(\alpha + \theta\alpha))$ is *one*.

Up to multiplication by a scalar, the determinant of a matrix representing $\langle \cdot, \cdot \rangle_{\lambda_t}$ on the $\lambda_t - n(\alpha + \theta\alpha)$ weight space of $M(\lambda_t)$ is

$$\prod_{m=1}^{\infty} \prod_{\beta \in \Delta^+(\mathfrak{g}, \mathfrak{h})} ((\lambda_t + \rho, \beta^\vee) - m)^{P(n(\alpha+\theta\alpha) - m\beta)}.$$

The only factors which are zero when $t = 0$ are those corresponding to the pairs (α, n) and $(\theta\alpha, n)$ as $\alpha + \theta\alpha$ is not a root. Observe that $P(n\alpha) = P(n\theta\alpha)$ as θ is a bijection from $\Delta^+(\mathfrak{g}, \mathfrak{h})$ to itself. Combining this with $(\lambda_t + \rho, \alpha^\vee) = (\lambda_t + \rho, (\theta\alpha)^\vee)$, we see that the determinant does not change as t changes from positive to negative. In other words, $L(\lambda_0 - n(\alpha + \theta\alpha))$ must be contained in an *even* level of the filtration. We have:

Proposition 3.7. *In the setup of this section, suppose α is complex and α and $\theta\alpha$ are orthogonal. Then for $t_1 \in (0, \varepsilon)$ and $t_2 \in (-\varepsilon, 0)$,*

$$ch_s M(\lambda_{t_1}) = e^{\lambda_{t_1} - \lambda_{t_2}} \cdot ch_s M(\lambda_{t_2}).$$

If α and $-\bar{\alpha} = \theta\alpha$ are not orthogonal: Now α and $-\bar{\alpha} = \theta\alpha$ have the same length. If α and $\theta\alpha$ are not orthogonal, then either $(\alpha, (\theta\alpha)^\vee) = \pm 1$ or $(\theta\alpha, \alpha^\vee) = \pm 1$, whence either $\alpha + \theta\alpha$ or $\alpha - \theta\alpha$ is a root. Observe that α and $\theta\alpha$ have the same height as θ applied to an expression for α as a sum of indecomposable roots gives an expression for $\theta\alpha$ as a sum of indecomposable roots (we will see such an argument again in Section 6). We conclude that $\alpha - \theta\alpha$ cannot be a root. Thus $\alpha + \theta\alpha$ must be a root and α and $\theta\alpha$ generate a subroot system of type A_2. Now

$$\begin{array}{rrcl} & (\lambda_0 + \rho, \alpha^\vee) = (\lambda_0 + \rho, (\theta\alpha)^\vee) & = & n \\ \Rightarrow \quad (\lambda_0 + \rho, (\alpha + \theta\alpha)^\vee) \quad = & (\lambda_0 + \rho, \alpha^\vee) + (\lambda_0 + \rho, (\theta\alpha)^\vee) & = & 2n. \end{array}$$

It follows that $M(\lambda_0 - 2n(\alpha + \theta\alpha))$ is a submodule of $M(\lambda_0)$. From

$$\begin{array}{rl} (\lambda_0 - n\alpha + \rho, (\theta\alpha)^\vee) = 2n & \qquad (\lambda_0 - n\alpha - 2n\theta\alpha + \rho, \alpha^\vee) = n \\ (\lambda_0 - n\alpha + \rho, (\alpha + \theta\alpha)^\vee) = n & \qquad (\lambda_0 - n\alpha - 2n\theta\alpha + \rho, (\theta\alpha)^\vee) = 0 \end{array}$$

and from symmetry, we observe the following containment of Verma modules:

$$\begin{array}{ccccc} & & M(\lambda_0) & & \\ & \diagup & & \diagdown & \\ M(\lambda_0 - n\alpha) & & & & M(\lambda_0 - n\theta\alpha) \\ & \diagdown & & \diagup & \\ & & M(\lambda_0 - n\alpha) \cap M(\lambda_0 - n\theta\alpha) & & \\ & \diagup & & \diagdown & \\ M(\lambda_0 - n\alpha - 2n\theta\alpha) & & & & M(\lambda_0 - 2n\alpha - n\theta\alpha) \\ & \diagdown & & \diagup & \\ & & M(\lambda_0 - 2n(\alpha + \theta\alpha)) & & \end{array}$$

As in the subcase where α and $\theta\alpha$ are orthogonal, the only composition factor which may make a contribution to the change in signature character as t changes sign is $L(\lambda_0 - 2n(\alpha + \theta\alpha)) = M(\lambda_0 - 2n(\alpha + \theta\alpha))$, and its multiplicity is one.

We study the determinant of $\langle \cdot, \cdot \rangle_{\lambda_t}$ on the $\lambda_t - 2n(\alpha + \theta\alpha)$ weight space of $M(\lambda_t)$:

$$\prod_{m=1}^{\infty} \prod_{\beta \in \Delta^+(\mathfrak{g}, \mathfrak{h})} ((\lambda_t + \rho, \beta^\vee) - m)^{P(2n(\alpha + \theta\alpha) - m\beta)}.$$

The pairs (β, m) for which the corresponding factor is zero at $t = 0$ are (α, n), $(\theta\alpha, n)$, and $(\alpha + \theta\alpha, 2n)$. Again, since $(\lambda_t + \rho, \alpha) = (\lambda_t + \rho, \theta\alpha)$, $P(n\alpha + 2n\theta\alpha)$ and $P(2n\alpha + n\theta\alpha)$ are equal, and $P(0) = 1$, therefore $L(\lambda_0 - 2n(\alpha + \theta\alpha))$ must be contained in an *odd* level of the filtration. We obtain:

Proposition 3.8. *In the setup of this section, suppose α is complex and α and $\theta\alpha$ are not orthogonal so that $\alpha + \theta\alpha$ is an imaginary root. Then for $t_1 \in (0, \varepsilon)$ and $t_2 \in (-\varepsilon, 0)$,*

$$ch_s M(\lambda_{t_1}) = e^{\lambda_{t_1} - \lambda_{t_2}} \cdot ch_s M(\lambda_{t_2}) \pm 2 ch_s M(\lambda_{t_1} - n(\alpha + \theta\alpha)).$$

Remark 3.9. This is compatible with Proposition 3.2.

4. A preliminary formula for the signature character

In this and the subsequent section, we will assume that $\mathfrak{h}$ is a compact Cartan subalgebra–that is, $\mathfrak{h} = \mathfrak{t}$ and $\mathfrak{a} = 0$. Then *all* roots are imaginary.

Definition 4.1. According to Theorem 2.10, there are constants c_μ for $\mu \in \Lambda_r^+$ so that

$$R(\lambda) := \sum_{\mu \in \Lambda_r^+} c_\mu e^{\lambda - \mu}$$

is the signature character of the Shapovalov form $\langle \cdot, \cdot \rangle_\lambda$ when $\lambda + \rho$ lies in the Wallach region.

Consider $A_0 = \{\lambda+\rho \,|\, (\lambda+\rho,\alpha^\vee) < 0 \;\forall\, \alpha \in \Pi,\; (\lambda+\rho,\widetilde{\alpha}^\vee) > -1\}$, which we call the **fundamental alcove**. Reflections through the walls of the fundamental alcove generate the affine Weyl group, W_a. The action of the affine Weyl group defines alcoves which have walls of the form $H_{\alpha,n}$. (See [4].) Note that the signature of the Shapovalov form does not change within each of these alcoves.

Definition 4.2. For an alcove A, there are constants c^A_μ for $\mu \in \Lambda_r^+$ such that

$$R^A(\lambda) := \sum_{\mu\in\Lambda_r^+} c^A_\mu e^{\lambda-\mu}$$

is the signature character of the Shapovalov form $\langle\cdot,\cdot\rangle_\lambda$ when $\lambda+\rho$ lies in A.

Lemma 4.3. *If wA_0 and $w'A_0$ are adjacent alcoves separated by the hyperplane $H_{\alpha,n}$, then*

$$R^{wA_0}(\lambda) = R^{w'A_0}(\lambda) + 2\varepsilon(wA_0,w'A_0)R^{wA_0-n\alpha}(\lambda-n\alpha) \tag{4.1}$$

where $\varepsilon(wA_0,w'A_0)$ is zero if $H_{\alpha,n}$ is not a reducibility hyperplane and plus or minus one otherwise.

Proof. This is just Proposition 3.2. □

Remark 4.4. Calculating ε is difficult and will be the subject of the following section.

Remark 4.5. Observe that $\varepsilon(wA_0,w'A_0) = -\varepsilon(w'A_0,wA_0)$.

Recall that the reflections through the walls of A_0 generate W_a. These reflections are denoted by $s_{\alpha,0}$ for each simple root α and $s_{\widetilde{\alpha},-1}$. If we omit $s_{\widetilde{\alpha},-1}$, we generate the Weyl group W as a subgroup of W_a. These generators are compatible with reflection through the walls of the fundamental Weyl chamber $\mathfrak{C}_0$, which we choose to be the Weyl chamber which contains A_0: $\mathfrak{C}_0 = \bigcap_{\alpha\in\Pi} H^-_{\alpha,0}$. Observe that for each $s \in W$, sA_0 lies in the Wallach region so that $R^{sA_0} = R$.

We will define two maps $\bar{\cdot}$ and $\tilde{\cdot}$ from the affine Weyl group to the Weyl group as follows:
If $w = ts$ where s is an element of the Weyl group and t is translation by an element of the root lattice, then $\overline{w} = s$. We let $\widetilde{w}$ be such that wA_0 lies in the Weyl chamber $\widetilde{w}\mathfrak{C}_0$. Observe that $\bar{\cdot}$ is a group homomorphism while $\tilde{\cdot}$ is not. Furthermore, $\overline{s_{\alpha,n}} = s_\alpha$. Observe that we can rewrite (4.1) as

$$\begin{aligned} R^{wA_0}(\lambda) &= R^{w'A_0}(\lambda) + 2\varepsilon(wA_0,w'A_0)R^{s_{\alpha,0}s_{\alpha,n}wA_0}(s_{\alpha,0}s_{\alpha,n}\lambda) \\ &= R^{w'A_0}(\lambda) + 2\varepsilon(wA_0,w'A_0)R^{\overline{s_{\alpha,n}}w'A_0}(\overline{s_{\alpha,n}}s_{\alpha,n}\lambda). \end{aligned} \tag{4.2}$$

For w in the affine Weyl group, let $wA_0 = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \cdots \xrightarrow{r_\ell} C_\ell = \widetilde{w}A_0$ be a (not necessarily reduced) path from wA_0 to $\widetilde{w}A_0$. Applying (4.2) ℓ times, we obtain

$$\begin{aligned} R^{wA_0}(\lambda) &= R^{\widetilde{w}A_0}(\lambda) + \sum_{j=1}^{\ell} \varepsilon(C_{j-1},C_j)2R^{\overline{r_j}C_j}(\overline{r_j}r_j\lambda) \\ &= R(\lambda) + 2\sum_{j=1}^{\ell} \varepsilon(C_{j-1},C_j)R^{\overline{r_j}C_j}(\overline{r_j}r_j\lambda). \end{aligned}$$

Observe that a path from $\overline{r_j}C_j$ to $\overline{r_j}C_\ell$ is $\overline{r_j}C_j \xrightarrow{\overline{r_j}r_{j+1}\overline{r_j}} \overline{r_j}C_{j+1} \xrightarrow{\overline{r_j}r_{j+2}\overline{r_j}} \cdots \xrightarrow{\overline{r_j}r_\ell\overline{r_j}} \overline{r_j}C_\ell$. Applying induction on path length, we arrive at the following:

Theorem 4.6. *Recall* $R : \lambda \mapsto \sum_{\mu \in \Lambda_r^+} c_\mu e^{\lambda - \mu}$ *and also* $R^{wA_0} : \lambda \mapsto \sum_{\mu \in \Lambda_r^+} c_\mu^{wA_0} e^{\lambda-\mu}$ *which were defined to agree with the signature character of the Shapovalov form in the Wallach region and in the alcove* wA_0*, respectively.*
Let $wA_0 = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \cdots \xrightarrow{r_\ell} C_\ell = \widetilde{w}A_0$ *be a (not necessarily reduced) path from* wA_0 *to* $\widetilde{w}A_0$*. Then*

$$\begin{aligned} R^{wA_0}(\lambda) &= \sum_{I=\{i_1<\cdots<i_k\}\subset\{1,\ldots,\ell\}} \varepsilon(I) 2^{|I|} R^{\overline{r_{i_1}}\cdots\overline{r_{i_k}}\widetilde{w}A_0}\left(\overline{r_{i_1}}\,\overline{r_{i_2}}\cdots\overline{r_{i_k}} r_{i_k} r_{i_{k-1}}\cdots r_{i_1}\lambda\right) \\ &= \sum_{I=\{i_1<\cdots<i_k\}\subset\{1,\ldots,\ell\}} \varepsilon(I) 2^{|I|} R\left(\overline{r_{i_1}}\,\overline{r_{i_2}}\cdots\overline{r_{i_k}} r_{i_k} r_{i_{k-1}}\cdots r_{i_1}\lambda\right) \end{aligned}$$

where $\varepsilon(\emptyset) = 1$ *and*

$$\varepsilon(I) = \varepsilon(C_{i_1-1}, C_{i_1})\varepsilon(\overline{r_{i_1}}C_{i_2-1}, \overline{r_{i_1}}C_{i_2})\cdots\varepsilon(\overline{r_{i_1}}\cdots\overline{r_{i_{k-1}}}C_{i_k-1}, \overline{r_{i_1}}\cdots\overline{r_{i_{k-1}}}C_{i_k}).$$

We will determine $\varepsilon(C, C')$ using the principle that in a closed loop, the changes introduced by crossing reducibility hyperplanes must sum to zero. We know R by Wallach's work (Theorem 2.10). Theorem 4.6 will therefore give an explicit formula for the signature character of the Shapovalov form on $M(\lambda)$, where $\lambda + \rho$ lies in wA_0. This solves the problem of calculating the signature for all irreducible Verma modules which admit an invariant Hermitian form in the case where the Cartan subalgebra $\mathfrak{h}$ is compact.

5. Calculating ε

The strategy for computing ε is as follows:

- We show that for a fixed hyperplane $H_{\alpha,n}$, the value of ε for crossing from $H_{\alpha,n}^+$ to $H_{\alpha,n}^-$ depends only on the Weyl chamber to which the point of crossing belongs.
- We consider irreducible rank 2 root systems generated by simple roots α_1 and α_2, and calculate the value of ε by calculating changes that occur at the Weyl chamber walls. It is trivial to show by considering appropriate weight vectors in the Verma module that ε for a hyperplane corresponding to a simple root is constant and does not depend on Weyl chambers in any way. However, we prove this in a manner that does not depend on simplicity of the α_i.
- For an arbitrary positive root γ in a generic irreducible root system which is not type G_2, we develop a formula for ε inductively by replacing the α_i from the previous step with appropriate roots. Key in the induction is the independence of our rank 2 arguments from the simplicity of the α_i.

5.1. **Dependence on Weyl chambers.** We begin by refining Theorem 4.6: if we take an arbitrary C_ℓ, the formula becomes

$$R^{wA_0}(\lambda) = \sum_{I=\{i_1<\cdots<i_k\}\subset\{1,\ldots,\ell\}} \varepsilon(I) 2^{|I|} R^{\overline{r_{i_1}}\cdots\overline{r_{i_k}}C_\ell}\left(\overline{r_{i_1}}\,\overline{r_{i_2}}\cdots\overline{r_{i_k}} r_{i_k} r_{i_{k-1}}\cdots r_{i_1}\lambda\right).$$

If we choose in particular $C_\ell = C_0$, we have

$$R^{C_0}(\lambda) = \sum_{\substack{I=\{i_1<\cdots<i_k\}\\ \subset\{1,\ldots,\ell\}}} \varepsilon(I) 2^{|I|} R^{\overline{r_{i_1}}\cdots\overline{r_{i_k}}C_0}\left(\overline{r_{i_1}}\cdots\overline{r_{i_k}} r_{i_k}\cdots r_{i_1}\lambda\right). \tag{5.1.1}$$

For the following two subsections, our paths will have this property.

Proposition 5.1.1. *Suppose α is a positive root and $n \in \mathbb{Z}^+$ and suppose $H_{\alpha,n}$ separates adjacent alcoves wA_0 and $w'A_0$, with $wA_0 \subset H^+_{\alpha,n}$ and $w'A_0 \subset H^-_{\alpha,n}$. The value of $\varepsilon(w, w')$ depends only on $H_{\alpha,n}$ and on $\widetilde{w}$.*

Proof. We begin by proving the proposition for types A_2 and B_2 in the case where $wA_0 = C_i$ and $w'A_0 = C_{i+1}$ as described in the following figure.

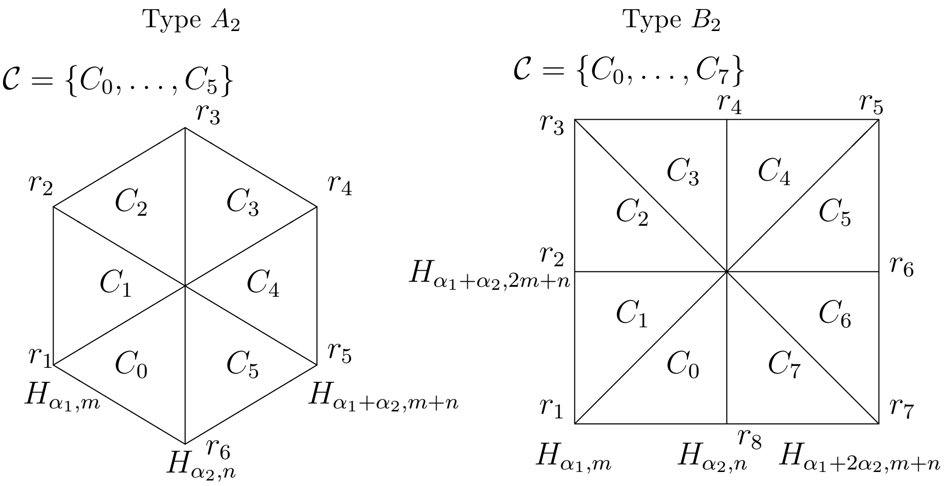


FIGURE 1. Classical rank 2 systems

As we may cover any hyperplane with overlapping translates of $\mathcal{C}$, it suffices to show that $\varepsilon(C_i, C_{i+1}) + \varepsilon(C_{i+\ell/2}, C_{i+1+\ell/2}) = 0$ for $i = 0, 1, \ldots, \ell/2 - 1$ in these rank 2 cases. To show this, we need the following result:

Lemma 5.1.2. *Let $\mathcal{C} = \{C_i\}_{i=0,\ldots,\ell-1}$ be a set of alcoves that lie in the interior of some Weyl chamber and suppose the reflections $\{r_j\}_{j=1,\cdots,k}$ preserve $\mathcal{C}$. If w and v are elements of W_a generated by the r_j, then*

$$\overline{w}^{-1}w = \overline{v}^{-1}v \iff w = v.$$

Proof. $\Rightarrow$: By simple transitivity of the action of W_a on the alcoves, $\overline{w}^{-1}w = \overline{v}^{-1}v$ if and only if $\overline{w}^{-1}wC = \overline{v}^{-1}vC$ for any alcove C. Choose in particular $C = C_i$. The alcoves $\overline{w}^{-1}wC_i$ and $\overline{v}^{-1}vC_i$ belong to the same Weyl chamber as they are the same alcove. As the r_j's preserve $\mathcal{C}$ which lies in the interior of some Weyl chamber, wC_i and vC_i belong to the same Weyl chamber $s\mathfrak{C}_0$, say. Therefore the Weyl chamber containing $\overline{w}^{-1}wC_i = \overline{v}^{-1}vC_i$ may be expressed both as $\overline{w}^{-1}s\mathfrak{C}_0$ and as $\overline{v}^{-1}s\mathfrak{C}_0$. It follows that $\overline{w}^{-1} = \overline{v}^{-1}$, whence $w = v$. The other direction is trivial. □

We return to proving $\varepsilon(C_i, C_{i+1}) + \varepsilon(C_{i+\ell/2}, C_{i+1+\ell/2}) = 0$ for $i = 0, 1, \ldots, \ell/2 - 1$ in our rank 2 cases.

Definition 5.1.3. For $I = \{i_1 < \cdots < i_k\}$, we define $w_I = r_{i_k} r_{i_{k-1}} \cdots r_{i_1}$.

We rewrite (5.1.1) as

$$\sum_{\emptyset \neq I \subset \{1,\ldots,\ell\}} 2^{|I|}\varepsilon(I) R^{\overline{w_I}^{-1}C_0}\left(\overline{w_I}^{-1}w_I\lambda\right) = 0. \tag{5.1.2}$$

Definition 5.1.4. We will use T_μ to denote translation by $-\mu$: $T_\mu(\lambda) = \lambda - \mu$.

Our rank 2 cases satisfy the conditions for Lemma 5.1.2. Using Lemma 5.1.2 and the partial ordering on Λ_r on (5.1.2), we obtain

$$\sum_{\substack{\emptyset \neq I \subset \{1,\ldots,\ell\} \\ \overline{w_I}^{-1} w_I = T_\mu}} 2^{|I|} \varepsilon(I) = 0 \tag{5.1.3}$$

for every $\mu \in \Lambda_r$.

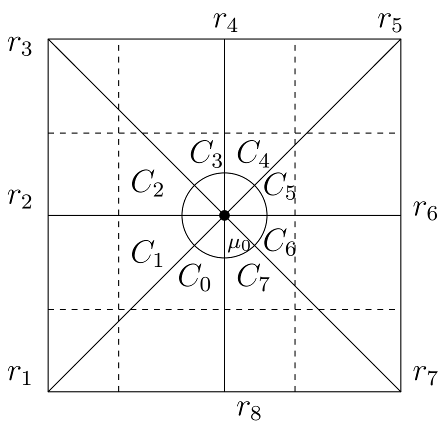
Suppose $\mu = m\alpha_1$. The subsets I such that $|I| < 3$ for which $\overline{w_I}^{-1} w_I = T_{m\alpha_1}$ are $I = \{1\}, \{1 + \ell/2\}$. By considering equation (5.1.3) modulo 8, we obtain

$$\varepsilon(C_0, C_1) + \varepsilon(C_{\ell/2}, C_{\ell/2+1}) = 0,$$

which gives the desired result for $H_{\alpha_1,m}$. The same proof can be used for the other hyperplanes. (Note that this proof works for type G_2 also.)

To extend the proof of this proposition to the general case where $\Delta(\mathfrak{g}, \mathfrak{h})$ is any irreducible root system other than G_2, we consider an arbitrary positive root α. There exists some positive β distinct from α such that $(\alpha, \beta) \neq 0$. Then α and β generate a rank 2 root subsystem of type A_2 or B_2. Consider two-dimensional affine planes of the form $P = \text{span}\{\alpha, \beta\} + \mu_0$. We may choose μ_0 to lie in the intersection of the hyperplanes $H_{\alpha,n}$ and $H_{\beta,m}$. The intersection of $H_{\alpha,n}$ and $H_{\beta,m}$ with P looks like Figure 1, with the possible inclusion of additional affine hyperplanes.

Consider roots δ that do not belong to the subsystem generated by α and β. If δ is orthogonal to α and to β, then $P \subset H_{\delta,k}$ if $(\mu_0, \delta^\vee) = k$, and $P \cap H_{\delta,k} = \emptyset$ otherwise. We restrict our attention for now to the case where P has trivial intersection with reducibility hyperplanes corresponding to roots orthogonal to α and to β. For a root δ for which $(\delta, \alpha) \neq 0$ or $(\delta, \beta) \neq 0$, $H_{\delta,k}$ intersects P in a line. Whenever we have an intersection of reducibility hyperplanes in a point μ_0 in P that does not lie in any Weyl chamber wall, we may take the alcoves C_i and the reflections r_i to correspond to a circular path in P around μ_0 of suitably small radius, and we take $\mathcal{C} \supset \{C_i\}$ to be the set of alcoves containing μ_0 in their boundaries, so that r_i preserves $\mathcal{C}$. Then, the conditions of Lemma 5.1.2 are satisfied, so we may argue as before and conclude that the signs corresponding to alcoves in the circular path agree with the proposition.



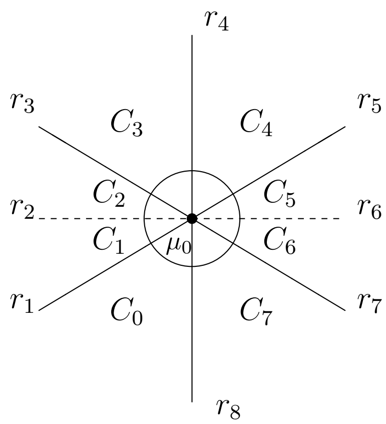
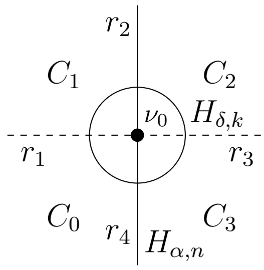


FIGURE 2. Some examples

In the previous diagrams, solid lines correspond to roots in the subsystem generated by α and β; dotted lines correspond to various δ.

We partition a given Weyl chamber into regions by hyperplanes $H_{\delta,k}$ for positive integers k and positive roots δ orthogonal to α and to β. We conclude from our discussion above that for any pair of adjacent alcoves wA_0 and $w'A_0$ belonging to a given region, the value of $\varepsilon(wA_0, w'A_0)$ is the same, provided the alcoves are separated by $H_{\alpha,n}$, $wA_0 \subset H^+_{\alpha,n}$, and $w'A_0 \subset H^-_{\alpha,n}$.

To obtain our result for the entire Weyl chamber, consider a reducibility hyperplane $H_{\delta,k}$ for which δ is orthogonal to both α and β. Take ν_0 in the intersection of $H_{\delta,k}$ with $H_{\alpha,n}$ such that ν_0 lies in the Weyl chamber under consideration and $(\nu_0, \gamma^\vee)$ is not an integer for roots γ not equal to plus or minus α or δ. Then, taking a circular path in $\mathrm{span}\{\alpha, \delta\} + \nu_0$ around ν_0 of suitably small radius, we may argue as above to conclude that the value for ε corresponding to crossing $H_{\alpha,n}$ in the region bounded by $H_{\delta,k-1}$ and $H_{\delta,k}$ is the same as the value for ε corresponding to crossing $H_{\alpha,n}$ in the region bounded by $H_{\delta,k}$ and $H_{\delta,k+1}$.



5.2. **Calculating ε for the rank 2 cases.** Some of the results of this section arise from the structure as dihedral groups of the Weyl groups corresponding to the rank 2 simple root systems.

For a given hyperplane, in order to calculate the value of ε in any given Weyl chamber, we need to calculate the value in one particular Weyl chamber and then to calculate the changes that occur as we cross Weyl chamber walls. We work in the setup of the following three figures for the remainder of this subsection. Alcoves C_i have been labelled by $w \in W_a$ such that $C_i = wC_0$ and by corresponding translations $\overline{w}^{-1}w = T_\mu$.

Recall that we used ℓ to denote path length, which is 6 for type A_2, 8 for type B_2, and 12 for type G_2. Recall that $C_0 = C_\ell$.

Lemma 5.2.1. *Suppose $\mathcal{C}$ intersects k reducibility hyperplanes. Then those reducibility hyperplanes correspond to $r_1, r_2, \ldots, r_k$ if $H_{\alpha_1,m}$ is a reducibility hyperplane, or they correspond to $r_\ell, r_{\ell-1}, \ldots, r_{\ell-k+1}$ if $H_{\alpha_2,n}$ is a reducibility hyperplane.*

Proof. Recall that the fundamental Weyl chamber $\mathfrak{C}_0$ was defined so that $-\rho \in \mathfrak{C}_0$ and the fundamental alcove A_0 was defined so that $A_0 \subset \mathfrak{C}_0$. This lemma may be proved by observation. □

Lemma 5.2.2. *Suppose $\mathcal{C}$ intersects one Weyl chamber wall: $H_{\alpha,0}$. Then*

$$\overline{w_I}^{-1}w_I = \overline{w_J}^{-1}w_J \iff w_I = w_J \text{ or } w_I = s_\alpha w_J.$$

Proof. $\Rightarrow$: $\mathcal{C}$ intersects two Weyl chambers. Suppose $s\mathfrak{C}_0$ is one of them. As s_α preserves $\mathcal{C}$, therefore $s_\alpha s\mathfrak{C}_0$ is the other. Since possible values for $\widetilde{w_I}$ and $\widetilde{w_J}$ are s and $s_\alpha s$, either $\widetilde{w_I} = \widetilde{w_J}$ or $\widetilde{w_I} = s_\alpha\widetilde{w_J}$. Applying $\widetilde{\cdot}$ to both sides of $\overline{w_I}^{-1}w_I = \overline{w_J}^{-1}w_J$, we have $\overline{w_I}^{-1} = \overline{w_J}^{-1}$ or $\overline{w_I}^{-1} = \overline{w_J}^{-1}s_\alpha$. Substituting this into $\overline{w_I}^{-1}w_I = \overline{w_J}^{-1}w_J$, we have $w_I = w_J$ or $w_I = s_\alpha w_J$. The other direction is trivial. □

$\mathcal{C} = \{C_0, \ldots, C_5\}$

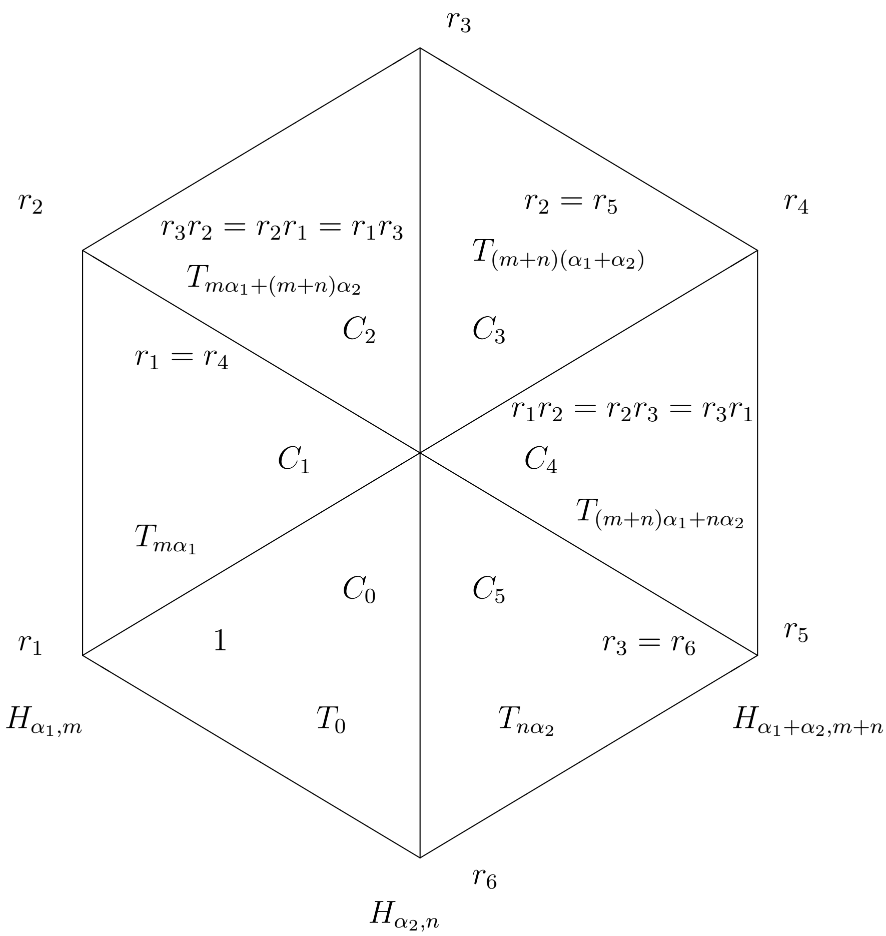


FIGURE 3. Type A_2

We will need some results concerning the Weyl group. For $s \in W$, we have the following definitions (see 1.6 of [4]):

$$\begin{aligned} \Delta(s) &= \Delta^+ \cap s^{-1}(\Delta^-) \\ n(s) &= \#\Delta(s) \end{aligned}$$

The product $s = s_{i_1} \cdots s_{i_k} \in W$, where $s_{i_j} = s_{\alpha_{i_j}}$ and the α_{i_j} are simple roots, is a **reduced expression** for s if k is minimal. The **length** of s is defined to be $\ell(s) = k$. We have $\ell(s) = n(s) = \ell(s^{-1})$ (see Lemma 10.3 A of [3]). We note that $\Delta(s) = \{s^{-1}(-\alpha) \mid \alpha \in \Delta^+ \text{ and } s^{-1}(-\alpha) > 0\}$. We may rewrite this as $\Delta(s) = \{\alpha \in \Delta^+ \mid s\alpha < 0\}$. Also, if $s = s_{i_1} \cdots s_{i_k}$ is a reduced expression for $s \in W$, then

$$\Delta(s^{-1}) = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \ldots, s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k}\} \tag{5.2.1}$$

$\mathcal{C} = \{C_0, \ldots, C_7\}$

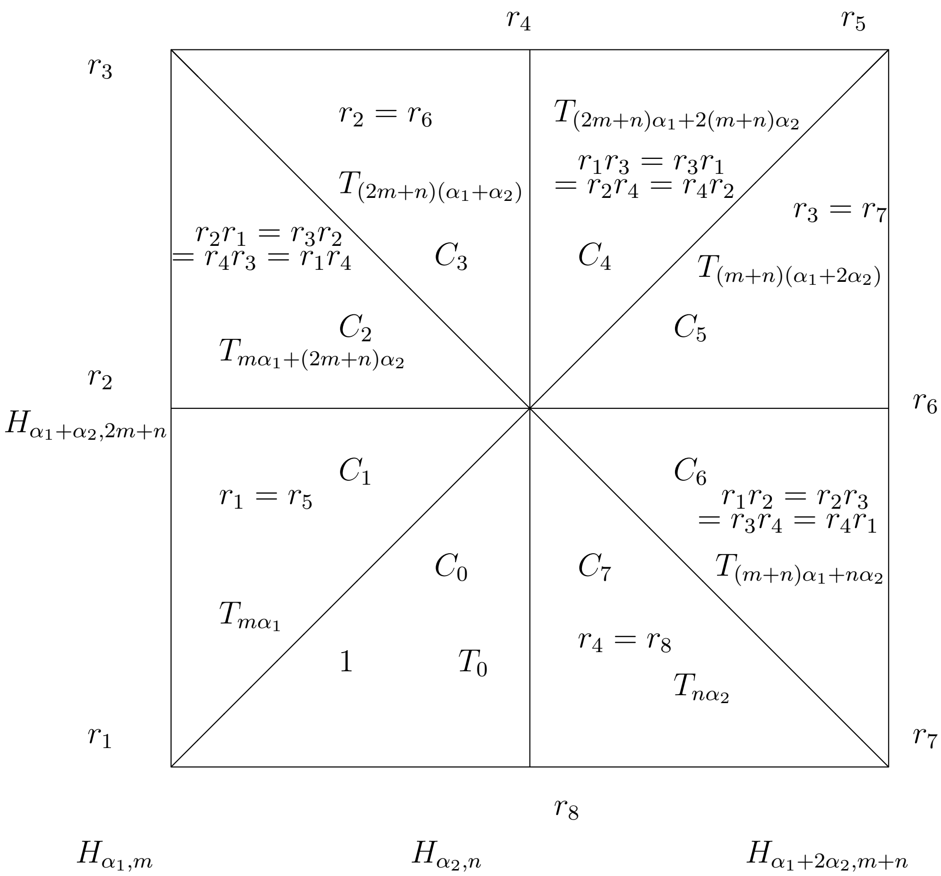


FIGURE 4. Type B_2

(see the proof of Corollary 1.7 of [4]).

Lemma 5.2.3. *Recall that we defined the fundamental Weyl chamber $\mathfrak{C}_0$ so that $-\rho \in \mathfrak{C}_0$. Let $s \in W$ and $\alpha \in \Delta^+$. If the α hyperplanes are positive in $s\mathfrak{C}_0$, then*

$$\#\left\{\beta \in \Delta^+ \;\middle|\; \begin{array}{c}\beta \text{ hyperplanes are}\\ \text{positive in } s\mathfrak{C}_0\end{array}\right\} > \#\left\{\beta \in \Delta^+ \;\middle|\; \begin{array}{c}\beta \text{ hyperplanes are}\\ \text{positive in } s_\alpha s\mathfrak{C}_0\end{array}\right\}.$$

Proof. Note that as

$$\begin{array}{rcl}\left\{\beta \in \Delta^+ \;\middle|\; \begin{array}{c}\beta \text{ hyperplanes are}\\ \text{positive in } s\mathfrak{C}_0\end{array}\right\} & = & \{\beta \in \Delta^+ \mid (\beta, s(-\rho)) > 0\} \\ \text{by invariance of Killing form} & = & \{\beta \in \Delta^+ \mid s^{-1}\beta < 0\} \\ & = & \Delta(s^{-1}) \quad \text{by definition,}\end{array}$$

we only need to show that $\ell(s^{-1}) = \ell(s) > \ell(s_\alpha s) = \ell(s^{-1}s_\alpha)$ if the hypotheses for s and α are satisfied. By (5.2.1), we may assume that $\alpha = s_{i_1} \cdots s_{i_{j-1}}\alpha_{i_j}$ for

$\mathcal{C} = \{C_0, \dots, C_{11}\}$

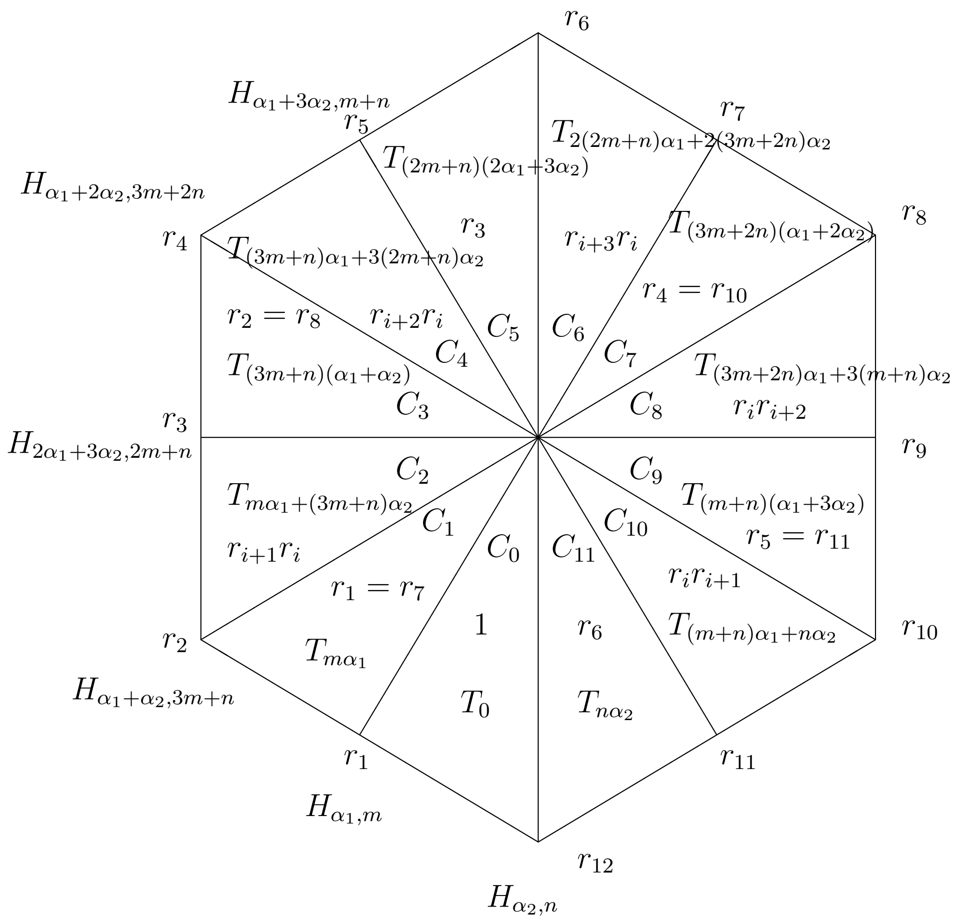


FIGURE 5. Type G_2

some $j \in \{1, \dots, k\}$. Then $s_\alpha = s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_1}$ by Proposition 1.2 of [4]. Therefore $s_\alpha s = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_k}$, whence $\ell(s) > \ell(s_\alpha s)$. $\square$

Lemma 5.2.4. *Suppose $I = \{i_1 < i_2 < \cdots < i_k\}$ satisfies $\varepsilon(I) \neq 0$. Then, letting μ_j be such that $\overline{w_{\{i_1,\dots,i_j\}}}^{-1} w_{\{i_1,\dots,i_j\}} = T_{\mu_j}$, we have*

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k.$$

Proof. We prove this by induction on k. As the reducibility hyperplanes correspond to positive roots and positive integers, therefore the lemma holds for $k = 1$. We wish to prove the lemma for $k = N \geq 2$, assuming that it holds for $k \leq N - 1$. If $\varepsilon(I) \neq 0$, then $\varepsilon(\{i_1, \dots, i_{N-1}\}) \neq 0$. To prove the lemma for $k = N$, by our induction hypothesis, it suffices to show that $\mu_{N-1} < \mu_N$. Now

$$\begin{aligned}\overline{w_I}^{-1} w_I &= \overline{r_{i_1} r_{i_2}} \cdots \overline{r_{i_N}} r_{i_N} \cdots r_{i_2} r_{i_1} \\ &= (\overline{r_{i_1}} \cdots \overline{r_{i_N}} r_{i_N} \overline{r_{i_{N-1}}} \cdots \overline{r_{i_1}})(\overline{r_{i_1}} \cdots \overline{r_{i_{N-1}}} r_{i_{N-1}} \cdots r_{i_1}) \\ &= \overline{r_{i_1}} \cdots \overline{r_{i_N}} r_{i_N} \overline{r_{i_{N-1}}} \cdots \overline{r_{i_1}} T_{\mu_{N-1}}\end{aligned}$$

As $\varepsilon(\overline{r_{i_1}} \cdots \overline{r_{i_{N-1}}} C_{i_N-1}, \overline{r_{i_1}} \cdots \overline{r_{i_{N-1}}} C_{i_N}) \neq 0$, therefore the corresponding hyperplane is a reducibility hyperplane. It follows that $\overline{r_{i_1}} \cdots \overline{r_{i_N}} r_{i_N} \overline{r_{i_{N-1}}} \cdots \overline{r_{i_1}} = T_\nu$ for some $\nu > 0$, whence $\mu_N = \nu + \mu_{N-1} > \mu_{N-1}$. □

Proposition 5.2.5. *Suppose $\mathcal{C}$ intersects a Weyl chamber wall and k reducibility hyperplanes, where $k \geq 1$. Then the Weyl chamber wall corresponds to r_{k+1} if $H_{\alpha_1,m}$ is a reducibility hyperplane, or to $r_{\ell-k}$ if $H_{\alpha_2,n}$ is a reducibility hyperplane. Among the $\ell/2$ possible values for μ where $\overline{w_I}^{-1} w_I = T_\mu$, k are positive. They correspond to*

$$\overline{r_1} r_1 = T_{\mu_1}, \quad \overline{r_1 r_2} r_2 r_1 = T_{\mu_2}, \quad \ldots \quad , \quad \overline{r_1} \cdots \overline{r_k} r_k \cdots r_1 = T_{\mu_k}$$

in the case where $H_{\alpha_1,m}$ is a reducibility hyperplane; or to

$$\overline{r_\ell} r_\ell = T_{\mu_1}, \quad \overline{r_\ell r_{\ell-1}} r_{\ell-1} r_\ell = T_{\mu_2}, \quad \ldots \quad , \quad \overline{r_\ell} \cdots \overline{r_{\ell-k+1}} r_{\ell-k+1} \cdots r_\ell = T_{\mu_k}$$

in the case where $H_{\alpha_2,n}$ is a reducibility hyperplane. Furthermore,

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k.$$

Proof. The first statement follows from Lemma 5.2.1. Since the alcoves of $\mathcal{C}$ are in one-to-one correspondence with possible values for w_I, therefore by Lemma 5.2.2, there are $\ell/2$ possible values for μ. These correspond to $\overline{r_1} r_1$, $\overline{r_2} r_2$, …, $\overline{r_{\ell/2}} r_{\ell/2}$ by Lemma 5.2.2 since $r_i \neq r_j$ for $1 \leq i < j \leq \ell/2$ and certainly $r_i = s_\alpha r_j$ is not possible as r_i is an affine reflection while $s_\alpha r_j$ is not. It now follows by Lemma 5.2.1 that there are k positive values for μ.

To prove the remainder of the proposition, by Lemma 5.2.4 and symmetry, it suffices to assume that $H_{\alpha_1,m}$ is a reducibility hyperplane and to show that $\varepsilon(\{1, 2, \ldots, k\}) \neq 0$.

By Lemma 5.2.1, there exists a product of simple reflections $s = s_{i_1} \cdots s_{i_{k+1}}$ such that

$$\begin{gathered} r_1 \text{ corresponds to } \alpha_{i_1} \\ r_2 \text{ corresponds to } s_{i_1} \alpha_{i_2} \\ \vdots \\ r_{k+1} \text{ corresponds to } s_{i_1} \cdots s_{i_k} \alpha_{i_{k+1}}. \end{gathered}$$

(See Lemma 5.2.3 and the material preceding it.) Observing that $C_j = r_j \cdots r_2 r_1 C_0$, we also have

$$\overline{r_j r_{j-1}} \cdots \overline{r_1} = s_{i_1} \cdots s_{i_j} \text{ for } 1 \leq j \leq k.$$

We need to show that $\varepsilon(\overline{r_1 r_2} \cdots \overline{r_{j-1}} C_{j-1}, \overline{r_1 r_2} \cdots \overline{r_{j-1}} C_j) \neq 0$ for $2 \leq j \leq k$. The hyperplane separating C_{j-1} and C_j corresponds to the root $s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}$. Therefore the hyperplane separating $\overline{r_1 r_2} \cdots \overline{r_{j-1}} C_{j-1}$ and $\overline{r_1 r_2} \cdots \overline{r_{j-1}} C_j$ corresponds to the root $\overline{r_1} \cdots \overline{r_{j-1}} s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j} = s_{i_{j-1}} \cdots s_{i_1} s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j} = \alpha_{i_j}$. Alcoves of $\mathcal{C}$ lie in $s_{i_1} \cdots s_{i_{k+1}} \mathfrak{C}_0$ and $s_{i_1} \cdots s_{i_k} \mathfrak{C}_0$. Therfore $\overline{r_1 r_2} \cdots \overline{r_{j-1}} C_{j-1}$ and $\overline{r_1 r_2} \cdots \overline{r_{j-1}} C_j$ lie in $s_{i_j} s_{i_{j+1}} \cdots s_{i_k} \mathfrak{C}_0$ or in $s_{i_j} s_{i_{j+1}} \cdots s_{i_{k+1}} \mathfrak{C}_0$. In both of those

Weyl chambers, α_{i_j} hyperplanes are reducibility hyperplanes, which implies that $\varepsilon(\overline{r_1 r_2} \cdots \overline{r_{j-1}} C_{j-1}, \overline{r_1 r_2} \cdots \overline{r_{j-1}} C_j) \neq 0$. Therefore $\varepsilon(\{1, 2, \dots, k\}) \neq 0$. □

Proposition 5.2.6. *For a fixed hyperplane $H_{\alpha_i,N}$ separating two adjacent alcoves $wA_0 \subset H^+_{\alpha_i,N}$ and $w'A_0 \subset H^-_{\alpha_i,N}$, the value of $\varepsilon(wA_0, w'A_0)$ does not depend on w and w'.*

Proof. By symmetry, it suffices to suppose $\mathcal{C}$ intersects a Weyl chamber wall and to show that if $H_{\alpha_1,m}$ is a reducibility hyperplane, then

$$\varepsilon(C_0, C_1) + \varepsilon(C_{\ell/2}, C_{\ell/2+1}) = 0.$$

Suppose $\mathcal{C}$ intersects k reducibility hyperplanes. Then by Proposition 5.2.5, the only positive values for μ in $\overline{w_I}^{-1} w_I = T_\mu$ are

$$\overline{r_1} r_1 = T_{\mu_1}, \quad \overline{r_1 r_2} r_2 r_1 = T_{\mu_2}, \quad \dots \quad , \quad \overline{r_1} \cdots \overline{r_k} r_k \cdots r_1 = T_{\mu_k}$$

and $0 < \mu_1 < \mu_2 < \cdots < \mu_k$.

Consider (5.1.2). We may rewrite the equation as

$$\sum_{j=1}^{k} \sum_{\substack{\emptyset \neq I \subset \{1,\dots,\ell\} \\ \overline{w_I}^{-1} w_I = T_{\mu_j}}} 2^{|I|} \varepsilon(I) R^{\overline{w_I}^{-1} C_0} \left(\overline{w_I}^{-1} w_I \lambda\right) = 0.$$

Here, we observe that as $\mathcal{C}$ does not lie entirely within a single Weyl chamber, Lemma 5.1.2 may not be used as it was in the proof of Proposition 5.1.1 to obtain equation (5.1.3). However, by *minimality* of μ_1,

$$\sum_{\substack{\emptyset \neq I \subset \{1,\dots,\ell\} \\ \overline{w_I}^{-1} w_I = T_{\mu_1}}} 2^{|I|} \varepsilon(I) = 0.$$

Indices I of size one in the above sum are $\{1\}$ and $\{\ell/2+1\}$. Because μ_1 is minimal, by Lemma 5.2.4 $\varepsilon(I) = 0$ if $\overline{w_I}^{-1} w_I = T_{\mu_1}$ and $|I| \geq 2$, whence

$$\varepsilon(C_0, C_1) + \varepsilon(C_{\ell/2}, C_{\ell/2+1}) = 0.$$

In fact, we see that

$$\sum_{\substack{\emptyset \neq I \subset \{1,\dots,\ell\} \\ \overline{w_I}^{-1} w_I = T_{\mu_1}}} 2^{|I|} \varepsilon(I) R^{\overline{w_I}^{-1} C_0} \left(\overline{w_I}^{-1} w_I \lambda\right) = 0. \tag{5.2.2}$$

□

Remark 5.2.7. Note that we did not use simplicity of α_1 and α_2 in the proof above or in any of the lemmas quoted.

Definition 5.2.8. Recall that $C_k = r_k \cdots r_2 r_1 C_0$. Define $\mathfrak{C}_k$ to be $\overline{r_k} \cdots \overline{r_2 r_1} \mathfrak{C}_0$.

Proposition 5.2.9. *Suppose $\mathcal{C}$ intersects $\mathfrak{C}_k$ and $\mathfrak{C}_{k+1}$. If $2 \leq k < \ell/2$, then*

$$\varepsilon(C_{k-1}, C_k) + \varepsilon(C_{\ell/2+k-1}, C_{\ell/2+k}) + 2\varepsilon(C_0, C_1)(\overline{r_1} C_1, \overline{r_1} C_2) = 0.$$

Symmetrically, if $\ell/2 \leq k < \ell - 2$, then

$$\varepsilon(C_{k+1-\ell/2}, C_{k+2-\ell/2}) + \varepsilon(C_{k+1}, C_{k+2}) + 2\varepsilon(C_{\ell/2-1}, C_{\ell/2}) \varepsilon(\overline{r_{\ell/2}} C_{\ell-2}, \overline{r_{\ell/2}} C_{\ell-1}) = 0.$$

Proof. It suffices to prove the first statement. By (5.2.2), (5.1.2) may be written as

$$\sum_{j=2}^{k} \sum_{\substack{\emptyset \neq I \subset \{1,\ldots,\ell\} \\ \overline{w_I}^{-1} w_I = T_{\mu_j}}} 2^{|I|} \varepsilon(I) R^{\overline{w_I}^{-1} C_0} \left(\overline{w_I}^{-1} w_I \lambda \right) = 0.$$

With the μ_1 terms removed, μ_2 is now minimal. Therefore

$$\sum_{\substack{\emptyset \neq I \subset \{1,\ldots,\ell\} \\ \overline{w_I}^{-1} w_I = T_{\mu_2}}} 2^{|I|} \varepsilon(I) = 0.$$

Now $T_{\mu_2} = \overline{r_1 r_2} r_2 r_1 = \overline{r_k r_{k+1}} r_{k+1} r_k = \overline{r_k} r_k$, whence $\{k\}$ and $\{k + \ell/2\}$ are the indices of size one appearing in the sum.

By Lemma 5.2.4, $\varepsilon(I) = 0$ for $|I| \geq 3$ satisfying $\overline{w_I}^{-1} w_I = T_{\mu_2}$. Furthermore, if $I = \{i_1, i_2\}$ is such that $\varepsilon(I) \neq 0$ and $\overline{w_I}^{-1} w_I = T_{\mu_2}$, then $\overline{r_{i_1}} r_{i_1} = T_{\mu_1}$. Therefore $\overline{r_{i_1}} r_{i_1} = \overline{r_1} r_1$ and $\overline{r_{i_1} r_{i_2}} r_{i_2} r_{i_1} = \overline{r_1 r_2} r_2 r_1$. It follows that $\{1, 2\}$, $\{1, \ell/2 + 2\}$, and $\{\ell/2 + 1, \ell/2 + 2\}$ are the indices of size two appearing in the sum. By Proposition 5.2.6, $\varepsilon(\{1, \ell/2 + 2\}) + \varepsilon(\{\ell/2 + 1, \ell/2 + 2\}) = 0$. We conclude that

$$\varepsilon(C_{k-1}, C_k) + \varepsilon(C_{\ell/2+k-1}, C_{\ell/2+k}) + 2\varepsilon(C_0, C_1)(\overline{r_1} C_1, \overline{r_1} C_2) = 0.$$

□

Proposition 5.2.10. *If $\mathcal{C}$ contains one Weyl chamber wall, then for $2 \leq i \leq \ell/2$*

$$\varepsilon(C_{i-1}, C_i) + (-1)^{N(i,\mathcal{C})} \varepsilon(C_{\ell/2+i-1}, C_{\ell/2+i}) = 0,$$

where $N(i, \mathcal{C}) = \# \left\{ I = \{i_1 < i_2\} \subset \{1, \ldots, \ell\} \mid \varepsilon(I) \neq 0 \text{ and } \overline{w_I}^{-1} w_I = \overline{r_i} r_i \right\}$. In fact, if $\varepsilon(C_{i-1}, C_i) \neq 0$ then $(-1)^{N(i,\mathcal{C})} = (-1)^{n(i,\mathcal{C})-1}$ where $T_{\mu_{n(i,\mathcal{C})}} = \overline{r_i} r_i$ in the language of Proposition 5.2.5.

Proof. The statement is trivial if r_i does not correspond to a reducibility hyperplane. We may work in the setup of Proposition 5.2.5 and assume that there is i_0 such that $\overline{r_i} r_i = T_{\mu_{i_0}}$. We may rewrite (5.1.2) using

$$R^{wA_0}(\lambda) = R^{w'A_0}(\lambda) + 2\varepsilon(wA_0, w'A_0) R^{\overline{s_{\alpha,n}} w' A_0}(\overline{s_{\alpha,n}} s_{\alpha,n} \lambda)$$

so that μ_{i_0} is minimal. We obtain an equation of the form

(5.2.3)

$$\sum_{j=i_0}^{k} \sum_{\substack{\emptyset \neq I \subset \{1,\ldots,\ell\} \\ \overline{w_I}^{-1} w_I = T_{\mu_j}}} 2^{|I|} \varepsilon(I) R^{\overline{w_I}^{-1} C_0} \left(\overline{w_I}^{-1} w_I \lambda \right) + \sum_{w \in S \subset W_a} c_S R^{\overline{w}^{-1} C_0} \left(\overline{w}^{-1} w \lambda \right) = 0,$$

where the c_S are constant integers. For $w \in S$, $\overline{w}^{-1} w = T_\mu$ where $\mu \in \{\mu_{i_0}, \ldots, \mu_k\}$.

We first show that $c_S \equiv 0 \mod 8$ by elaborating on the procedure by which we obtain the above equation. We may remove from (5.1.2) pairs of the form $\{j\}, \{j + \ell/2\}$ corresponding to $\mu_1, \ldots, \mu_{i_0-1}$ such that $\varepsilon(\{j\}) + \varepsilon(\{j + \ell/2\}) = 0$. Summands arising from pairs $\{j\}, \{j+\ell/2\}$ for which $\varepsilon(\{j\}) + \varepsilon(\{j+\ell/2\}) \neq 0$ may be rewritten as $4\varepsilon(\{j\}) R^{\overline{r_j} C_0}(\overline{r_j} r_j \lambda)$. Our equation (5.1.2) has now been rewritten so that any summand corresponding to $\mu_1, \ldots, \mu_{i_0-1}$ has coefficient divisible by 4. It now follows that applications of (4.2) result in integer coefficients divisible by 8.

The coefficients in (5.2.3) corresponding to μ_{i_0} must sum to zero. As ε must be 1 or -1, our first statement follows from considering that sum modulo 8.

To prove the second statement, we first note that if we define $I = \{i_1 < i_2\}$ and $J = \{j_1 < j_2\}$ to be equivalent if $r_{i_1} = r_{j_1}$ and $r_{i_2} = r_{j_2}$, then for each equivalence class, the set in the statement of this proposition contains either zero, one, or three representatives. Therefore the parity of $N(i, \mathcal{C})$ is the same as the parity of the number of possible values for r_{i_1} for $\{i_1 < i_2\}$ belonging to that set. That number is $i_0 - 1 = n(i, \mathcal{C}) - 1$ by Lemma 5.2.4. □

Lemma 5.2.11. *In the setting of Proposition 5.2.5, the equations*

$$\overline{r_1}r_1 = T_{\mu_1}, \quad \overline{r_1}\,\overline{r_2}r_2r_1 = T_{\mu_2}, \quad \cdots \quad , \quad \overline{r_1}\cdots\overline{r_k}r_k\cdots r_1 = T_{\mu_k}$$

may be rewritten as

$$\overline{r_1}r_1 = T_{\mu_1}, \quad \overline{r_k}r_k = T_{\mu_2}, \quad \overline{r_2}r_2 = T_{\mu_3}, \quad \overline{r_{k-1}}r_{k-1} = T_{\mu_4}, \quad \cdots$$

and

$$\overline{r_\ell}r_\ell = T_{\mu_1}, \quad \overline{r_\ell}\,\overline{r_{\ell-1}}r_{\ell-1}r_\ell = T_{\mu_2}, \quad \ldots \quad , \quad \overline{r_\ell}\cdots\overline{r_{\ell-k+1}}r_{\ell-k+1}\cdots r_\ell = T_{\mu_k}$$

may be rewritten as

$$\overline{r_\ell}r_\ell = T_{\mu_1}, \quad \overline{r_{\ell-k+1}}r_{\ell-k+1} = T_{\mu_2}, \quad \overline{r_{\ell-1}}r_{\ell-1} = T_{\mu_3}, \quad \overline{r_{\ell-k+2}}r_{\ell-k+2} = T_{\mu_4}, \quad \cdots .$$

Proof. We make repeated use of $r_ir_{i+a} = r_jr_{j+a}$ and $\overline{r_{k+1}}r_{k+1} = 1$ in the case where $\mathcal{C}$ intersects $\mathfrak{C}_k$ and $\mathfrak{C}_{k+1}$, and $\overline{r_{\ell-k}}r_{\ell-k} = 1$ in the case where $\mathcal{C}$ intersects $\mathfrak{C}_{\ell-k-1}$ and $\mathfrak{C}_{\ell-k}$. □

Proposition 5.2.12. *In the setting of Proposition 5.2.10, if $\mathcal{C}$ intersects $\mathfrak{C}_k$ and $\mathfrak{C}_{k+1}$ and $\varepsilon(C_{i-1}, C_i) \neq 0$, then*

$$(-1)^{N(i,\mathcal{C})} = \begin{cases} 1 & \text{if } 2 \le i \le \lfloor \frac{k+1}{2} \rfloor \text{ and } k < \ell/2 \\ -1 & \text{if } \ell/2 \ge i > \lfloor \frac{k+1}{2} \rfloor \text{ and } k < \ell/2 \\ 1 & \text{if } \ell/2 \ge i \ge \ell/2 - \lfloor \frac{k-1}{2} \rfloor \text{ and } k \ge \ell/2 \\ -1 & \text{if } 2 \le i < \ell/2 - \lfloor \frac{k-1}{2} \rfloor \text{ and } k \ge \ell/2. \end{cases}$$

Proof. Use the previous lemma and proposition. □

We combine the propositions of this subsection and record our computations in the following tables:

Theorem 5.2.13. Type A_2:

Weyl chamber walls in $\mathcal{C}$	**Equations for Type A_2**
$H_{\alpha_1,0}$	$\varepsilon(C_2, C_3) + \varepsilon(C_5, C_6) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_2, C_3)\varepsilon(\overline{r_3}C_4, \overline{r_3}C_5) = 0$
$H_{\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_0, C_1)\varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) = 0$
$H_{\alpha_1+\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) = 0$ $\varepsilon(C_2, C_3) + \varepsilon(C_5, C_0) = 0$

Type B_2:

Weyl chamber walls in $\mathcal{C}$	**Equations**
$H_{\alpha_1,0}$	$\varepsilon(C_2,C_3)+\varepsilon(C_6,C_7)=0$ $\varepsilon(C_3,C_4)+\varepsilon(C_7,C_0)=0$ $\varepsilon(C_1,C_2)+\varepsilon(C_5,C_6)+2\varepsilon(C_3,C_4)\varepsilon(\overline{r_4}C_6,\overline{r_4}C_7)=0$
$H_{\alpha_2,0}$	$\varepsilon(C_0,C_1)+\varepsilon(C_4,C_5)=0$ $\varepsilon(C_1,C_2)+\varepsilon(C_5,C_6)=0$ $\varepsilon(C_2,C_3)+\varepsilon(C_6,C_7)+2\varepsilon(C_0,C_1)\varepsilon(\overline{r_1}C_1,\overline{r_1}C_2)=0$
$H_{\alpha_1+\alpha_2,0}$	$\varepsilon(C_0,C_1)+\varepsilon(C_4,C_5)=0$ $\varepsilon(C_3,C_4)+\varepsilon(C_7,C_0)=0$ $\varepsilon(C_2,C_3)+\varepsilon(C_6,C_7)+2\varepsilon(C_3,C_4)\varepsilon(\overline{r_4}C_6,\overline{r_4}C_7)=0$
$H_{\alpha_1+2\alpha_2,0}$	$\varepsilon(C_0,C_1)+\varepsilon(C_4,C_5)=0$ $\varepsilon(C_3,C_4)+\varepsilon(C_7,C_0)=0$ $\varepsilon(C_1,C_2)+\varepsilon(C_5,C_6)+2\varepsilon(C_0,C_1)\varepsilon(\overline{r_1}C_1,\overline{r_1}C_2)=0$

Type G_2:

Weyl chamber walls in $\mathcal{C}$	**Equations**
$H_{\alpha_1,0}$	$\varepsilon(C_1,C_2)+\varepsilon(C_7,C_8)+2\varepsilon(C_5,C_6)\varepsilon(\overline{r_6}C_{10},\overline{r_6}C_{11})=0$ $\varepsilon(C_2,C_3)-\varepsilon(C_8,C_9)=0$ $\varepsilon(C_3,C_4)+\varepsilon(C_9,C_{10})=0$ $\varepsilon(C_4,C_5)+\varepsilon(C_{10},C_{11})=0$ $\varepsilon(C_5,C_6)+\varepsilon(C_{11},C_0)=0$
$H_{\alpha_1+\alpha_2,0}$	$\varepsilon(C_0,C_1)+\varepsilon(C_6,C_7)=0$ $\varepsilon(C_2,C_3)+\varepsilon(C_8,C_9)+2\varepsilon(C_5,C_6)\varepsilon(\overline{r_6}C_{10},\overline{r_6}C_{11})=0$ $\varepsilon(C_3,C_4)-\varepsilon(C_9,C_{10})=0$ $\varepsilon(C_4,C_5)+\varepsilon(C_{10},C_{11})=0$ $\varepsilon(C_5,C_6)+\varepsilon(C_{11},C_0)=0$
$H_{2\alpha_1+3\alpha_2,0}$	$\varepsilon(C_0,C_1)+\varepsilon(C_6,C_7)=0$ $\varepsilon(C_1,C_2)+\varepsilon(C_7,C_8)+2\varepsilon(C_0,C_1)\varepsilon(\overline{r_1}C_1,\overline{r_1}C_2)=0$ $\varepsilon(C_3,C_4)+\varepsilon(C_9,C_{10})+2\varepsilon(C_5,C_6)\varepsilon(\overline{r_6}C_{10},\overline{r_6}C_{11})=0$ $\varepsilon(C_4,C_5)+\varepsilon(C_{10},C_{11})=0$ $\varepsilon(C_5,C_6)+\varepsilon(C_{11},C_0)=0$
$H_{\alpha_1+2\alpha_2,0}$	$\varepsilon(C_0,C_1)+\varepsilon(C_6,C_7)=0$ $\varepsilon(C_1,C_2)+\varepsilon(C_7,C_8)=0$ $\varepsilon(C_2,C_3)+\varepsilon(C_8,C_9)+2\varepsilon(C_0,C_1)\varepsilon(\overline{r_1}C_1,\overline{r_1}C_2)=0$ $\varepsilon(C_4,C_5)+\varepsilon(C_{10},C_{11})+2\varepsilon(C_5,C_6)\varepsilon(\overline{r_6}C_{10},\overline{r_6}C_{11})=0$ $\varepsilon(C_5,C_6)+\varepsilon(C_{11},C_0)=0$
$H_{\alpha_1+3\alpha_2,0}$	$\varepsilon(C_0,C_1)+\varepsilon(C_6,C_7)=0$ $\varepsilon(C_1,C_2)+\varepsilon(C_7,C_8)=0$ $\varepsilon(C_2,C_3)-\varepsilon(C_8,C_9)=0$ $\varepsilon(C_3,C_4)+\varepsilon(C_9,C_{10})+2\varepsilon(C_0,C_1)\varepsilon(\overline{r_1}C_1,\overline{r_1}C_2)=0$ $\varepsilon(C_5,C_6)+\varepsilon(C_{11},C_0)=0$

Type G_2 cont'd:

Weyl chamber walls in $\mathcal{C}$	**Equations**
$H_{\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_6, C_7) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_7, C_8) = 0$ $\varepsilon(C_2, C_3) + \varepsilon(C_8, C_9) = 0$ $\varepsilon(C_3, C_4) - \varepsilon(C_9, C_{10}) = 0$ $\varepsilon(C_4, C_5) + \varepsilon(C_{10}, C_{11}) + 2\varepsilon(C_0, C_1)\varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) = 0$

Remark 5.2.14. None of our arguments referred to simplicity of the α_i.

Definition 5.2.15. Fix a hyperplane $H_{\gamma,N}$ and $s \in W$. We let $\varepsilon(H_{\gamma,N}, s)$ be the value of any $\varepsilon(wA_0, w'A_0)$, where $H_{\gamma,N}$ separates the adjacent alcoves wA_0 and $w'A_0$, $wA_0 \subset H^+_{\gamma,N}$ and $w'A_0 \subset H^-_{\gamma,N}$, and $wA_0 \subset s\mathfrak{C}_0$ (and hence $w'A_0 \subset s\mathfrak{C}_0$ also). By Proposition 5.1.1, this is well-defined.

Definition 5.2.16. Given a root α, let δ_α be -1 if α is noncompact, and 1 if it is compact.

Lemma 5.2.17. *If α is simple and n is positive, then $\varepsilon(H_{\alpha,n}, s) = \delta^n_\alpha$ if α hyperplanes are positive on $s\mathfrak{C}_0$.*

Proof. Choose a standard triple $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha = [X_\alpha, Y_\alpha] \in \mathfrak{h}$ satisfying $\mu(H_\alpha) = (\mu, \alpha^\vee)\ \forall\, \mu \in \mathfrak{h}^*$. We have the relations

$$[H_\alpha, X_\alpha] = 2X_\alpha, \qquad [H_\alpha, Y_\alpha] = -2Y_\alpha, \qquad [X_\alpha, Y_\alpha] = H_\alpha,$$

$$\alpha(H_\alpha) = (\alpha, \alpha^\vee) = 2.$$

Taking complex conjugates, multiplying by -1, and using anti-commutativity,

$$\left[-\bar{H}_\alpha, \bar{X}_\alpha\right] = -2\bar{X}_\alpha, \qquad \left[-\bar{H}_\alpha, \bar{Y}_\alpha\right] = 2\bar{Y}_\alpha, \qquad \left[\bar{Y}_\alpha, \bar{X}_\alpha\right] = -\bar{H}_\alpha,$$

$$\bar{\alpha}(\bar{H}_\alpha) = (\bar{\alpha}, \bar{\alpha}^\vee) = 2.$$

If α is imaginary, then $\bar{X}_\alpha \in \mathfrak{g}_{-\alpha}$ and $\bar{Y}_\alpha \in \mathfrak{g}_\alpha$. Also, $-\bar{H}_\alpha = H_\alpha$. The above relations give $(\bar{Y}_\alpha, \bar{X}_\alpha, -\bar{H}_\alpha) = (cX_\alpha, c^{-1}Y_\alpha, H_\alpha)$ for some non-zero scalar c. $B(X, \bar{X})$ is positive for non-zero $X \in \mathfrak{p}$ and negative for non-zero $X \in \mathfrak{k}$. By Lemma 2.18a) of [7], if α is compact, then $c < 0$ and if α is noncompact, then $c > 0$. We may arrange for c to be ± 1. We have:

$$-\bar{Y}_\alpha = \delta_\alpha X_\alpha.$$

The $\lambda - n\alpha$ weight space of $M(\lambda)$ is one-dimensional and spanned by the vector $Y^n_\alpha v_\lambda$. We know that

$$\begin{aligned}\langle Y^n_\alpha v_\lambda, Y^n_\alpha v_\lambda\rangle_\lambda &= \delta^n_\alpha \langle v_\lambda, X^n_\alpha Y^n_\alpha v_\lambda\rangle_\lambda \\ &= \delta^n_\alpha n!\, \langle v_\lambda, H_\alpha(H_\alpha - 1)\cdots(H_\alpha - (n-1))v_\lambda\rangle_\lambda\end{aligned}$$

from $\mathfrak{sl}_2$ theory. As $\lambda(H_\alpha) - j$ is positive for $j < n-1$ and $\lambda \in H^-_{\alpha,n} \cap H^+_{\alpha,n-1}$, negative for $j = n-1$ and $\lambda \in H^-_{\alpha,n} \cap H^+_{\alpha,n-1}$, while it is positive for $j = n-1$ and $\lambda \in H^+_{\alpha,n}$, we conclude that $\varepsilon(H_{\alpha,n}, s) = \delta^n_\alpha$. ∎

Theorem 5.2.18. *In the setting of figures 3, 4, and 5, we have the following tables of values for* $\varepsilon(H_{\alpha,N}, s)$*:*

Type A_2**:**

Hyperplane	**Weyl Chamber** $s\mathfrak{C}_0$		
$H_{\alpha_1,N}$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$
	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$
$H_{\alpha_1+\alpha_2,N}$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$
	$\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$-\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$\delta^N_{\alpha_1}\delta^N_{\alpha_2}$
$H_{\alpha_2,N}$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$
	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$

Type B_2**:**

Hyperplane	**Weyl Chamber** $s\mathfrak{C}_0$			
$H_{\alpha_1,N}$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$
	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$
$H_{\alpha_1+\alpha_2,N}$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$
	$\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$-\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$-\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$\delta^N_{\alpha_1}\delta^N_{\alpha_2}$
$H_{\alpha_1+2\alpha_2,N}$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$
	$\delta^N_{\alpha_1}$	$-\delta^N_{\alpha_1}$	$-\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$
$H_{\alpha_2,N}$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$	$\mathfrak{C}_7$
	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$

Type G_2**:**

Hyperplane	**Weyl Chamber** $s\mathfrak{C}_0$					
$H_{\alpha_1,N}$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$
	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$
$H_{\alpha_1+\alpha_2,N}$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$	$\mathfrak{C}_7$
	$\delta^N_{\alpha_2}\delta^N_{\alpha_1}$	$-\delta^N_{\alpha_2}\delta^N_{\alpha_1}$	$-\delta^N_{\alpha_2}\delta^N_{\alpha_1}$	$-\delta^N_{\alpha_2}\delta^N_{\alpha_1}$	$-\delta^N_{\alpha_2}\delta^N_{\alpha_1}$	$\delta^N_{\alpha_2}\delta^N_{\alpha_1}$
$H_{2\alpha_1+3\alpha_2,N}$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$	$\mathfrak{C}_7$	$\mathfrak{C}_8$
	$\delta^N_{\alpha_2}$	$-\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$	$-\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$
$H_{\alpha_1+2\alpha_2,N}$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$	$\mathfrak{C}_7$	$\mathfrak{C}_8$	$\mathfrak{C}_9$
	$\delta^N_{\alpha_1}$	$-\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$	$-\delta^N_{\alpha_1}$	$\delta^N_{\alpha_1}$
$H_{\alpha_1+3\alpha_2,N}$	$\mathfrak{C}_5$	$\mathfrak{C}_6$	$\mathfrak{C}_7$	$\mathfrak{C}_8$	$\mathfrak{C}_9$	$\mathfrak{C}_{10}$
	$\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$-\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$-\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$-\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$-\delta^N_{\alpha_1}\delta^N_{\alpha_2}$	$\delta^N_{\alpha_1}\delta^N_{\alpha_2}$
$H_{\alpha_2,N}$	$\mathfrak{C}_6$	$\mathfrak{C}_7$	$\mathfrak{C}_8$	$\mathfrak{C}_9$	$\mathfrak{C}_{10}$	$\mathfrak{C}_{11}$
	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$	$\delta^N_{\alpha_2}$

Proof. Combine the previous theorem and lemma. □

5.3. **Using induction to obtain the general case.**

Lemma 5.3.1. *Let* γ *be a positive non-simple root. There exists some simple root* α *such that* $(\gamma, \alpha) > 0$ *and* $s_\alpha\gamma > 0$*.*

Proof. The first statement follows from Lemma 10.2 A of [3] and the second from Lemma 10.2 B. □

Proposition 5.3.2. *Let* γ *be a positive non-simple root. Let* α *and* $\beta = s_\alpha\gamma$ *be the roots provided by Lemma 5.3.1. If* α *and* γ *do not, along with other roots in*

$\Delta(\mathfrak{g},\mathfrak{h})$, *form a type* G_2 *root system, then:*
If $|\gamma| = |\alpha|$*:*

$$\varepsilon(H_{\gamma,N}, s) = \begin{cases} -\delta_\alpha^N \varepsilon(H_{\beta,N}, s_\alpha s) & \text{if } \alpha \text{ and } \beta \text{ hyperplanes are positive on } s\mathfrak{C}_0 \\ \delta_\alpha^N \varepsilon(H_{\beta,N}, s_\alpha s) & \text{otherwise.} \end{cases}$$

If $2|\gamma|^2 = |\alpha|^2$*:*

$$\varepsilon(H_{\gamma,N}, s) = \begin{cases} -\delta_\alpha^N \varepsilon(H_{\beta,N}, s_\alpha s) & \text{if } \alpha \text{ and } \alpha+2\beta = s_\beta\alpha \text{ hyperplanes are} \\ & \text{positive on } s\mathfrak{C}_0 \\ \delta_\alpha^N \varepsilon(H_{\beta,N}, s_\alpha s) & \text{otherwise.} \end{cases}$$

If $|\gamma|^2 = 2|\alpha|^2$*:*

$$\varepsilon(H_{\gamma,N}, s) = \begin{cases} -\varepsilon(H_{\beta,N}, s_\alpha s) & \text{if } \alpha \text{ and } \alpha+\beta = s_\beta\alpha \text{ hyperplanes are positive} \\ & \text{on } s\mathfrak{C}_0 \\ \varepsilon(H_{\beta,N}, s_\alpha s) & \text{otherwise.} \end{cases}$$

Proof. Consider a two-dimensional slice $P = \operatorname{span}\{\alpha,\gamma\} + \mu_0$ through $s\mathfrak{C}_\circ$, where μ_0 lies in the intersection of $H_{\gamma,N}$ and $H_{\alpha,k}$ for some integer k, and $(\mu_0, \delta^\vee)$ is not an integer for any root δ that does not lie in the root subsystem generated by α and γ. We are in the leftmost situation of Figure 2. If we take a suitably small circular path around μ_0 in P, due to Remark 5.2.14, the proof of Theorem 5.2.13 still applies with α and γ corresponding to a suitable choice of the roots in the root system generated by α_1 and α_2. This choice must be made so that α corresponds to some α_i. Further, an appropriate analogue of Proposition 5.2.6 still holds, so that for α and some root δ in our root system generated by α and γ so that $\{\alpha,\delta\}$ corresponds to $\{\alpha_1,\alpha_2\}$, the values for ε for the hyperplanes $H_{\alpha,k}$ and $H_{\delta,k}$ do not change as we cross Weyl chamber walls along a path restricted to P.

This procedure handles all three cases. We will illustrate this in detail by applying it to the first case and discuss the remaining cases briefly.
Case $|\gamma| = |\alpha|$**:** First, we work in the setup of Figure 1 when $m = 0$. Our equation from Theorem 5.2.13 gives:

$$\underset{H_{\alpha_1+\alpha_2,m+n}}{\overset{-\ \ +}{\varepsilon(C_1,C_2)}} + \underset{H_{\alpha_1+\alpha_2,m+n}}{\overset{+\ \ -}{\varepsilon(C_4,C_5)}} + 2\underset{H_{\alpha_2,n}}{\overset{-\ \ +}{\varepsilon(C_2,C_3)}}\underset{H_{\alpha_1,m+n}}{\overset{+\ \ -}{\varepsilon(\overline{r_3}C_4,\overline{r_3}C_5)}} = 0$$

As $\varepsilon = \pm 1$, letting $n = N$ we may rewrite this equation as

$$\begin{aligned} \varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_2}s_{\alpha_1}) &= -\varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}) \\ &= \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1})\varepsilon(H_{\alpha_1,N}, s_{\alpha_1}) \\ \text{by Lemma 5.2.17} \quad &= \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}). \end{aligned}$$

Theorem 5.2.13 indicates that $\varepsilon(H_{\alpha_1+\alpha_2,N}, s)$ changes sign as we cross the hyperplane $H_{\alpha_2,0}$, so we also have

$$\varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}) = \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}) = \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}).$$

Writing $\alpha > 0$ on $s\mathfrak{C}_\circ$ to mean that α hyperplanes are positive on $s\mathfrak{C}_\circ$, we have

$$\varepsilon(H_{\alpha_1+\alpha_2,N}, s) = \begin{cases} \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s) & \text{if } \alpha_1 < 0, \alpha_2 > 0 \text{ on } s\mathfrak{C}_\circ, \\ -\delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s) & \text{if } \alpha_1 > 0, \alpha_2 > 0 \text{ on } s\mathfrak{C}_\circ, \\ \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_2}s) & \text{if } \alpha_1 > 0, \alpha_2 < 0 \text{ on } s\mathfrak{C}_\circ. \end{cases}$$

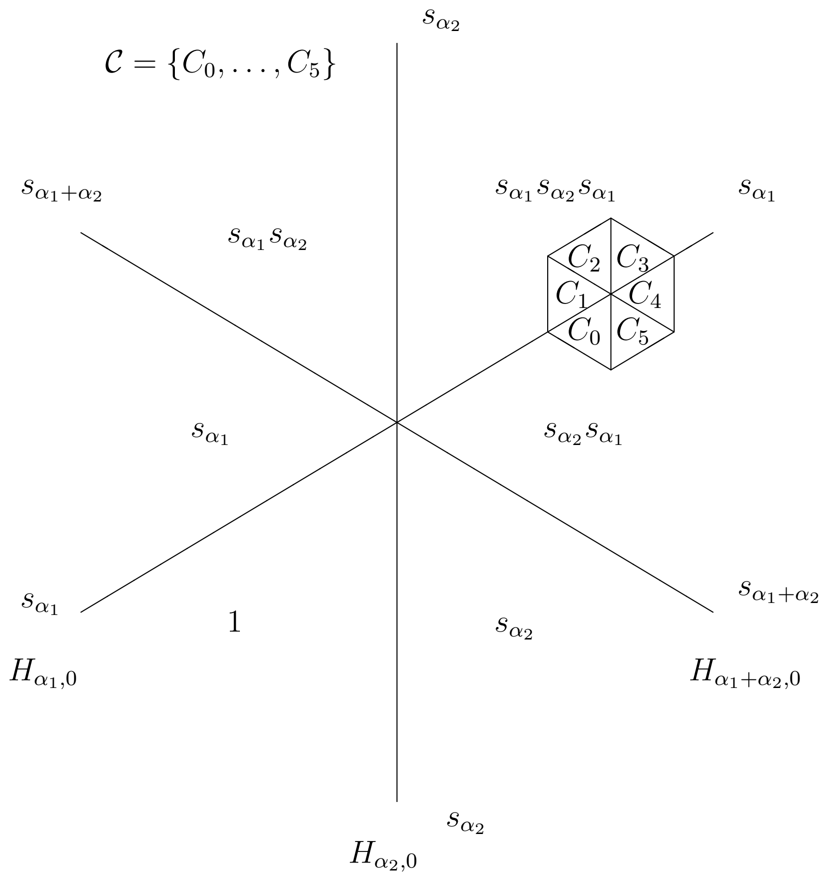


FIGURE 6. Type A_2: calculating ε for $H_{\alpha_1+\alpha_2,N}$

Note that $\alpha_1+\alpha_2$ hyperplanes are positive on $s\mathfrak{C}_\circ$ if and only if $\alpha_2 = s_{\alpha_1}(\alpha_1+\alpha_2)$ hyperplanes are positive on $s_{\alpha_1}s\mathfrak{C}_\circ$. By Theorem 5.2.13, $\varepsilon(H_{\alpha_2,N}, s_{\alpha_1 s})$, $\varepsilon(H_{\alpha_2,N}, s)$, and $\varepsilon(H_{\alpha_2,N}, s_{\alpha_2 s})$ are all equal. We may rewrite the previous equation as:

$$\varepsilon(H_{\alpha_1+\alpha_2,N}, s) = \begin{cases} -\delta^N_{\alpha_1}\varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s) & \text{if } \alpha_1, \alpha_2 > 0 \text{ on } s\mathfrak{C}_\circ, \\ \delta^N_{\alpha_1}\varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s) & \text{otherwise.} \end{cases}$$

In the case where $|\gamma| = |\alpha|$, the root subsystem generated by α and γ is type A_2 as $(\gamma, \alpha) \neq 0$. We assign $\alpha_1 + \alpha_2 = \gamma$ and $\alpha_1 = \alpha$, without loss of generality. The first formula in the proposition now follows from our initial remarks in the proof of this proposition and Remark 5.2.14.

Case $|\alpha|^2 = 2|\gamma|^2$: In using the setup of Figure 1 in this case, note that the roots γ and α generate a root system of type B_2 and they must correspond to $\alpha_1 + \alpha_2$ and α_1, respectively.

Case $|\gamma|^2 = 2|\alpha|^2$: The roots γ and α generate a root system of type B_2 and they must correspond to $\alpha_1 + 2\alpha_2$ and α_2, respectively, in the setup of Figure 1. Observe that the formula we wish to prove for this case is of a different form than the formulas for the previous two cases. This is because in substituting $m = N$

into the equation

$$\begin{array}{ccccccc} - \ + & & + \ - & & - \ + & - \ + & \\ \varepsilon(C_2,C_3) & + & \varepsilon(C_6,C_7) & + & 2\varepsilon(C_0,C_1) & \varepsilon(\overline{r_1}C_1,\overline{r_1}C_2) & = 0 \\ H_{\alpha_1+2\alpha_2,m+n} & & H_{\alpha_1+\alpha_2,m+n} & & H_{\alpha_1,m} & H_{\alpha_2,2m+n} & \end{array}$$

for $n = 0$ from Theorem 5.2.13, $2m + n = 2N \Rightarrow \varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) = -\delta_\alpha^{2N} = -1$. □

Lemma 5.3.3. *For a positive root α and $\beta = s_{i_1} \cdots s_{i_k}\alpha$ where for $j = 1, \ldots, k$ we have $\mathrm{ht}(s_{i_j} \cdots s_{i_k}\alpha) > \mathrm{ht}(s_{i_{j+1}} \cdots s_{i_k}\alpha)$, $s_{i_1} \cdots s_{i_k}$ is a reduced expression.*

Proof. Suppose $s_{i_1} \cdots s_{i_k}$ is not reduced. By the deletion condition (see [4], Theorem 1.7), there are indices $j_1 < j_2$ such that $s_{i_1} \cdots s_{i_k} = s_{i_1} \cdots \hat{s_{i_{j_1}}} \cdots \hat{s_{i_{j_2}}} \cdots s_{i_k}$. It suffices to consider the case where $j_1 = 1$ and $j_2 = k$. Again, from the deletion condition, $\alpha_{i_1} = s_{i_2} \cdots s_{i_{k-1}}\alpha_{i_k}$. Now

$$(\alpha_{i_1}, \beta) = (s_{i_2} \cdots s_{i_{k-1}}\alpha_{i_k}, s_{i_1} \cdots s_{i_k}\alpha) = (s_{i_2} \cdots s_{i_{k-1}}\alpha_{i_k}, s_{i_2} \ldots s_{i_{k-1}}\alpha) = (\alpha_{i_k}, \alpha).$$

Since applying s_{i_1} to β decreases the height while applying s_{i_k} to α increases the height, therefore $(\alpha_{i_1}, \beta) > 0$ while $(\alpha_{i_k}, \alpha) < 0$, which gives us a contradiction. □

Theorem 5.3.4. *Let γ be a positive root that does not form a type G_2 root system with other roots in $\Delta(\mathfrak{g}, \mathfrak{h})$, and let $\gamma = s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k}$ be such that $\mathrm{ht}(s_{i_j} \cdots s_{i_{k-1}}\alpha_{i_k})$ decreases as j increases. Let $w_\gamma = s_{i_1} \cdots s_{i_k}$. If γ hyperplanes are positive on $s\mathfrak{C}_{\mathfrak{o}}$, then*

$$\begin{aligned} \varepsilon(H_{\gamma,N}, s) = \quad & (-1)^{N\#\{noncompact\ \alpha_{i_j}\,:\,|\alpha_{i_j}|\geq|\gamma|\}} \\ \times \quad & (-1)^{\#\{\beta\in\Delta(w_\gamma^{-1})\,:\,|\beta|=|\gamma|,\ \beta\neq\gamma,\ and\ \beta,s_\beta\gamma\in\Delta(s^{-1})\}} \\ \times \quad & (-1)^{\#\{\beta\in\Delta(w_\gamma^{-1})\,:\,|\beta|\neq|\gamma|\ and\ \beta,-s_\beta s_\gamma\beta\in\Delta(s^{-1})\}}. \end{aligned}$$

Proof. Note that $s_{i_1} \cdots s_{i_{k-1}}$ must be reduced, by Lemma 5.3.3. Combined with the fact that $\gamma = s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k} > 0$, we deduce that $s_{i_1} \cdots s_{i_{k-1}}s_{i_k}$ must also be reduced by Lemma 1.6 of [4]. By (5.2.1), $\Delta(w_\gamma^{-1}) = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \ldots, s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k}\}$.

Let $t_j = (s_{i_1} \cdot \cdots \cdot s_{i_j})^{-1}s \in W$ and let $\gamma_j = (s_{i_1} \cdot \cdots \cdot s_{i_j})^{-1}\gamma$ for $j = 0, 1, \ldots, k-1$. Note that $t_0 = s$ and $\gamma_0 = \gamma$. Also, $t_j = s_{i_j}t_{j-1}$ and $\gamma_j = s_{i_j}\gamma_{j-1}$. Observe that γ is positive on $s\mathfrak{C}_0$ if and only if γ_j is positive on $t_j\mathfrak{C}_0$. As $\mathrm{ht}(\gamma_j) > \mathrm{ht}(\gamma_{j+1})$, therefore $(\gamma_j, \alpha_{i_{j+1}}) > 0$. Thus by Proposition 5.3.2,
If $|\gamma_j| = |\alpha_{i_{j+1}}|$:

$$\varepsilon(H_{\gamma_j,N}, t_j) = \begin{cases} -\delta^N_{\alpha_{i_{j+1}}}\varepsilon(H_{\gamma_{j+1},N}, t_{j+1}) & \text{if } \alpha_{i_{j+1}} > 0 \text{ and } \ \gamma_{j+1} > 0 \text{ on } t_j\mathfrak{C}_0, \\ \delta^N_{\alpha_{i_{j+1}}}\varepsilon(H_{\gamma_{j+1},N}, t_{j+1}) & \text{otherwise.} \end{cases}$$

If $2|\gamma_j|^2 = |\alpha_{i_{j+1}}|^2$:

$$\varepsilon(H_{\gamma_j,N}, t_j) = \begin{cases} -\delta^N_{\alpha_{i_{j+1}}}\varepsilon(H_{\gamma_{j+1},N}, t_{j+1}) & \text{if } \alpha_{i_{j+1}} + 2\gamma_{j+1} = s_{\gamma_{j+1}}\alpha_{i_{j+1}} > 0 \\ & \text{and } \alpha_{i_{j+1}} > 0 \text{ on } t_j\mathfrak{C}_0, \\ \delta^N_{\alpha_{i_{j+1}}}\varepsilon(H_{\gamma_{j+1},N}, t_{j+1}) & \text{otherwise.} \end{cases}$$

If $|\gamma_j|^2 = 2|\alpha_{i_{j+1}}|^2$:

$$\varepsilon(H_{\gamma_j,N}, t_j) = \begin{cases} -\varepsilon(H_{\gamma_{j+1},N}, t_{j+1}) & \text{if } \alpha_{i_{j+1}} + \gamma_{j+1} = s_{\gamma_{j+1}}\alpha_{i_{j+1}} > 0 \\ & \text{and } \alpha_{i_{j+1}} > 0 \text{ on } t_j\mathfrak{C}_0, \\ \varepsilon(H_{\gamma_{j+1},N}, t_{j+1}) & \text{otherwise.} \end{cases}$$

We make the following observations:

(1) As the Weyl group preserves length, $|\gamma_j| = |\gamma|$.
(2) The Killing form is invariant under the action of the Weyl group. Therefore $\alpha_{i_{j+1}}$ and $\gamma_{j+1} = (s_{i_1} \cdots s_{i_{j+1}})^{-1}\gamma$ hyperplanes are positive on the Weyl chamber $t_j\mathfrak{C}_0 = (s_{i_1} \cdots s_{i_j})^{-1}s\mathfrak{C}_0$ if and only if $s_{i_1} \cdots s_{i_j}\alpha_{i_{j+1}}$ and $s_{i_1} \cdots s_{i_j}s_{i_{j+1}}s_{i_j} \cdots s_{i_1}\gamma$ are positive on $s\mathfrak{C}_0$.
(3) The reflection corresponding to $s_{i_1} \cdots s_{i_j}\alpha_{i_{j+1}}$ is $s_{i_1} \cdots s_{i_j}s_{i_{j+1}}s_{i_j} \cdots s_{i_1}$.
(4) Using Proposition 1.2 of [4], since $\gamma_{j+1} = (s_{i_1} \cdots s_{i_j}s_{i_{j+1}})^{-1}\gamma$, therefore

$$s_{\gamma_{j+1}} = s_{i_{j+1}} \cdots s_{i_1}s_\gamma s_{i_1} \cdots s_{i_{j+1}}.$$

(5) From our previous observation, we may conclude that

$$s_{i_1} \cdots s_{i_j}s_{\gamma_{j+1}}\alpha_{i_{j+1}} = -(s_{i_1} \cdots s_{i_j}s_{i_{j+1}}s_{i_j} \cdots s_{i_1})s_\gamma(s_{i_1} \cdots s_{i_j}\alpha_{i_{j+1}}).$$

From these observations, $k-1$ applications of our equations above and an application of Lemma 5.2.17 give the desired result. □

6. Extending to non-compact $\mathfrak{h}$

According to our results from Section 3, in some sense, the only reducibility hyperplanes we should worry about in computing the signature character are those corresponding to imaginary roots.

Let $\Delta_i(\mathfrak{g}, \mathfrak{h})$ be the imaginary roots in $\Delta(\mathfrak{g}, \mathfrak{h})$. We observe that it satisfies the axioms of a root system, hence it is a semisimple root subsystem of $\Delta(\mathfrak{g}, \mathfrak{h})$. Let $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$ be the intersection of $\Delta_i(\mathfrak{g}, \mathfrak{h})$ with $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Observe that if we replace W_a and W with the affine Weyl group and Weyl group corresponding to $\Delta_i(\mathfrak{g}, \mathfrak{h})$ in our arguments in sections 4 and 5, our arguments carry through to the non-compact Cartan subalgebra case. The remaining difficulty is to determine the set of simple roots corresponding to $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$ and to calculate ε for hyperplanes corresponding to those simple roots (recall δ_α).

We begin with the observation that as $\theta\Delta^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{g}, \mathfrak{h})$, if there are complex roots, then θ is a non-trivial automorphism of the corresponding Dynkin diagram. The only Dynkin diagrams which have a non-trivial automorphism are those of types A_n, D_n, and E_6. The vertices of the Dynkin diagram fixed by θ correspond to the imaginary simple roots, and the others to the complex simple roots.

Let $\Pi_i = \{\alpha \in \Pi \,|\, \alpha \text{ is imaginary}\}$ and $\Pi_{\mathbb{C}} = \{\alpha \in \Pi \,|\, \alpha \text{ is complex}\}$.

Proposition 6.1. *The set of simple roots corresponding to* $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$ *is*

$$\Pi^i := \Pi_i \cup \{\alpha^i \,|\, \alpha \in \Pi_{\mathbb{C}}\}$$

where α^i *is defined to be* $\alpha + \alpha_{i_1} + \cdots \alpha_{i_m} + \theta\alpha$ *if the segment of the Dynkin diagram from* α *to* $\theta\alpha$ *is* $(\alpha) - (\alpha_{i_1}) - \cdots - (\alpha_{i_m}) - (\theta\alpha)$.

Proof. It is clear that $\alpha \in \Pi_i$ is indecomposable as a sum of positive imaginary roots.

Note that $\alpha, \alpha_{i_1}, \ldots, \alpha_{i_m}, \theta\alpha$ all have the same length. Since θ flips the segment of the Dynkin diagram $(\alpha) - (\alpha_{i_1}) - \cdots - (\alpha_{i_m}) - (\theta\alpha)$, therefore $\theta\alpha_{i_k} = \alpha_{i_{m+1-k}}$. From knowledge of type A_n root systems, we see that α^i cannot be decomposed into the sum of two positive imaginary roots.

Listing the roots in root systems of types A_n, D_n, and E_6 and possible θ, we see that we have found all the roots in $\Delta_i(\mathfrak{g}, \mathfrak{h})$ which are indecomposable. □

Now we compute ε for $H_{\alpha^i,n}$ where $\alpha \in \Pi_{\mathbb{C}}$. We assume λ to be imaginary and $(\lambda+\rho, (\alpha^i)^\vee) = n$.

We may assume that $\mathfrak{g}$ is type A_m, $\alpha = \alpha_1$, and that the Dynkin diagram is $(\alpha_1) - (\alpha_2) - \cdots - (\alpha_m)$ so that $\alpha^i = \alpha_1 + \alpha_2 + \cdots + \alpha_m$.

Recall the definition of $X_{\alpha_j}, Y_{\alpha_j}$, and H_{α_j} from Lemma 5.2.17. The definition is unique up to multiplication of X_{α_j} by c and Y_{α_j} by c^{-1} for some non-zero scalar c. In the case where α_j is complex, $\bar{\mathfrak{g}}_{\alpha_j} = \mathfrak{g}_{-\theta\alpha_j}$ and $\bar{\mathfrak{g}}_{-\alpha_j} = \mathfrak{g}_{\theta\alpha_j}$, whence complex conjugation preserves $\mathfrak{g}_{\alpha_j} + \mathfrak{g}_{-\alpha_j} + \mathfrak{g}_{\theta\alpha_j} + \mathfrak{g}_{-\theta\alpha_j}$. We may choose c so that

$$-\bar{Y}_{\alpha_j} = X_{\theta\alpha_j} \qquad \text{and} \qquad -\bar{Y}_{\theta\alpha_j} = X_{\alpha_j} \qquad \text{when } j \neq \frac{m+1}{2}.$$

In order to compute ε, we use concepts introduced in [9]. Let $g_1, \cdots, g_m \in \mathfrak{g}$ be linearly independent. In [9], the authors give meaning to some monomials

$$g_{i_1}^{\gamma_1} \cdots g_{i_N}^{\gamma_N}, \tag{6.1}$$

where the γ_j are complex numbers, by associating them with appropriate elements of the universal enveloping algebra. Let $J_u = \{1 \leq j \leq m \,|\, i_j = u\}$ and let $\gamma^u = \sum_{j \in J_u} \gamma_j$. In the case where $\gamma_1, \ldots, \gamma_N$ are non-negative integers, by using appropriate commutation relations, we have

$$g_{i_1}^{\gamma_1} \cdots g_{i_N}^{\gamma_N} = \sum_{j_1,\ldots,j_m=0}^{\infty} P_{j_1\ldots j_m}(g_1,\ldots,g_m) g_1^{\gamma^1 - j_1} \cdots g_m^{\gamma^m - j_m} \tag{6.2}$$

for some elements $P_{j_1\ldots j_m}(g_1,\ldots,g_m)$ of $U([\mathfrak{g},\mathfrak{g}]) \subset U(\mathfrak{g})$. The $P_{j_1\ldots j_m}(g_1,\ldots,g_m)$ are polynomial in the γ_j, and thus we may extend the $P_{j_1\ldots j_m}$ to all possible γ_j and not just non-negative integral γ_j.

Definition 6.2. If the following conditions are satisfied:

(1) All γ^u are non-negative integers.
(2) If $j_u > \gamma^u$, then $P_{j_1\ldots j_m}(g_1,\ldots,g_m) = 0$.

then the monomial (6.1) is said to **make sense**. If the monomial makes sense, then the right side of equation (6.2) is an element of $U(\mathfrak{g})$ and we may say that (6.1) is equal to it.

Given $w = s_{i_N} \cdots s_{i_1} \in W$ and $\lambda \in \mathfrak{h}^*$, we define $\lambda_0, \lambda_1, \ldots, \lambda_N \in \mathfrak{h}^*$ by:

$$\lambda_j + \rho = s_{i_j} \cdots s_{i_1}(\lambda+\rho).$$

As $s_\beta\mu - \mu$ is a multiple of β for any $\beta, \mu \in \mathfrak{h}^*$, we may define the scalars γ_j for $1 \leq j \leq N$ so that $\lambda_j - \lambda_{j-1} = \gamma_j \alpha_{i_j}$.

Definition 6.3. Using the notation defined above and letting $Y_j = Y_{\alpha_j}$, we define

$$F(w;\lambda) = Y_{i_N}^{-\gamma_N} \cdots Y_{i_1}^{-\gamma_1}.$$

Lemma 6.4. *(Malikov-Feigin-Fuks,[9]) If $F(w;\lambda)$ makes sense, then $F(w;\lambda)v_\lambda$ is a singular vector of the Verma module $M(\lambda)$.*

Theorem 6.5. *(Malikov-Feigin-Fuks,[9]) If $(\lambda+\rho, \alpha^\vee) = n$ where α is a positive root and n is a positive integer, then $F(s_\alpha;\lambda)$ makes sense and $F(s_\alpha;\lambda)v_\lambda$ is a singular vector of the Verma module $M(\lambda)$ of weight $\lambda - n\alpha$.*

Remark 6.6. Here, we must make the observation that for a complex semisimple Lie algebra viewed as a Kac-Moody algebra, all roots are real roots.

We return to the setup where $\alpha^i = \alpha_1 + \cdots + \alpha_m$. We define

$$\begin{aligned} c_1 &= (\lambda+\rho, \alpha_1^\vee) \\ c_2 - c_1 &= (\lambda+\rho, \alpha_2^\vee) \\ &\vdots \\ c_{m-1} - c_{m-2} &= (\lambda+\rho, \alpha_{m-1}^\vee). \\ \text{Then} \quad n - c_{m-1} &= (\lambda+\rho, \alpha_m^\vee). \end{aligned}$$

If we use $s_1 s_2 \cdots s_m \cdots s_2 s_1$ as a reduced expression for s_{α^i}, then

$$F(s_{\alpha^i}; \lambda) = Y_1^{n-c_1} Y_2^{n-c_2} \cdots Y_{m-1}^{n-c_{m-1}} Y_m^n Y_{m-1}^{c_{m-1}} \cdots Y_2^{c_2} Y_1^{c_1}.$$

If we use $s_m s_{m-1} \cdots s_1 \cdots s_{m-1} s_m$ as a reduced expression for s_{α^i} instead, we get

$$F(s_{\alpha^i}; \lambda) = Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \cdots Y_2^{c_1} Y_1^n Y_2^{n-c_1} \cdots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}}.$$

Lemma 6.7. *In our setup above,*

$$F(s_1 s_2 \cdots s_m \cdots s_2 s_1; \lambda) = F(s_m s_{m-1} \cdots s_1 \cdots s_{m-1} s_m; \lambda).$$

Proof. We prove this by induction on m. If $m = 1$, this is clear. If $m = 2$:

$$\begin{aligned} Y_1^{n-c_1} Y_2^n Y_1^{c_1} &= Y_1^{n-c_1} \sum_j \binom{c_1}{j}\binom{n}{j} Y_1^{c_1-j} Y_2^{n-j} [Y_2, Y_1]^j \\ &= \sum_j \binom{c_1}{j}\binom{n}{j} Y_1^{n-j} Y_2^{n-j} \, [Y_2, Y_1]^j \\ &= \sum_j \binom{c_1}{j}\binom{n}{j} Y_1^{n-j} Y_2^{c_1-j} \, [Y_2, Y_1]^j \, Y_2^{n-c_1} \\ &= Y_2^{c_1} Y_1^n Y_2^{n-c_1} \end{aligned}$$

by Proposition 2.2 (2) of [9]. Now assume $m > 2$. Let $\lambda' = s_1(\lambda+\rho) - \rho$ and $\alpha' = \alpha_2 + \alpha_3 + \cdots + \alpha_m$. Then

$$\begin{aligned} c_2 &= (\lambda+\rho, \alpha_2^\vee) \\ c_3 - c_2 &= (\lambda+\rho, \alpha_3^\vee) \\ &\vdots \\ c_{m-1} - c_{m-2} &= (\lambda+\rho, \alpha_{m-1}^\vee) \\ n - c_{m-1} &= (\lambda+\rho, \alpha_m^\vee). \end{aligned}$$

Note that $(\lambda' + \rho, (\alpha')^\vee) = n$. Applying our induction hypothesis for $m-1$ to α' and λ',

$$\begin{aligned} F(s_{\alpha'}, \lambda') &= Y_2^{n-c_2} Y_3^{n-c_3} \cdots Y_{m-1}^{n-c_{m-1}} Y_m^n Y_{m-1}^{c_{m-1}} \cdots Y_3^{c_3} Y_2^{c_2} \\ &= Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \cdots Y_3^{c_2} Y_2^n Y_3^{n-c_2} \cdots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}}. \end{aligned}$$

Thus, using our knowledge of type A_m,

$$
\begin{aligned}
& F(s_1 \cdots s_m \cdots s_1;\lambda) \\
(6.3) \quad = \; & Y_1^{n-c_1} Y_2^{n-c_2} \cdots Y_{m-1}^{n-c_{m-1}} Y_m^n Y_{m-1}^{c_{m-1}} \cdots Y_2^{c_2} Y_1^{c_1} \\
= \; & Y_1^{n-c_1} Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \cdots Y_3^{c_2} Y_2^n Y_3^{n-c_2} \cdots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}} Y_1^{c_1} \\
= \; & Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \cdots Y_3^{c_2} (Y_1^{n-c_1} Y_2^n Y_1^{c_1}) Y_3^{n-c_2} \cdots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}} \\
(6.4) \quad = \; & Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \cdots Y_2^{c_1} Y_1^n Y_2^{n-c_1} \cdots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}} \\
= \; & F(s_m s_{m-1} \cdots s_1 \cdots s_{m-1} s_m;\lambda).
\end{aligned}
$$

□

We wish to compute $(F(s_{\alpha^i};\lambda))^* F(s_{\alpha^i};\lambda)$. Recall that $-\bar{Y}_j = X_{m+1-j}$ when $j \neq \frac{m+1}{2}$. Apply the expression (6.3) for $F(s_{\alpha^i};\lambda)$ to the left factor. We get

$$(F(s_{\alpha^i};\lambda))^* = X_m^{c_1} X_{m-1}^{c_2} \cdots X_2^{c_{m-1}} X_1^n X_2^{n-c_{m-1}} \cdots X_{m-1}^{n-c_2} X_m^{n-c_1}$$

if m is even. Note that $n - c_{m-j} = c_j$ from $\theta(\alpha_1 + \cdots + \alpha_j) = \alpha_m + \cdots + \alpha_{m-j+1}$. This give us

$$(F(s_{\alpha^i};\lambda))^* = X_m^{n-c_{m-1}} X_{m-1}^{n-c_{m-2}} \cdots X_2^{n-c_1} X_1^n X_2^{c_1} \cdots X_{m-1}^{c_{m-2}} X_m^{c_{m-1}}.$$

When m is odd, we have to multiply the expression by $\delta^n_{\alpha_{\frac{m+1}{2}}}$. Now apply (6.4) to the right factor. We see that the element of the universal enveloping algebra corresponding to $(F(s_{\alpha^i};\lambda))^* F(s_{\alpha^i};\lambda)$ obtained is precisely what you would get from applying equation (6.4) to *both* factors of $(F(s_{\alpha^i};\lambda))^* F(s_{\alpha^i};\lambda)$ in the case where all of the α_j are imaginary and compact for $j \neq \frac{m+1}{2}$ and $\alpha_{\frac{m+1}{2}}$ is left as it is. As $(\alpha_j, \alpha^i) = (\alpha_j, \alpha_1 + \cdots + \alpha_m) = 0$ for $2 \leq j \leq m-1$, it does no harm to assume that for each of those j, $(\lambda + \rho, \alpha_j^\vee)$ is a small positive number so that $(\lambda + \rho, \alpha_1)$ and $(\lambda + \rho, \alpha_m)$ are positive.

Let $s \in W$ be such that $\lambda + \rho \in s\mathfrak{C}_0$. Letting ϵ be the value of $\varepsilon(H_{\alpha_1 + \cdots + \alpha_m, n}, s)$ where all of the α_j, even $\alpha_{\frac{m+1}{2}}$, are imaginary and compact,

$$\varepsilon(H_{\alpha^i,n}, \cdot\,) = \begin{cases} \epsilon & \text{if } m \text{ is even,} \\ \delta^n_{\alpha_{\frac{m+1}{2}}} \epsilon & \text{if } m \text{ is odd.} \end{cases}$$

Calculating ϵ using Proposition 5.3.2, we obtain the following:

Proposition 6.8. *For $\alpha \in \Pi_{\mathbb{C}}$ for which the segment from α to $\theta\alpha$ in the Dynkin digram has m vertices,*

$$\varepsilon(H_{\alpha^i,n}, \cdot\,) = \begin{cases} (-1)^{m-1} = -1 & \textit{if } m \textit{ is even,} \\ (-1)^{m-1} \delta^n_{\alpha_{\frac{m+1}{2}}} = \delta^n_{\alpha_{\frac{m+1}{2}}} & \textit{if } m \textit{ is odd.} \end{cases}$$

Remark 6.9. Here, we observe that we could have arrived at the above answer using Theorem 5.3.4 without adjusting λ so that $(\lambda + \rho, \alpha_j^\vee) > 0$ for all j as follows:

(1) Set $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_m$, and choose s so that $\lambda + \rho \in s\mathfrak{C}_0$. Since $\gamma = s_1 s_2 \cdots s_{m-1} \alpha_m$, therefore $\Delta(w_\gamma^{-1}) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_m\}$. By θ-invariance of $(\cdot,\cdot)$, $(\lambda + \rho, (\alpha_{j+1} + \cdots + \alpha_m)^\vee) = (\lambda + \rho, (\alpha_1 + \cdots + \alpha_{m-j})^\vee)$.
(2) $s_{\alpha_1 + \cdots + \alpha_j} \gamma = \alpha_{j+1} + \cdots + \alpha_m$.
(3) $s_{s_{\alpha_1 + \cdots \alpha_j} \gamma} \gamma = \alpha_1 + \cdots + \alpha_j$.

(4) Thus $\beta \in \Delta(w_\gamma^{-1})$ such that $|\beta| = |\gamma|$, $\beta \neq \gamma$, and $\beta, s_\beta\gamma \in \Delta(s^{-1})$ occur in pairs $\alpha_1 + \cdots + \alpha_j$ and $\alpha_1 + \cdots + \alpha_{m-j}$, except for the root $\alpha_1 + \cdots + \alpha_{m/2}$ which is paired with itself in the case where m is even. In this case,

$$(\lambda + \rho, (\alpha_1 + \cdots + \alpha_{m/2})^\vee) = \frac{n}{2} > 0.$$

(5) Use the formula from Theorem 5.3.4.

Now $\mathfrak{g}_{\alpha^i} = \mathbb{C}\,[X_1, [X_2, [\cdots [X_{m-1}, X_m]\,]\cdots]\,]$ is θ-stable as α^i is imaginary. We have

$$\begin{aligned} \theta\,[X_1, [X_2, [\cdots [X_{m-1}, X_m]\,]\cdots]\,] &= [\theta X_1, \theta\,[X_2, [\cdots [X_{m-1}, X_m]\,]\cdots]\,] \\ &\vdots \\ &= [\theta X_1, [\theta X_2, [\cdots [\theta X_{m-1}, \theta X_m]\,]\cdots]\,]\,. \end{aligned}$$

As $\theta\alpha_j = \alpha_{m+1-j}$, we may arrange for $\theta X_j = X_{m+1-j}$ if $j \neq \frac{m+1}{2}$, and for $\theta X_{\frac{m+1}{2}} = \delta_{\alpha_{\frac{m+1}{2}}} X_{\frac{m+1}{2}}$. Using the Jacobi identity, induction on m, and type A_m commutation relations, we may show that

$$[X_m, [X_{m-1}, [\cdots [X_2, X_1]\,]\cdots]\,] = (-1)^{m-1}\,[X_1, [X_2, [\cdots [X_{m-1}, X_m]\,]\cdots]\,]\,.$$

It follows that

$$\theta\,[X_1, [X_2, [\cdots [X_{m-1}, X_m]\,]\cdots]\,] = \begin{cases} -\,[X_1, [X_2, [\cdots [X_{m-1}, X_m]\,]\cdots]\,] \\ \qquad \text{if m is even,} \\ \delta_{\alpha_{\frac{m+1}{2}}}\,[X_1, [X_2, [\cdots [X_{m-1}, X_m]\,]\cdots]\,] \\ \qquad \text{if } m \text{ is odd,} \end{cases}$$

whence:

Lemma 6.10. *Let α and m be as defined in the previous Proposition. α^i is compact if m is odd and $\alpha_{\frac{m+1}{2}}$ is compact, and noncompact otherwise.*

Theorem 6.11. *Let $\alpha \in \Pi_{\mathbb{C}}$ and let m be as defined in the previous proposition.*

$$\varepsilon(H_{\alpha^i,n}) = \begin{cases} -1 & \text{if } \alpha^i \text{ is noncompact and } m \text{ is even,} \\ (-1)^n & \text{if } \alpha^i \text{ is noncompact and } m \text{ is odd,} \\ 1 & \text{if } \alpha^i \text{ is compact.} \end{cases}$$

We may adjust Theorem 5.3.4 to obtain an analogous formula for the noncompact Cartan setting. Note that in the case where α^i is noncompact and m is even, none of the roots in Π are imaginary and the simple roots corresponding to $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$ are orthogonal to one another. In this case, ε is always -1 and $\Delta_i(\mathfrak{g}, \mathfrak{h})$ is a disjoint union of copies of A_1. In the remaining cases, Theorem 5.3.4 holds if we replace the ambient root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ with $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$.

In summary, we have the following formula:

Theorem 6.12. *Subscripts or superscripts i will refer to objects associated with $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$. Everything within this theorem (simple roots, reducibility hyperplanes, reduced expressions) has this association.*

The action of W_a^i, the affine Weyl group corresponding to $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$, partitions $i\mathfrak{h}_0^*$ into alcoves in which the signature character of the Shapovalov form is constant.

Choose the fundamental alcove A_0 *of* W_a^i *and the fundamental chamber* $\mathfrak{C}_0$ *of* W_i *to contain* ρ_i*, both of which are contained in the Wallach region*

$$\left(\bigcap_{\alpha\in\Pi^i} H^-_{\alpha,1}\right)\bigcap H^-_{\widetilde{\alpha}_i,1}.$$

a) *If* λ *is imaginary and* $\lambda+\rho$ *lies in the Wallach region, then*

$$\begin{aligned} ch_s M(\lambda)|_{\mathfrak{a}_0} &= e^{\lambda|_{\mathfrak{a}_0}} \qquad \textit{and} \\ ch_s M(\lambda)|_{\mathfrak{t}_0} &= \frac{e^{\lambda|_{\mathfrak{t}_0}}}{\prod_{\alpha\in\Delta^+(\mathfrak{p},\mathfrak{t})}(1-e^{-\alpha})\prod_{\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})}(1+e^{-\alpha})}. \end{aligned}$$

Let the constants c_μ *and* c_μ^A *for imaginary* $\mu\in\Lambda_r^+$ *be such that*

$$R(\lambda) := \sum_{\substack{\mu\in\Lambda_r^+ \\ \mu\ imaginary}} c_\mu e^{\lambda-\mu}$$

is the signature character of the Shapovalov form $\langle\cdot,\cdot\rangle_\lambda$ *when* $\lambda+\rho$ *lies in the Wallach region, and*

$$R^A(\lambda) := \sum_{\substack{\mu\in\Lambda_r^+ \\ \mu\ imaginary}} c_\mu^A e^{\lambda-\mu}$$

is the signature character of $\langle\cdot,\cdot\rangle_\lambda$ *when* $\lambda+\rho$ *lies in the alcove* A.

b) *For* $w\in W_a^i$*, let* $wA_0 = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \cdots \xrightarrow{r_\ell} C_\ell = \widetilde{w}A_0$ *be a (not necessarily reduced) path from* wA_0 *to* $\widetilde{w}A_0$*. Then*

$$\begin{aligned} R^{wA_0}(\lambda) &= \sum_{I=\{i_1<\cdots<i_k\}\subset\{1,\ldots,\ell\}} \varepsilon(I)2^{|I|}R^{\overline{r_{i_1}}\cdots\overline{r_{i_k}}\widetilde{w}A_0}\left(\overline{r_{i_1}}\,\overline{r_{i_2}}\cdots\overline{r_{i_k}}r_{i_k}r_{i_{k-1}}\cdots r_{i_1}\lambda\right) \\ &= \sum_{I=\{i_1<\cdots<i_k\}\subset\{1,\ldots,\ell\}} \varepsilon(I)2^{|I|}R\left(\overline{r_{i_1}}\,\overline{r_{i_2}}\cdots\overline{r_{i_k}}r_{i_k}r_{i_{k-1}}\cdots r_{i_1}\lambda\right) \end{aligned}$$

where $\varepsilon(\emptyset)=1$ *and*

$$\varepsilon(I)=\varepsilon(C_{i_1-1},C_{i_1})\varepsilon(\overline{r_{i_1}}C_{i_2-1},\overline{r_{i_1}}C_{i_2})\cdots\varepsilon(\overline{r_{i_1}}\cdots\overline{r_{i_{k-1}}}C_{i_k-1},\overline{r_{i_1}}\cdots\overline{r_{i_{k-1}}}C_{i_k}).$$

The $\varepsilon(C,C')$*, where* C *and* C' *are adjacent alcoves defined by the action of* W_a^i *separated by some hyperplane* $H_{\gamma,N}$ *where* $\gamma\in\Delta_i^+(\mathfrak{g},\mathfrak{h})$ *and* $N\in\mathbb{Z}$*, take the values* $0,\pm1$ *as follows:*

- $\varepsilon(C,C')=-\varepsilon(C',C)$
- $\varepsilon(C,C')=0$ *if* $N\le 0$
- $\varepsilon(C,C')=\varepsilon(H_{\gamma,N},s)$ *if* $N>0$*, where* $C\subset s\mathfrak{C}_0$.

Choose $\gamma=s_{i_1}\cdots s_{i_{k-1}}\alpha_{i_k}$ *where the* $\alpha_{i_j}\in\Pi^i$ *so that* $\mathrm{ht}_i(s_{i_j}\cdots s_{i_{k-1}}\alpha_{i_k})$ *decreases as* j *increases. (Proposition 6.1 provides a precise description of* Π^i*. Recall Definition 5.2.15.) Let* $w_\gamma = s_{i_1}\cdots s_{i_k}$*. If* $N>0$ *then:*

- *If* θ *does not fix any element of* Π*, then:*

$$\varepsilon(H_{\gamma,N},s)=-1.$$

- *If θ fixes some element of Π and γ does not form a type G_2 root system with other roots in $\Delta_i(\mathfrak{g},\mathfrak{h})$, then:*

$$\begin{aligned}\varepsilon(H_{\gamma,N},s) = \quad & (-1)^{N\#\{noncompact\ \alpha_{i_j}\,:\,|\alpha_{i_j}|\geq|\gamma|\}} \\ \times \quad & (-1)^{\#\{\beta\in\Delta(w_\gamma^{-1})\,:\,|\beta|=|\gamma|,\ \beta\neq\gamma,\ and\ \beta,s_\beta\gamma\in\Delta(s^{-1})\}} \\ \times \quad & (-1)^{\#\{\beta\in\Delta(w_\gamma^{-1})\,:\,|\beta|\neq|\gamma|\ and\ \beta,-s_\beta s_\gamma\beta\in\Delta(s^{-1})\}}.\end{aligned}$$

- *If γ, along with other roots in $\Delta_i(\mathfrak{g},\mathfrak{h})$, forms a type G_2 root system, then the value of $\varepsilon(H_{\gamma,N},s)$ can be found in Theorem 5.2.18.*

7. Historical context

In this section, we will expand on the historical context of the problem solved in this paper.

Let $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{h}\subset\mathfrak{l}$ be a Cartan subalgebra. Let L be the normalizer of $\mathfrak{q}$ in G. We define $\rho(\mathfrak{u})$ to be $\frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{u},\mathfrak{h})}\alpha$. We make these definitions in the context of our setup from previous sections.

Recall the definition of the production functor, $\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}:\mathcal{C}(\mathfrak{l},L\cap K)\to\mathcal{C}(\mathfrak{g},L\cap K)$:

$$\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}V=\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}),V)_{L\cap K-\mathrm{finite}}.$$

We define $\mathscr{R}^i:\mathcal{C}(\mathfrak{l},L\cap K)\to\mathcal{C}(\mathfrak{g},K)$ by

$$\mathscr{R}^iV=\Gamma^i\,\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(V\otimes\wedge^{\mathrm{top}}\mathfrak{u}).$$

In [10], Vogan conjectured:

Conjecture 7.1. *For an irreducible, unitarizable $(\mathfrak{l},L\cap K)$-module V with infinitesimal character $\lambda\in\mathfrak{h}^*$, if*

$$\mathrm{Re}(\alpha,\lambda-\rho(\mathfrak{u}))\geq 0\qquad\forall\,\alpha\in\Delta(\mathfrak{u},\mathfrak{h})$$

$$and\ if\ m=\dim\mathfrak{u}\cap\mathfrak{k},$$

then $\mathscr{R}^mV$ is also unitarizable.

In [11], Vogan gave a proof of this conjecture, the fundamental idea of which was to couple the theory of minimal K-types with knowledge of a large family of well understood unitary representations which were studied by Harish-Chandra: the tempered unitary representations. The following will describe some work leading up to and inspired by this result.

Important to the study of unitarizability is a duality theorem for cohomological induction functors. In his 1978 IAS lectures, Zuckerman proved what was equivalent to the following duality theorem for the right derived functors Γ^i and Γ^{2m-i}:

Theorem 7.2. *For $0\leq i\leq 2m$, $X\mapsto\Gamma^iX$ and $X\mapsto(\Gamma^{2m-i}(X^h))^h$ are naturally equivalent on the subcategory of admissible $(\mathfrak{k},\mathfrak{l}\cap\mathfrak{k})$-modules.*

In [2], Enright and Wallach show that since the forgetful functor is additive, covariant, exact, takes injectives to injectives, and commutes with Γ, one can prove the following (stronger) duality theorem (see Theorem 4.3 in [2]):

Theorem 7.3. *If X is in fact an admissible $(\mathfrak{g}, \mathfrak{k}\cap\mathfrak{l})$ − module, then the $\mathfrak{k}$-module isomorphism*

$$\Gamma^i \mathscr{F} X \simeq (\Gamma^{2m-i}(\mathscr{F} X^h))^h,$$

where $\mathscr{F}$ denotes the forgetful functor, induces a $\mathfrak{g}$-module isomorphism

$$\Gamma^i X \simeq (\Gamma^{2m-i}(X^h))^h.$$

We will discuss the implementation of the $\mathfrak{k}$-module isomorphism.

For every $\gamma \in \hat{\mathfrak{k}}$ with corresponding representation F_γ, there is a positive definite $\mathfrak{k}$-invariant Hermitian form $\langle\cdot,\cdot\rangle_\gamma$ on F_γ and its dual pairing on F_γ^*. The pairing of $\Gamma^i X$ with $\Gamma^{2m-i}(X^h)$ uses the natural isomorphism

$$\Gamma^i X \simeq \bigoplus_{\gamma\in\hat{\mathfrak{k}}} H^i(\mathfrak{k}, \mathfrak{k}\cap\mathfrak{l}; X\otimes F_\gamma^*)\otimes F_\gamma$$

as $\mathfrak{k}$-modules, where the action on the right is on the last term.

We may pair $H^i(\mathfrak{k}, \mathfrak{k}\cap\mathfrak{l}; X\otimes F_\gamma^*)$ with $H^{2m-i}(\mathfrak{k}, \mathfrak{k}\cap\mathfrak{l}; X^h\otimes F_\gamma^*)$ by pairing spaces C^i and C^{2m-i} in the cochain complexes using the identification

$$C^i(X\otimes F_\gamma^*) = \mathrm{Hom}_{\mathfrak{k}\cap\mathfrak{l}}(\wedge^i(\mathfrak{k}/\mathfrak{k}\cap\mathfrak{l}), X\otimes F_\gamma^*) \simeq \left[\wedge^i(\mathfrak{k}/\mathfrak{k}\cap\mathfrak{l})\right]^* \otimes X\otimes F_\gamma^*.$$

Using $\langle\cdot,\cdot\rangle$, a Hermitian pairing of $\wedge^i(\mathfrak{k}/\mathfrak{k}\cap\mathfrak{l})$ and $\wedge^{2m-i}(\mathfrak{k}/\mathfrak{k}\cap\mathfrak{l})$ defined in Section 1 of [12], we define the following $\mathfrak{k}$-invariant pairing of $C^i(\mathfrak{k}, \mathfrak{k}\cap\mathfrak{l}; X\otimes F_\gamma^*)$ and $C^{2m-i}(\mathfrak{k}, \mathfrak{k}\cap\mathfrak{l}; X^h\otimes F_\gamma^*)$:

$$\langle \omega_1\otimes v\otimes f_1^*, \omega_2\otimes v'\otimes f_2^*\rangle = \langle\omega_1,\omega_2\rangle\,\langle v, v'\rangle\,\langle f_1^*, f_2^*\rangle_\gamma\,. \tag{7.1}$$

This produces a $\mathfrak{k}$-invariant pairing at the level of cohomology by the standard proof of Poincaré duality. From the tensor product pairing of $H^i(\mathfrak{k}, \mathfrak{k}\cap\mathfrak{l}; X\otimes F_\gamma^*)\otimes F_\gamma$ and $H^{2m-i}(\mathfrak{k}, \mathfrak{k}\cap\mathfrak{l}; X^h\otimes F_\gamma^*)\otimes F_\gamma$, we obtain a $\mathfrak{k}$-invariant pairing of $\Gamma^i(X)$ and $\Gamma^{2m-i}(X^h)$ which induces the $\mathfrak{g}$-invariant pairing of the duality theorem of Enright and Wallach.

In the case where X has an invariant Hermitian form and $i = m$, this implies that there is a $\mathfrak{g}$-invariant isomorphism $\Gamma^m X \simeq (\Gamma^m X)^h$, and so $\Gamma^m X$ has a $\mathfrak{g}$-invariant Hermitian form.

Define $\mathscr{L}^i : \mathcal{C}(\mathfrak{l}, L\cap K) \to \mathcal{C}(\mathfrak{g}, K)$ by

$$\mathscr{L}^i V = \Gamma^i \operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V\otimes\wedge^{\mathrm{top}}\mathfrak{u}).$$

As $\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V)^h \simeq \operatorname{pro}_{\mathfrak{q}}^{g}(V^h)$, if V is an $(\mathfrak{l}, L\cap K)$-module, then $(\mathscr{L}^i V)^h \simeq \mathscr{R}^{2m-i}(V^h)$. When studying $\mathscr{R}^i V$ where V admits an invariant Hermitian form, we are essentially looking at the duality theorem in the case where X has an invariant Hermitian form and is the generalized Verma module $\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V)$.

The general outline of the proof of Conjecture 7.1 in [11] is as follows:

- Any representation which admits an invariant Hermitian form may be obtained from a tempered unitary representation via analytic continuation through representations admitting invariant Hermitian forms.
- Jantzen filtration arguments lead us to conclude that for a $(\mathfrak{g}, K)$-module of finite length admitting a non-degenerate invariant Hermitian form $\langle\cdot,\cdot\rangle$, there exists some finite collection of tempered irreducible $(\mathfrak{g}, K)$-modules

$Z_1, \dots, Z_p$ of formal K-character $\Theta(Z_i)$ and integers $r_1^+, \dots, r_p^+$, $r_1^-, \dots, r_p^-$ such that the signature of $\langle \cdot, \cdot \rangle$ is $\left(\sum_{i=1}^{p} r_i^+ \Theta(Z_i), \sum_{j=1}^{p} r_j^- + \Theta(Z_j) \right)$.

- The tempered characters in this expression for the signature in the case where the $(\mathfrak{g}, K)$-module is $\mathscr{R}V$ must have lowest K-types in the bottom layer of $\mathscr{R}V$. It follows that unitarizability of $\mathscr{R}V$ is equivalent to the form being definite on the bottom layer.
- Using the ideas of Enright and Wallach (including the construction (7.1) of the invariant Hermitian pairing) described above, one may compare the signatures of the invariant Hermitian forms for V and $\mathscr{R}V$ on the bottom layer of K-types.

Wallach gives an alternate proof to Vogan's conjecture in [12] that does not use the complicated machinery of K-types and tempered unitary representations.

For $\lambda \in \mathfrak{z}(\mathfrak{l})$ and V an admissible, finitely generated $(\mathfrak{l}, L \cap K)$-module, Wallach defines $\mathbb{C}_\lambda$ to be the one-dimensional $\mathfrak{l}$-module corresponding to λ and he defines V_λ to be $\mathbb{C}_\lambda \otimes V$, which may be extended to a $\mathfrak{q}$-module by allowing $\mathfrak{u}$ to act trivially. $M(\mathfrak{q}, \lambda, V)$ denotes the generalized Verma module $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{q})} V_\lambda$. From an irreducible V which admits an $\mathfrak{l}$-invariant Hermitian form $\langle \cdot, \cdot \rangle$, one constructs an invariant Hermitian form on $M(\mathfrak{q}, \lambda, V)$ analogous to the Shapovalov form that we constructed.

Wallach defines $(V, \lambda, \langle\rangle)$ to be *well placed* if for some $\xi \in (\mathfrak{z}(\mathfrak{l}) \cap \mathfrak{k})^*$ that is purely imaginary valued on $\mathfrak{z}(\mathfrak{l}) \cap \mathfrak{k}_0$, $(\xi, \alpha) < 0$ for $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$ and $M(\mathfrak{q}, \lambda + t\xi, V)$ is irreducible for all $t \geq 0$. (Here, we note the connection between the definitions of well placed and Wallach region.)

In the case that $(V, \lambda, \langle\rangle)$ is well placed,

$$ch_s(M(\mathfrak{q}, \lambda + t\xi, V)) = e^{t\xi} ch_s(M(\mathfrak{q}, \lambda, V)).$$

As discussed previously in Section 2, an asymptotic argument as t goes to infinity gives us a formula for the signature character of $M(\mathfrak{q}, \lambda, V, \langle\rangle)$ in terms of the signature character of $(V, \langle\rangle)$. A similar argument gives us a formula for the character of the generalized Verma module in terms of the character of V.

As in Vogan's proof, the construction (7.1) of an invariant Hermitian form on $\Gamma^m X$ is an instrumental component in discussing unitarizability. From the construction, it is clear that

$$ch_s(\Gamma^i X \oplus \Gamma^{2m-i} X) = 0 \tag{7.2}$$

for $i \neq m$, whence

$$ch_s \Gamma^{\cdot} X = ch_s \Gamma^m X. \tag{7.3}$$

Furthermore, the signature character of $\Gamma^m X$ can be expressed in terms of signatures of the forms on the $H^m(\mathfrak{g}, \mathfrak{k}; X \otimes F_\gamma^*)$ and in terms of characters chF_γ:

$$ch_s \Gamma^m X = \sum_{\gamma \in \hat{\mathfrak{k}}} \mathrm{sgn}(H^m(\mathfrak{g}, \mathfrak{k}; X \otimes F_\gamma^*))\, chF_\gamma \tag{7.4}$$

where $\mathrm{sgn}(Y)$ is $p - q$ if (p, q) is the signature on Y.

For $X = M(\mathfrak{q}, \lambda, V, \langle\rangle)$, Wallach uses the above equation to calculate the coefficient of chF_γ in $(-1)^m ch_s \Gamma^m(M(\mathfrak{q}, \lambda, V, \langle\rangle))$ in the case where $(V, \lambda, \langle\rangle)$ is well placed and $\langle\rangle$ is positive definite . He then calculates the coefficient of chF_γ in $\sum_{i=0}^{2m}(-1)^i ch\,\Gamma^i(M(\mathfrak{q},\lambda,V,\langle\rangle))$. The first expression obtained involves the signature character $ch_sM(\mathfrak{q},\lambda,V,\langle\rangle)$ while the second involves the character $chM(\mathfrak{q},\lambda,V,\langle\rangle)$. Manipulating these expressions using the formulas calculated using asymptotic arguments mentioned above, he shows that the two expressions are in fact equal. It follows that

$$(-1)^m ch_s\Gamma^m(M(\mathfrak{q},\lambda,V,\langle\rangle) = \sum_{i=0}^{2m}(-1)^i ch\,\Gamma^i(M(\mathfrak{q},\lambda,V,\langle\rangle)).$$

Since $\Gamma^i M(\mathfrak{q},\lambda,V) = 0$ for $i \neq m$ and $(V,\lambda,\langle\rangle)$ well placed, therefore

$$ch_s\Gamma^m(M(\mathfrak{q},\lambda,V,\langle\rangle)) = ch\,\Gamma^m(M(\mathfrak{q},\lambda,V,\langle\rangle)),$$

whence $\Gamma^m(M(\mathfrak{q},\lambda,V,\langle\rangle))$ is unitarizable or zero.

Wallach shows that for $(V,\langle\rangle)$ satisfying the conditions of Conjecture 7.1, with necessary adjustments to accommodate usage of $\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}$ instead of $\operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}}$, $(V,0,\langle\rangle)$ is well placed. It follows that $\Gamma^m \operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}} V$ is unitarizable, proving the conjecture.

8. Final Remarks

As discussed in the introduction, the motivation behind the problem solved in this paper is the utilization of a formula for the Shapovalov form on an arbitrary generalized Verma module when it exists in the study of unitarizability of cohomologically induced modules. We observe that many formulas such as (7.2), (7.3), and (7.4) still hold outside of the Wallach region, which is encouraging.

In order to extend the approach of this paper to arrive at a formula for the signature of the Shapovalov form on an arbitrary *generalized* Verma module, we must begin by determining when generalized Verma modules are irreducible. This is regarded to be a difficult open problem in the most general case.

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