RESEARCH STATEMENT: WAI LING YEE

My research interests are in representation theory: specifically, the unitarizability of representations of Lie groups, Category $\mathcal{O}$, and combinatorial aspects of representation theory. In the following, I will describe past and current research projects and future plans.

1. Size of Characters of Representations of Compact Lie Groups and Singularity of Orbital Measures

[1] Hare, Kathryn E., Wilson, David C., and Yee, Wai Ling, *Pointwise Estimates of the Size of Characters of Compact Lie Groups*, Journal of the Australian Mathematics Society Series A **69** (2000), no. 1, 61–84.

[2] Hare, Kathryn E. and Yee, Wai Ling, *The Singularity of Orbital Measures on Compact Lie Groups*, Revista Math. Iberoamericana **20** (2004), no. 2, 517–530.

Broadly, this research could be described as the usage of representation theory as a tool in understanding the smoothing behaviour of convolutions on orbital measures. The main theorem in [1] gives sharp pointwise estimates for the size of characters of representations of compact, connected, classical, simple Lie groups expressed as a function of the type and rank of the Lie group:

Theorem 1.1. *Let G be a compact, connected, classical, simple Lie group. For every g not in the centre of G there is a constant $c(g)$ such that*

$$\left|\frac{Tr\,\lambda}{deg\,\lambda}\right| \leq c(g)(deg\,\lambda)^{-s}$$

for all representations λ if and only if

$$s \leq \begin{cases} 1/(n-1) & \text{if } G \text{ is type } A_{n-1} or D_n; \\ 1/(2n-1) & \text{if } G \text{ is type } B_n; \\ 2/(2n-1) & \text{if } G \text{ is type } C_n, n \neq 3; \\ 1/3 & \text{if } G \text{ is type } C_3. \end{cases}$$

Sufficiency of the bounds is a consequence of the structure of associated root systems and their root subsystems. The prime motivation was to use these estimates to study the singularity of central, continuous measures.

It is well known that there are many continuous, singular measures on any compact abelian group, all of whose convolutions remain singular to L^1. In contrast, Ragozin in [Rag72] proved the striking fact that if G was a compact, connected, simple Lie group and μ was any central, continuous measure on G, then $\mu^{\dim G} \in L^1(G)$. We used the optimal approximations of Theorem 1.1 in conjunction with the Peter-Weyl theorem to determine the *minimal* integer k such that any continuous orbital measure convolved with itself k times belongs to L^2. As our bound was $O(n)$ where $n = \operatorname{Rank} G$ whereas $\dim G$ is $O(n^2)$, this was a significant improvement upon previously known results. Subsequently, in *The Singularity of Orbital Measures on Compact Lie Groups*, we generalized the result for continuous orbital measures by identifying the minimal $k \in \mathbb{R}$ for which the k^{th} power of the Fourier transform of any continuous orbital measure belongs to ℓ^2.

In recognition of this work on the size of characters and the singularity of orbital measures, I was awarded the Honourable Mention for the 2000 AMS-MAA-SIAM Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student.

2. The Unitary Dual Problem

Classically, the fundamental concept of Fourier analysis was that an essentially arbitrary function could be expanded as a linear combination of exponentials. The more recent development of ideas in group theory has illuminated the dependence of results in Fourier analysis on group-theoretic concepts, resulting in the movement from Euclidean spaces to the more general setting of locally compact groups. Results such as the Peter-Weyl Theorem give us a means of decomposing function spaces of a compact group G into an orthogonal direct sum of subspaces expressed in terms of characters of irreducible unitary representations of G. Equipped with this decomposition and knowledge of these simpler subspaces, one can reformulate problems in analysis in more tractable settings. In fact, Fourier analysis on groups is just one incarnation of I.M. Gelfand's broad programme in abstract harmonic analysis, introduced in the 1930s, which would permit the transfer of difficult problems in areas as distinct from analysis as topology to more tractable problems in algebra.

An unresolved component in Gelfand's programme is the classification of the irreducible unitary representations of a group, known as the unitary dual problem. Computing the unitary dual is the main goal of the NSF-funded Atlas of Lie Groups and Representations research group of which I am a member. In the case of a real reductive Lie group, the problem is equivalent to identifying all irreducible Harish-Chandra modules which admit a postive definite invariant Hermitian form. As Harish-Chandra modules may be constructed via an algebraic method introduced by Zuckerman in 1978 known as cohomological induction, it is of interest to study signatures of invariant Hermitian forms on cohomologically induced modules and to understand how positivity can fail. The following theorem, due to Vogan, is an important result in this direction:

Theorem 2.1. [Vog84] *Let G be a real reductive Lie group and K a maximal compact subgroup of G with corresponding Cartan involution θ. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the corresponding Lie algebras and $\mathfrak{g}$ and $\mathfrak{k}$ their complexifications. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. Let L be the normalizer of $\mathfrak{q}$ in G.*

For an irreducible unitarizable $(\mathfrak{l}, L \cap K)$-module V with infinitesimal character $\lambda \in \mathfrak{h}^*$, if

$$Re(\alpha, \lambda - \rho(\mathfrak{u})) \geq 0 \qquad \forall \alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$$
$$\text{and if } m = \dim \mathfrak{u} \cap \mathfrak{k},$$

then $\mathscr{R}^m V = \Gamma^m \mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(V \otimes \wedge^{top}\mathfrak{u})$ is also unitarizable.

The proof of this theorem goes deeply into the representation theory of real reductive groups, the fundamental idea of which is to couple the theory of minimal K-types with knowledge of a large family of well understood unitary representations which were studied by Harish-Chandra: the tempered unitary representations.

We obtain the cohomological induction functors by composing the induction functor with the Zuckerman functors Γ^i. The intermediate module in cohomological induction is a generalized Verma module which carries an invariant Hermitian form, the Shapovalov form, if the initial $(\mathfrak{l}, L \cap K)$-module carries an invariant Hermitian form. In [Wal84], Wallach gave a more elementary proof of Theorem 2.1 by computing the signature character [1] of the Shapovalov form on Hermitian generalized Verma modules with corresponding restrictions to the infinitesimal character of the module to which the cohomological inductions functors are applied. Using his formula for the signature character for this intermediate module in the process of cohomological induction, he is able to compute the signature character for the form on the cohomologically induced module. With a formula for the signature of the Shapovalov form *regardless of the value of the infinitesimal character*, it may be possible to compute signatures of cohomologically induced modules constructed from representations of *arbitrary* infinitesimal character. This is discussed in Sections 7 and 8 of [3]. Having such a tool at one's disposal would be a significant step in solving the problem of classifying the Harish-Chandra modules which admit a positive definite invariant Hermitian form.

I hope to achieve the objective of computing signature characters of invariant Hermitian forms on cohomologically induced modules through the following steps:

(1) computing the signature character of the Shapovalov form on any irreducible Verma module
(2) computing the signature character of the invariant Hermitian form induced by the Shapovalov form on any irreducible highest weight module
(3) determining reducibility criteria for generalized Verma modules
(4) determining the composition factors of reducible generalized Verma modules and their multiplicities
(5) computing the signature character of the Shapovalov form on any generalized Verma module
(6) computing the signature character of the invariant Hermitian form on any cohomologically induced $(\mathfrak{g}, K)$-module in relation to the signature of the invariant Hermitian form on the starting module

[1] Thinking of the signature as a generalization of the dimension, one may generalize the notion of a character to a signature character.

We now discuss these steps in detail.

2.1. The Signature of the Shapovalov Form on Irreducible Verma Modules.

[3] Yee, Wai Ling, *The Signature of the Shapovalov Form on Irreducible Verma Modules*, Representation Theory **9** (2005), 638–677.

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, θ a Cartan involution of $\mathfrak{g}_0$, and drop the subscript 0 to denote complexification. A Hermitian form $\langle \cdot, \cdot \rangle$ on a $\mathfrak{g}$-module V is *invariant* if it satisfies

$$\langle Xv, w\rangle + \langle v, \bar{X}w\rangle = 0$$

for every $X \in \mathfrak{g}$ and every $v, w \in V$, where $\bar{X}$ denotes the complex conjugate of X with respect to the real form $\mathfrak{g}_0$. Such forms on a Verma module $M(\lambda)$ are unique up to a real scalar when they exist. They exist when $\lambda \in i\mathfrak{h}_0^*$, where $\mathfrak{h}_0$ is a θ-stable Cartan subalgebra and $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is also θ-stable. This is equivalent to $\mathfrak{h}$ being maximally compact so that there are no real roots. The Shapovalov form is the invariant Hermitian form for which $\langle v_\lambda, v_\lambda\rangle = 1$, where v_λ is the canonical generator. We denote the Shapovalov form on $M(\lambda)$ by $\langle \cdot, \cdot \rangle_\lambda$.

The radical of the Shapovalov form is the unique maximal submodule of $M(\lambda)$, hence the form is non-degenerate precisely for the irreducible Verma modules.

By invariance, the Shapovalov form pairs the $\lambda - \mu$ weight space with the $\lambda + \bar{\mu} = \lambda - \theta\mu$ weight space. Further, the dimension of each weight space of $M(\lambda)$ is finite, hence by restricting our attention to each pair of weight spaces of weights $\lambda - \mu$ and $\lambda - \theta\mu$ individually, we may discuss the determinant of the Shapovalov form. We are able to show, by suitably adapting Shapovalov's formula for the determinant of the classical Shapovalov form, that the determinant of our invariant Hermitian form on the $\lambda - \mu$ weight space is

$$c \prod_{\alpha \in \Delta^+} \prod_{n=1}^{\infty} \left((\lambda + \rho, \alpha^\vee) - n\right)^{P(\mu - n\alpha)}$$

in the case where $\theta\mu = \mu$ and the determinant on the $\lambda - \mu$ and $\lambda - \theta\mu$ weight spaces is

$$c \prod_{\alpha \in \Delta^+} \prod_{n=1}^{\infty} \left((\lambda + \rho, \alpha^\vee) - n\right)^{P(\mu - n\alpha)} \left((\lambda + \rho, \alpha^\vee) - n\right)^{P(\theta\mu - n\alpha)}$$

in the case where $\theta\mu \neq \mu$. Here, P denotes Kostant's partition function.

These determinant formulas indicate precisely where the Shapovalov form is degenerate, and consequently where $M(\lambda)$ is reducible: on the affine hyperplanes $H_{\alpha,n} := \{\lambda + \rho \,|\, (\lambda + \rho, \alpha^\vee) = n\}$ where α is a positive root and n is a positive integer. We conclude that *in any region avoiding these reducibility hyperplanes, the signature remains constant.* The largest of such regions, which we refer to as the Wallach region since it is the region for which Wallach computed the signature character, is the intersection of the negative open half spaces

$$\left(\bigcap_{\alpha \in \Pi} H^-_{\alpha,1}\right) \bigcap H^-_{\widetilde{\alpha},1}$$

with $i\mathfrak{h}_0^*$, where $\widetilde{\alpha}^\vee$ is the highest coroot and Π is the set of simple roots.

In [Wal84], Wallach shows that the diagonal entries in a matrix associated to the Shapovalov form $\langle \cdot, \cdot \rangle_{\lambda + t\xi}$ on a weight space have higher degree in t than the off-diagonal entries. Thus an asymptotic argument which examines the signs of the diagonal entries for large t yields a formula for the signature of the Shapovalov form within the entire Wallach region.

In this paper, I extend Wallach's result to all irreducible Verma modules which carry an invariant Hermitian form. The development of a formula for the signature of the Shapovalov form on irreducible Verma modules has four major components:

(1) We determine via our determinant formulas and a Jantzen filtration argument how the signature for the Shapovalov form changes as you cross a reducibility hyperplane $H_{\alpha,n}$. We get three answers: one for α imaginary, one for α complex with α and $\theta\alpha$ orthogonal, and one for α complex with α and $\theta\alpha$ generating a root subsystem of type A_2. For example, if α is imaginary, as λ traces a path from $H^+_{\alpha,n}$ to $H^-_{\alpha,n}$, the signature character changes by plus or minus twice the signature character of the Shapovalov form on $M(\lambda - n\alpha)$.

(2) The action of the affine Weyl group on $\mathfrak{h}^*$ defines alcoves with walls of the form $H_{\alpha,n}$ where α is a root and n is an integer. The signature of the Shapovalov form does not change within the interior of these alcoves, and we know from part (1) how signatures of adjacent alcoves are related, up to some signs. By defining a suitable notion of distance of an alcove to the Wallach region, we determine by induction a formula, up to the unknown signs from part (1), for the signature character of the Shapovalov form in terms of Wallach's formula in the case where the Cartan subalgebra is compact (i.e. the case where all roots are imaginary).
(3) We calculate the unknown signs for the case where our Cartan subalgebra is compact, first for simple rank 2 root systems, and then for general root systems by induction.
(4) We extend the results for compact Cartan subalgebras to non-compact maximally compact Cartan subalgebras using formulas for singular vectors and Dynkin diagram automorphisms.

2.2. **Signatures of Invariant Hermitian Forms on Irreducible Highest Weight Modules and the Kazhdan-Lusztig Conjecture.**

[4] Yee, Wai Ling, *Signatures of Invariant Hermitian Forms on Irreducible Highest Weight Modules*, 19 pages. Submitted (Duke Mathematical Journal).

The irreducible highest weight module $L(\lambda)$ is the quotient of $M(\lambda)$ by its unique maximal submodule. Since the unique maximal submodule is also the radical of the Shapovalov form, the Shapovalov form on $M(\lambda)$ descends to a non-degenerate invariant Hermitian form on $L(\lambda)$ which we also call the Shapovalov form. We would like to compute the signature character of this form on $L(\lambda)$.

A *composition series* for a module V is a series of invariant subspaces

$$V = V_0 \supset V_1 \supset V_2 \supset \cdots$$

such that V_i/V_{i+1} is irreducible. The irreducible V_i/V_{i+1} are called the composition factors of V. Verma modules are known to have finite composition series. Their composition factors are irreducible highest weight modules with highest weight linked to the highest weight of the Verma module. Although the set of singular vectors [2] of a Verma module of a fixed weight has dimension one, it is possible for composition factors of a Verma module to have multiplicity greater than one (cf. [DL77]). A number of partial results were published in the late 1970s concerning the multiplicities of composition factors in Verma modules (eg. [DL77], [Jan79]).

In [KL79], Kazhdan and Lusztig define polynomials $P_{x,y}$ for $x, y \in W$. They are called the Kazhdan-Lusztig polynomials. The Kazhdan-Lusztig Conjecture states that for $\lambda+\rho$ *antidominant* (henceforth, $\lambda+\rho$ will always be antidominant) and for x and y in the integral Weyl group W_λ,

$$[M(x\cdot\lambda) : L(y\cdot\lambda)] = P_{w_\lambda x, w_\lambda y}(1)$$

where w_λ is the long element of W_λ and $[V : X]$ denotes the multiplicity of X as a composition factor of V. Here $x\cdot\lambda = x(\lambda+\rho)-\rho$. The Kazhdan-Lusztig Conjecture provides more precise information than the multiplicity of each composition factor: the multiplicity of $L(y\cdot\lambda)$ in the j^{th} level of the Jantzen filtration [3] of the Verma module $M(x\cdot\lambda)$ is the coefficient of $q^{(\ell(x)-\ell(y)-j)/2}$ in $P_{w_\lambda x, w_\lambda y}(q)$.

A proof of the Kazhdan-Lusztig Conjecture was perhaps the most important problem in representation theory in the early 1980s. In [Vog79b], Vogan showed that semisimplicity of $U_\alpha L(x\cdot\lambda)$ where $U_\alpha L(x\cdot\lambda)$ is defined to be the cohomology of the complex $0 \to L(x\cdot\lambda) \to \theta_\alpha L(x\cdot\lambda) \to L(x\cdot\lambda) \to 0$ and θ_α is a coherent continuation functor, implies the Kazhdan-Lusztig Conjecture. In [GJ81], Gabber and Joseph proved that Jantzen's Conjecture implies Vogan's Conjecture. Brylinski- Kashiwara and Beilinson- Bernstein finished the proof of the Kazhdan-Lusztig Conjecture by proving Jantzen's Conjecture in [BK81] and [BB93]. It is widely regarded as a difficult proof, involving perverse sheaves and rings of twisted differential operators.

In terms of characters, the Kazhdan-Lusztig Conjecture may be stated:

$$ch\, M(x\cdot\lambda) = \sum_{y\le x} P_{w_\lambda x, w_\lambda y}(1) ch\, L(y\cdot\lambda).$$

[2] Singular vectors are the vectors annihilated by the nilpotent part of the Borel subalgebra used to define the Verma module.

[3] Think of the forms $\langle\cdot,\cdot\rangle_{x\cdot\lambda+\delta t}$ on $M(x\cdot\lambda+\delta t)$ where δ is regular and $t \in (-\varepsilon, \varepsilon)$ as a family of invariant Hermitian forms indexed by t on a common vector space. Roughly speaking, the j^{th} level of the Jantzen filtration is the quotient of the vectors which vanish at least to order j at $t = 0$ by those which vanish at least to order $j+1$.

The conjecture implies the inversion formula:

$$ch\, L(x\cdot\lambda)=\sum_{y\leq x}(-1)^{\ell(x)-\ell(y)}P_{y,x}(1)ch\, M(y\cdot\lambda).$$

Since $L(x\cdot\lambda)$ is the quotient of $M(x\cdot\lambda)$ by the radical of the Shapovalov form, the analogous formula for signature characters sheds no light on the problem of computing the signature character for $L(x\cdot\lambda)$: the signatures for $L(x\cdot\lambda)$ and for $M(x\cdot\lambda)$ differ by zero eigenvalues, and the signature character for reducible $M(x\cdot\lambda)$ is not known. However, I have shown:

Proposition 2.2. *Let $\lambda+\rho$ be imaginary and antidominant and let $\mathcal{C}$ be the set of alcoves which contain $x\cdot\lambda$ in their closures. Then the the signature of the Shapovalov form on $L(x\cdot\lambda)$ can be written as a linear combination of the signature characters of the Shapovalov forms corresponding to the alcoves in $\bigcup_{y\leq x} y\cdot\mathcal{C}$.*

Since we know the signature characters corresponding to these alcoves from [3], the remaining problem in describing the signature character of the Shapovalov form on $L(x\cdot\lambda)$ is to compute these linear combinations. When the rank of $\mathfrak{g}$ is at most two, the multiplicity of any composition factor in a composition series for $M(x\cdot\lambda)$ is one. This follows from the structure of the type A_2, B_2, and G_2 Weyl groups as dihedral groups. In fact, the dihedral group structure leads to a concise formula for the signature character. In general, we need a means of expressing the signature character for an alcove $A=wA_0+x\cdot\lambda$ containing $x\cdot\lambda$ in its closure in the form

$$\sum_{y\leq x}a_y^{x\cdot\lambda,w}ch_s\, L(y\cdot\lambda)$$

where the $a_y^{x\cdot\lambda,w}$ are integers. In the "multiplicity-free" cases, considering the form induced by the Shapovalov form and the Jantzen filtration on each of the composition factors of $M(x\cdot\lambda)$, it becomes apparent that the signs computed in step (3) of [3] determine the $a_y^{x\cdot\lambda,w}$. The $a_y^{x\cdot\lambda,w}$ are easy to determine when $\ell(x)\leq 2$, which suggests the existence of an inductive formula for the $a_y^{x\cdot\lambda,w}$. Indeed, an understanding of Gabber and Joseph's proof of Kazhdan and Lusztig's inductive formula for computing Kazhdan-Lusztig polynomials (cf. [GJ81]) at the level of coherent continuation functors, translation functors, symbols associated with contravariant forms, and their relations to contravariant forms on $M(x\cdot\lambda)$ and $M(xs_\alpha\cdot\lambda)$ where α is simple lead to an inductive formula for the $a_y^{x\cdot\lambda,w}$'s. The $a_y^{x\cdot\lambda,w}$'s are encoded in a generalization of the Kazhdan-Lusztig polynomials, which we call signed Kazhdan-Lusztig polynomials and which we may compute inductively.

The coherent continuation functors θ_α are exact functors on the Bernstein-Gelfand-Gelfand Category $\mathcal{O}$. They are compositions of projection functors onto blocks $D=W_\lambda\cdot\lambda$ (which may be thought of as the part of the module on which the centre of the universal enveloping algebra acts by central character χ_λ) with functors defined by tensoring with a finite-dimensional module. Projection onto blocks defines a primary decomposition of modules carrying a contravariant form into an orthogonal direct sum of submodules. As finite-dimensional modules are quotients of Verma modules and hence carry canonical contravariant forms induced by the Shapovalov form, $\theta_\alpha M$ naturally carries a contravariant form if M has a contravariant form F. Crucial to the proof of the inductive formulas for computing Kazhdan-Lusztig polynomials is the non-split exact sequence

$$0\to M(xs_\alpha\cdot\lambda)\to\theta_\alpha M(x\cdot\lambda)\to M(x\cdot\lambda)\to 0,$$

where α is simple and $x<xs_\alpha$, and the complex

$$0\to L(x\cdot\lambda)\to\theta_\alpha L(x\cdot\lambda)\to L(x\cdot\lambda)\to 0$$

whose cohomology is $U_\alpha L(x\cdot\lambda)$. Now $L(x\cdot\lambda)$ is the unique simple submodule of $\theta_\alpha L(x\cdot\lambda)$ and it is proper. The unique simple quotient is $L(x\cdot\lambda)$. Because the two are paired, $\theta_\alpha L(x\cdot\lambda)$ and $U_\alpha L(x\cdot\lambda)$ have the same signature character. The relation of U_α to the inductive formula for Kazhdan-Lusztig polynomials gives us an inductive formula for the $a_y^{x\cdot\lambda,w}$ which in turn gives us an inductive formula for $ch_s L(x\cdot\lambda)$:

$$ch_sL(x\cdot\lambda)=\sum_{y_1<\cdots<y_j=x}(-1)^{j-1}\left(\prod_{i=2}^{i=j}a_{y_{i-1}}^{y_i\lambda,w}\right)R^{y_1\lambda+wA_0}(y_1\lambda).$$

2.3. **Unitarity Testing.** An obvious future project is to use the signatures computed in [3] and [4] to test unitarizability of corresponding Harish-Chandra modules.

2.4. **Signed Kazhdan-Lusztig polynomials.** Another obvious future project is to develop the theory of signed Kazhdan-Lusztig polynomials.

2.5. **Generalized Verma Modules.** The difficulty in studying generalized Verma modules rather than Verma modules is that, in general, necessary and sufficient reducibility criteria are not known. Further, when generalized Verma modules are known to be reducible, one may not know their composition factors and their multiplicities. Reducibility of generalized Verma modules is a difficult open problem in the most general case. Special cases have been treated by first computing the determinant of the Shapovalov form (for example, see [KM99a], [KM99b]) so that, as for Verma modules, one can determine if the generalized Verma module is reducible by checking if the Shapovalov determinant is zero. Another approach using intertwining operators and Hecht-Schmid characters in the case of generalized principal series representations may be found in [SV80]. Speh and Vogan were particularly interested in understanding the composition series and the multiplicities of irreducible composition factors of generalized principal series representations as it would permit the determination of reducibility of any representation induced from a parabolic subgroup. Combining these approaches would be a good starting point for studying the composition series of generalized Verma modules. Another avenue to explore would be geometric constructions of Harish-Chandra modules and generalized Verma modules afforded by the theory of D-modules. I expect that if the composition series for any generalized Verma module were understood, then work in [3] and [4] on the Shapovalov form on Verma modules and highest weight modules can be adapted to generalized Verma modules. This leaves step (6) of the proposed programme for determining signatures of invariant Hermitian forms on cohomologically induced $(\mathfrak{g}, K)$ modules, which is discussed in Sections 7 and 8 of [3]. Since the signature characters from steps (1) to (5) would be expressed in terms of signature characters for the Wallach region, the hope is that since we know how the signature characters behave in the Wallach region under Zuckerman functors, addressing the effects of the Zuckerman functors elsewhere ought to be tractable.

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