July 15^{th}

Pigeon Hole Principle

(1) Let $y_1, y_2, \ldots, y_7 \in \mathbb{R}$. Prove that there are indices $i \neq j$ such that

$$0 \leq \frac{y_i - y_j}{1 + y_i y_j} \leq \sqrt{3}.$$

(2) There are 65 beetles on a 9×9 square board. At each move, beetles move one square horizontally or vertically. The beetles cannot move twice in a row vertically or twice in a row horizontally. Prove that at some point, there are at least two beetles occupying the same square.

Geometric Inequalities

In the following questions, a, b, c are lengths of the sides of a triangle corresponding to angles A, B, C respectively. P is the triangle's perimeter and S its area. Prove the following inequalities.

(1) (a) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{P}$
(b) $a^2 + b^2 + c^2 \geq \frac{P^2}{3}$
(c) $P^2 \geq 12\sqrt{3}S$
(d) $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$
(e) $a^3 + b^3 + c^3 \geq \frac{P^3}{9}$
(f) $a^3 + b^3 + c^3 \geq \frac{4\sqrt{3}}{3}SP$
(g) $a^4 + b^4 + c^4 \geq 16S^2$

(2) $\cos A + \cos B + \cos C \leq \dfrac{3}{2}$

(3) $\cos^2 A + \cos^2 B + \cos^2 C \geq \dfrac{3}{4}$

(4) $\dfrac{ab + ac + bc}{4S} \geq \sqrt{3}$