Problems on Inequalities

In the following problems, a, b and c will always represent positive real numbers. Some of these problems can be solved in a multitude of ways. See how many solutions you can come up with!

1. (a) Prove that $a^3 + b^3 + c^3 \geq 3abc$.

   (b) Prove that $(a + b + c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \geq 9$.

   (c) Prove that $\frac{ab}{c^2} + \frac{ac}{b^2} + \frac{bc}{a^2} \geq 3$.

2. Show that $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$.

3. Prove that $(a + b)(b + c)(c + a) \geq 8abc$.

4. For $0 \leq x \leq 4$, find the value of x that produces the maximum value of $f(x) = x^3(4 - x)$. What is this maximum value of $f(x)$?

5. Show that for all integers n greater than 1, the inequality $(\frac{n+1}{2})^n > n!$ holds.

6. Show that $(a+b+c)(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}) \geq \frac{9}{2}$. When does equality occur?

7. Show that for positive reals a, b, c we have $\frac{9}{a+b+c} \leq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

8. Prove that $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$ (1995 CMO, Question 2).

9. Show that $(\frac{a+nb}{n+1})^{n+1} > ab^n$, and use this to show that $a_n = (1 + \frac{1}{n})^n$ is an increasing sequence.

10. Let $x_1, \ldots, x_n$ be positive reals with sum 1. Prove that $\frac{x_1}{2-x_1} + \frac{x_2}{2-x_2} + \cdots + \frac{x_n}{2-x_n} \geq n(2n - 1)$.