Combinatorics

1. On a circle n points are selected and the chords joining them in pairs are drawn. Assuming that no three of these chords are concurrent (except at the endpoints), how many points of intersection are there?

2. The integers $1, 2, \ldots n$ are placed in order so that each value is strictly bigger than all the preceeding values or is strictly smaller than all the preceeding values. In how many ways can this be done?

3. Determine the number of solutions in positive integers $a_1, a_2, a_3, \ldots, a_{50}$ to the equation

$$a_1 + a_2 + \ldots + a_{50} = 1999.$$

   If we replaced *positive* integers with *non-negative* integers, how would this change the answer to our problem?

4. Prove that

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \ldots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

5. Prove that

$$\binom{r}{0}\binom{s}{n} + \binom{r}{1}\binom{s}{n-1} + \ldots + \binom{r}{n-1}\binom{s}{1} + \binom{r}{n}\binom{s}{0} = \binom{r+s}{n}.$$

6. Prove that the number of odd entries in the n^{th} row of Pascal's Triangle is a power of 2. How can you tell *which* power of 2 it is from the binary expansion of n?

7. In a movie theatre, one line consists of 8 seats. 4 married couples sit in these seats, but for each woman, only her husband or another woman may sit next to her. How many different ways are there for them to take their seats?

8. An unfair coin (probability p of showing heads) is tossed n times. What is the probability that the number of heads will be even?

9. Prove that for fixed m and n

$$\sum_{k_1+k_2+\cdots+k_m=n} k_1 k_2 \cdots k_m = \binom{m+n-1}{2m-1}$$

10. Prove that the number of partitions of n into unequal parts is equal to the number of its partitions into odd (unordered) parts. Realize the bijection.