University of Windsor Undergraduate Mathematics Contest: Solutions

October 16, 2007

1. If $\sin x = 3\cos x$, then what is $\sin x \cos x$?

   Solution:

$$\begin{aligned} \sin x &= 3\cos x \\ \sin^2 x &= 3\sin x\cos x \\ \frac{1}{3}\sin x\cos x &= \cos^2 x \\ 3\sin x\cos x + \frac{1}{3}\sin x\cos x &= \cos^2 x + \sin^2 x \\ \frac{10}{3}\sin x\cos x &= 1 \\ \sin x\cos x &= \frac{3}{10} \end{aligned}$$

   Alternate Solution:

$$\sin x = 3\cos x$$

   We have the identity:

$$\sin^2 x + \cos^2 x = 1$$

   So

$$\begin{aligned} 9\cos^2 x + \cos^2 x &= 1 \\ 10\cos^2 x &= 1 \\ \Rightarrow \cos^2 x &= 1/10 \\ \Rightarrow \cos x &= \pm 1/\sqrt{10} \end{aligned}$$

   We also have:

$$\begin{aligned} \sin x &= 3\cos x \\ \Rightarrow \sin x &= \pm 3(1/\sqrt{10}) = \pm 3/\sqrt{10} \end{aligned}$$

   Thus,

$$\sin x\cos x = (\pm 3/\sqrt{10})(\pm 1/\sqrt{10}) = 3/10$$

2. A 10 metre long rope goes from the ground to the top of a flag pole, wrapping around it five times in a parallel spiral. If the circumference of the flag pole is 60 centimetres, what is the height of the flag pole?

   Solution:
   Consider the rope as the hypotenuse of a right-triangle with the pole as the vertical side. The right-triangle has been wrapped around the pole five times so it's horizontal side is five times the circumference of the pole. So by Pythagoras...

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (5 \times 0.60)^2 + b^2 &= 10^2 \\ b^2 &= 100 - 9 \\ b &= \sqrt{91} \end{aligned}$$

   ...the height of the flag pole is $\sqrt{91}$.

   Similar Solution:
   Imagine the rope unravelled from the pole, retaining its shape so that the rope, pole and ground form a right-angled triangle. We are given that the rope wraps around the circumference of the pole five times. Convert 60 centimetres into 0.60 metres.
   Now we already know that the hypotenuse is 10 metres long and the base of this triangle is $5(0.60) = 3$ metres long.
   Simply use the Pythagorean Theorem to solve for the height of the pole:

$$10^2 = 3^2 + h^2$$

$$\Rightarrow 100 - 9 = h^2$$

$$\Rightarrow h = \sqrt{91} \approx 9.54$$

   Therefore, the height of the pole is $\sqrt{91} \approx 9.54$ metres.

3. The points $(0, 0), (a, 11)$, and $(b, 37)$ are vertices of an equilateral triangle. Find the value of ab.

   Solution:
   Since we have an equilateral triangle, let's consider each side of the triangle as a two-dimensional vector. Let
   $\bar{x} = [b - 0, 37 - 0] = [b, 37]$,
   $\bar{y} = [a - 0, 11 - 0] = [a, 11]$
   $\bar{z} = [b - a, 37 - 11] = [b - a, 26]$
   Since we are dealing with an equilateral triangle we know that $|x| = |y| = |z|$ with,
   $|x| = \sqrt{b^2 + 37^2}$
   $|y| = \sqrt{a^2 + 11^2}$
   $|z| = \sqrt{(b - a)^2 + 26^2}$
   Setting $|x| = |y|$ and squaring both sides gives,

$$b^2 + 37^2 = a^2 + 11^2$$

$$a^2 - b^2 = 1248 \ (1)$$

   Setting $|x| = |z|$ and squaring both sides gives,

$$b^2 + 37^2 = (b - a)^2 + 26^2$$

$$a^2 - 2ab = 693 \ (2)$$

Setting $|y| = |z|$ and squaring both sides gives,

$$a^2 + 11^2 = (b - a)^2 + 26^2$$
$$b^2 - 2ab = -555 \ (3)$$

Now we can re-write equation (1) as the difference of squares,

$$(a + b)(a - b) = 1248$$

Square both sides of this equations,

$$(a + b)^2(a - b)^2 = 1248^2$$

Expand inside the brackets,

$$(a^2 + 2ab + b^2)(a^2 - 2ab + b^2) = 1248^2$$

By strategically adding and subtracting abs we find that,

$$(a^2 + 2ab + b^2 + 4ab - 4ab)(a^2 - 2ab + b^2 + 2ab - 2ab) = 1248^2$$
$$([a^2 - 2ab] + [b^2 - 2ab] + 6ab)([a^2 - 2ab] + [b^2 - 2ab] + 2ab) = 1248^2$$

Sub in the numerical values for $a^2 - 2ab$ and $b^2 - 2ab$ from equations (2) and (3),

$$(693 - 555 + 6ab)(693 - 555 + 2ab) = 1248^2$$
$$(138 + 6ab)(138 + 2ab) = 1248^2$$

Expand and collect like terms,

$$0 = (ab)^2 + 92(ab) - 128205$$

Using the Quadratic Formula to solve for ab,

$$ab = \frac{-92 \pm \sqrt{92^2 - 4(-128205)}}{2}$$
$$ab = \frac{-92 \pm 722}{2}$$
$$ab = 315 \text{ or } ab = -407$$

Therefore $ab = 315$ (Why can't $ab = -407$? Try subbing this into equation (2) and solving for a.)

Alternate Solution: Consider the 60° rotation matrix. One of the following is true:

$$\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ 11 \end{pmatrix} = \begin{pmatrix} b \\ 37 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} b \\ 37 \end{pmatrix} = \begin{pmatrix} a \\ 11 \end{pmatrix}.$$

The first equation leads to $a = 21\sqrt{3}$, $b = 5\sqrt{3}$. The second equation leads to $b = -5\sqrt{3}$, $a = -21\sqrt{3}$. In both cases, $ab = 315$.

4. Find the last two digits of the product of the positive roots of

$$\sqrt{2007}x^{\log_{2007} x} = x^2$$

Solution: The trick is substituting $x = 2007^y$. Which we can do since 2007^y covers all the positive reals as y traverses all the reals. Start by simplifying.

$$\begin{aligned}
\frac{\sqrt{2007}x^{\log_{2007} x}}{x^2} &= 1 \\
x^{\log_{2007} x - 2} &= \frac{1}{\sqrt{2007}} \\
(2007^y)^{log_{2007} 2007^y - 2} &= 2007^{\frac{-1}{2}} \\
2007^{y(y-2)} &= 2007^{\frac{-1}{2}} \\
y(y-2) &= \frac{-1}{2} \\
y^2 - 2y + \frac{1}{2} &= 0 \\
(y - 1 - \frac{1}{\sqrt{2}})(y - 1 + \frac{1}{\sqrt{2}} &= 0
\end{aligned}$$

So $y = 1 + \frac{1}{\sqrt{2}}$ and $y = 1 - \frac{1}{\sqrt{2}}$. Substitute that into $x = 2007^y$ to find the coresponding values for x.

$$\begin{aligned}
root_1 &= 2007^{1+\frac{1}{\sqrt{2}}} \\
root_2 &= 2007^{1-\frac{1}{\sqrt{2}}} \\
root_1 \times root_2 &= 2007^{1+\frac{1}{\sqrt{2}}+1-\frac{1}{\sqrt{2}}} \\
root_1 \times root_2 &= 2007^2 \\
root_1 \times root_2 &= 4028049
\end{aligned}$$

$\Rightarrow$ the answer is 49.

Alternate Solution:
Recall the rules of logarithms, if

$$\log_a x = y \Rightarrow a^y = x$$

Let $y = \log_{2007} x \Rightarrow 2007^y = x$
Rewriting our equation we have,

$$2007^{\frac{1}{2}}(2007^y)^y = (2007^y)^2$$
$$2007^{\frac{1}{2}+y^2} = 2007^{2y}$$

Since the bases on both sides of this equation are the same, we can equate the exponents,

$$\frac{1}{2} + y^2 = 2y$$
$$0 = y^2 - 2y - \frac{1}{2}$$

Using the Quadratic Formula we find that,

$$y = \frac{2 \pm \sqrt{(-2)^2 - 4(\frac{1}{2})}}{2}$$

$$y = 1 \pm \frac{\sqrt{2}}{2}$$

Let $y_1 = 1 + \frac{\sqrt{2}}{2}$ and $y_2 = 1 - \frac{\sqrt{2}}{2}$
Then $x_1 = 2007^{y_1}$ and $x_2 = 2007^{y_2}$
Thus $x_1 x_2 = 2007^{y_1} 2007^{y_2} = 2007^{1+\frac{\sqrt{2}}{2}} 2007^{1-\frac{\sqrt{2}}{2}} = 2007^2$
Clearly, the last two digits of 2007^2 are 49.

5. Consider the number 123456789. In how many ways can you rearrange it's digits to get a number divisible 11?

Solution:
(This is similar to Casting Out Nines)
Let us label the first digit A, the second B, the third C, ..., the ninth I. To determine if this number is divisible by 11 consider:

$$\begin{aligned} ABCDEFGI &= A(10)^8 + B(10)^7 + C(10)^6 + D(10)^5 + E(10)^4 + F(10)^3 + G(10)^2 + H(10)^1 + I(10)^0 \\ &= A(11-1)^8 + B(11-1)^7 + C(11-1)^6 + ... + H(11-1)^1 + I(11-1)^0 \end{aligned}$$

Modulo 11 this equation becomes,

$$A(-1)^8 + B(-1)^7 + C(-1)^6 + D(-1)^5 + E(-1)^4 + F(-1)^3 + G(-1)^2 + H(-1)^1 + I(-1)^0$$
$$= A - B + C - D + E - F + G - H + I$$

Therefore if $A - B + C - D + E - F + G - H + I$ is congruent to 0 mod 11, the original number $ABCDEFGHI$ is divisible by 11.

To check if a number is divisible by 11, sum every other digit and subtract the rest. If 11 is a factor of the resulting number then the original number is also divisible by 11. Let the permutations of 123456789 be represented by $ABCDEFGHI$.

$$(A + C + E + G + I) - (B + D + F + H) = 11t, \; t \in \mathbf{Z}$$

We know that $A + B + C + D + E + G + H + I = 45$.

$$\begin{aligned} A + B + C + D + E + G + H + I &= 45 \\ (A + C + E + G + I) + (B + D + F + H) &= 45 \\ (A + C + E + G + I) - (B + D + F + H) &= 45 - 2(B + D + F + H) \\ 11t &= 45 - 2(B + D + F + H) \\ 2(B + D + F + H) &= 45 - 11t \\ B + D + F + H &= \frac{45 - 11t}{2} \end{aligned}$$

We know that $\frac{45-11t}{2}$ is an integer so t must be odd. The lowest number that $B + D + F + H$ could possibly be is

$$1 + 2 + 3 + 4 = 10$$

So, $\frac{45-11t}{2} \geq 10 \Rightarrow t = 1$

$$B + D + F + H = 17$$

The fastest way I know to find the number of ways to make 17 is to list them.

$$(6 + 5 + 2 + 4) = 17$$
$$(7 + 5 + 2 + 3) = 17$$
$$(7 + 5 + 1 + 4) = 17$$
$$(7 + 6 + 1 + 3) = 17$$
$$(8 + 4 + 2 + 3) = 17$$
$$(8 + 5 + 1 + 3) = 17$$
$$(8 + 6 + 2 + 1) = 17$$
$$(9 + 4 + 1 + 3) = 17$$

So there are 8 ways to sum to 17.

$$\Rightarrow answer = 8 \times 4! \times 5! = 23040$$

Note: when working modulo 2, $x \equiv -x \mod 2$ can be very useful. For example,
$A - B + C - D + E - F + G - H + I \equiv A + B + C + D + E + F + G + H + I \equiv 45 \equiv 1 \mod 2$
so $A - B + C - D + E - F + G - H + I = 11t$ is odd, and thus t is odd.

6. Let n be a fixed positive integer. Show that for only non-negative integers k, the diophantine equation:

$$x_1^3 + x_2^3 + ... + x_n^3 = y^{3k+2}$$

has infinitely many solutions in positive integers x_i and y.

Solution:
(Solutions from the 1995 CMO)
For any positive integer q, take,

$$x_1 = x_2 = ... = x_n = n^{2k+1}q^{3k+2}, y = n^2q^3$$

Then,

$$\sum_{i=1}^{n} x_i^3 = nn^{6k+3}q^{9k+6} = (n^2q^3)^{3k+2} = y^{3k+2}$$

Alternate Solution:
(Solution from the 1995 CMO)
If $n = 1$, take $x_1 = q^{3k+2}, y = q^3$.
For $n > 1$, we look for solutions of the form

$$x_1 = x_2 = ... = x_n = n^p, y = n^q$$

Then,

$$\sum_{i=1}^{n} x_i^3 = y^{3k+2} \Leftrightarrow n^{3p+1} = n^{(3k+2)q}$$

$$\Leftrightarrow 3p+1=(3k+2)q \Leftrightarrow (3k+2)q-3p=1$$

This last equation is satisfied if we take,

$$q=3t+2 \text{ and } p=(3k+2)t+(2k+1)$$

where t is and non-negative integer. Thus infinitely many solutions in positive integers given by

$$x_1=x_2=...=x_n=n^{(3k+2)t+(2k+1)}, y=n^{3t+2}$$