**University of Windsor Mathematics Practice Problems Solutions**

1. The function $f$ has the property that for each real number $x$,

$$f(x) + f(x-1) = x^2$$

If $f(19) = 94$ what is the remainder when $f(94)$ is divided by 1000?

*Solution:*

$$\begin{aligned} f(x) &= x^2 - f(x-1) \\ f(94) &= 94^2 - f(93) \\ &= 94^2 - 93^2 + f(92) \\ &= 94^2 - 93^2 + 92^2 - 91^2 + \ldots + 22^2 - 21^2 + 20^2 - f(19) \end{aligned}$$

We can pair up consecutive squares and write each pair as a difference of squares,

$$\begin{aligned} f(94) &= (94+93)(94-93) + (92+91)(92-91) + \ldots + (22+21)(22-21) + 20^2 - f(19) \\ &= 94 + 93 + 92 + \ldots + 22 + 21 + 20^2 - f(19) \\ &= (1 + 2 + 3 + \ldots + 74) + 74(20) + 20^2 - f(19) \end{aligned}$$

Recall,

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2}$$

So,

$$\begin{aligned} f(94) &= \frac{74(75)}{2} + 74(20) + 20^2 - 94 \\ &= 4561 \end{aligned}$$

Thus the remainder when $f(94) = 4561$ is divided by 1000 is 561.

2. Let $f(x) = \frac{9^x}{9^x+3}$. Evaluate the sum

$$f(\frac{1}{2007}) + f(\frac{2}{2007}) + f(\frac{3}{2007}) + ... + f(\frac{2006}{2007}).$$

*Solution:*
Notice that $\forall k$,

$$\begin{aligned} f(\frac{k}{2007}) + f(\frac{2007-k}{2007}) &= \frac{9^{\frac{k}{2007}}}{9^{\frac{k}{2007}}+3} + \frac{9^{\frac{2007-k}{2007}}}{9^{\frac{2007-k}{2007}}+3} \\ &= \frac{9^{\frac{k}{2007}} \times 9^{\frac{2007-k}{2007}} + 9^{\frac{k}{2007}} \times 3 + 9^{\frac{k}{2007}} \times 9^{\frac{2007-k}{2007}} + 3 \times 9^{\frac{2007-k}{2007}}}{(9^{\frac{k}{2007}}+3)(9^{\frac{2007-k}{2007}}+3)} \\ &= \frac{9 + 3 \times 9^{\frac{k}{2007}} + 9 + 3 \times 9^{\frac{2007-k}{2007}}}{(9^{\frac{k}{2007}}+3)(9^{\frac{2007-k}{2007}}+3)} \\ &= \frac{(9^{\frac{k}{2007}}+3)(9^{\frac{2007-k}{2007}}+3)}{(9^{\frac{k}{2007}}+3)(9^{\frac{2007-k}{2007}}+3)} \\ &= 1 \end{aligned}$$

Thus, the sum,

$$f(\frac{1}{2007})+f(\frac{2}{2007})+f(\frac{3}{2007})+...+f(\frac{2006}{2007}).$$

Rewritten as,

$$\begin{aligned}\sum_{k=1}^{2006} f(\frac{k}{2007}) &= \sum_{k=1}^{1003}[f(\frac{k}{2007})+f(\frac{2007-k}{2007})] \\ &= \sum_{k=1}^{1003} 1 \\ &= 1003\end{aligned}$$

Notice that $\frac{2007-1}{2}=\frac{2006}{2}=1003$.

3. Let $a$, $b$ and $c$ be positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

*Solution:*
We can equivalently prove that,

$$a^{3a}b^{3b}c^{3c} \geq (abc)^{a+b+c}$$

Or,

$$\frac{a^{3a}b^{3b}c^{3c}}{(abc)^{a+b+c}} \geq 1$$

WLOG, assume that $a \geq b \geq c$

$$\Rightarrow a-b \geq 0, \qquad a-c \geq 0, \qquad b-c \geq 0$$

and

$$\frac{a}{b} \geq 1, \qquad \frac{a}{c} \geq 1, \qquad \frac{b}{c} \geq 1$$

Thus we have,

$$\begin{aligned}\frac{a^{3a}b^{3b}c^{3c}}{(abc)^{a+b+c}} &= \frac{a^{3a}b^{3b}c^{3c}}{a^{a+b+c}b^{a+b+c}c^{a+b+c}} \\ &= a^{3a-(a+b+c)}b^{3b-(a+b+c)}c^{3c-(a+b+c)} \\ &= a^{2a-(b+c)}b^{2b-(a+c)}c^{2c-(a+b)} \\ &= a^{a-b}a^{a-c}b^{-(a-b)}b^{b-c}c^{-(a-c)}c^{-(b-c)} \\ &= (\frac{a}{b})^{a-b}(\frac{a}{c})^{a-c}(\frac{b}{c})^{b-c} \\ &\geq 1\end{aligned}$$

Thus we have

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$