Number Theory Practice Problems

1) i. ABC is divisible by 2 $\Rightarrow C$ is even.
   ii. ACB is divisible by 3 $\Rightarrow A+B+C$ is divisible by three $\Rightarrow$ every permutation of ABC is divisible by three.
   iii. BAC is divisible by 4 $\Rightarrow AC$ is divisible by 4.
   iv. BCA is divisible by 5 $\Rightarrow A=5$ or $A=0$.
   v. CAB is divisible by 6 $\Rightarrow B$ is even.
   vi. CBA is a divisor of 1995. $\Rightarrow CBA \in$ the set of three digit factors of 1995= $\{285, 105, 133, 399, 665\}$

   We can eliminate all the elements of the set based on our knowledge of A, B, and C with the exception of 285. Thus $ABC = 285$.
2) We know from the binomial theorem that if a number is divisible by m unique primes it is going to have 2^m factors.

$$\begin{aligned} 16 &= 2^m \\ 4 &= m \end{aligned}$$

   So, n is the product of the 4 smallest primes, namely $2 \times 3 \times 5 \times 7 = 210$.
3) The prime factorization of 225 is $3^2 5^2 = 9 \times 25 = 225$. For a number n to be divisible by 9 the sum of n's digits must be divisible by 9. Thus there must be 9 1's making up the number. For a number to be divisible by 25 it must end with the last two digits 25, 50, 75, or 00. The only one that is acceptable here is 00. Thus the smallest number made entirely of 0 and 1 digits is 11111111100.

4)

$$n^4 - 20n^2 + 4 = (n^2 - 4n - 2)(n^2 + 4n - 2)$$

   So, $\forall n \in \mathbb{Z}, n^4 - 20n^2 + 4$ is a composite number (not prime).

5) Let's start of with some facts:
   i) If the prime factorization of a number n is $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ then the number of factors of n is $(a_1+1)(a_2+1)\cdots(a_m+1)$.
   ii) $\gcd(p, p+1) = 1$. This is true because the gcd of p and $p+1$ divides both numbers, and hence it divides their difference $(p+1)-p = 1$.

   Thus, if the prime factorization of p is $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ and the prime factorization of $(p+1)$ is $q_1^{b_1} q_2^{b_2} \cdots q_k^{b_k}$ then the prime factorization of $p^3 + 2p^2 + p = p(p+1)^2$ is $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} q_1^{2b_1} q_2^{2b_2} \cdots q_k^{2b_k}$. The p_i's are distinct from the q_i's.
   We need:

$$(a_1+1)(a_2+1)\cdots(a_m+1)(2b_1+1)(2b_2+1)\cdots(2b_k+1) = 42 = 2 \times 3 \times 7.$$

   i) If $a_1 + 1 = 2$, $(2b_1+1) = 3$ and $(2b_2+1) = 7$:
   If we minimize p, we minimize $(p+1)^2$. The minimum possible value of $q_1^2 q_2^6$ is achieved with $q_1 = 3$, $q_2 = 2$. With these values, $(p+1) = 3 \times 2^3 = 24$ so that $p = 23$. The prime factorization of 23 is 23^1 so that $a_1 = 1$ which agrees with the condition that must be satisfied.
   ii) If $a_1 + 1 = 2$ and $2b_1 + 1 = 21$: $p + 1 = q_1^{10} \geq 2^{10} > 24$. We see from the previous

case that this will not minimize p.
iii) If $a_1 + 1 \geq 6$: (Note that $a_1 + 1$ is a factor of 42 and it cannot be 1 since the product of the $2b_i + 1$'s is odd.)
$p \geq p_1^{a_1} \geq 2^5 = 32 > 24$, and we see from the first case that this will not minimize p.
Therefore $p = 23$ is the smallest number that works.

6) We know that the number of tickets n is divisible by 7. We know that n divided by 5 gives a remainder of 4, which tells us that the last digit of n is either 4 or 9. We know that when n is divided by 3 it leaves a remainder of 1, telling us that the sum of the digits of $n - 1$ are divisible by three. Doing some quick search of the multiples of 7 the first number we come across that fits these criteria is 49. This is not a unique answer since a more comprehensive scan of the multiples of 7 brings us 154 which also fits the criteria.

7) The remainders of $3^n \div 7$ as n increases across the integers, follow the repeating pattern $"'3, 2, 6, 4, 5, 1, 3, 2, ..."'$. $56 \div 6$ leaves a remainder of 2, which coresponds to the second entry in the pattern. Thus the remainder of $3^{56} \div 7$ is 2.

8) Let $n =$ the number of sheep $=$ price per sheep.
$n^2 =$ the price of the flock.
$p =$ value of the pocket knife. $(0 < p \leq 5)$
$m \in \mathbb{Z}$ Lets say that the older brother took $10m$ dollars and the younger brother took $10(m-1) + (10 - 2p) = 10m - 2p$ dollars.

$$n^2 = 10m + 10m - 2p$$
$$n^2 = 20m - 2p$$
$$n^2 = 2(10m - p)$$

Since n^2 is clearly even we know that it must be divisible by $4 \Rightarrow (10m - p)$ must be even $\Rightarrow p$ must be even.

$$n^2 = 20m - 2p$$
$$2p = 20m - n^2$$
$$p = 10m - \frac{n^2}{2}$$

When $m = 1$ and $n = 4$, $p = 2$.
When $m = 2$ and $n = 6$, $p = 2$.
The proof that the only number that p can be is 2 is left as an exercise.