Number Theory

Here are some definitions to know:

Prime Number: a number with only 2 factors, 1 and the number itself
Composite Number: a number with more than 2 factors

Divisibility and Factors
The following statements mean the same thing:

1. $a|b$
2. a divides b
3. b is divisible by a
4. a is a divisor of b
5. a is a factor of b
6. $b = na$ for some integral n

However, normally the term factor refers only to positive factors. If we wish to refer to both positive and negative factors, we generally use the term integral factors. Also, if we say that a is a proper divisor of b, this means that a is a positive factor of b and a is not equal to b.

Infinitude of Primes
In The Elements, Euclid proved that there are an infinite number of primes. Dirichlet's Theorem states that there are an infinite number of primes in the arithmetic progression.
$a, a + d, a + 2d, a + 3d, \ldots$

Greatest Common Divisor (g.c.d): the largest possible number which divides into a set of numbers. Often the g.c.d. of a and b is denoted as (a,b).

Least Common Multiple (l.c.m): the smallest number which is a multiple of a set of numbers.

The best way of finding the greatest common divisor and least common multiple is to write all the numbers in prime factorization form. Let's take the numbers 36, 96, 250.

$36 = 2^2 \times 3^2 \times 5^0$
$96 = 2^5 \times 3^1 \times 5^0$
$250 = 2^1 \times 3^0 \times 5^3$

The find the g.c.d., we take the smallest exponent for each prime. Therefore, we have the g.c.d equal to $2^1 \times 3^0 \times 5^0$. To find the l.c.m., we take the largest exponent for each prime. Therefore we have the l.c.m. equal to $2^5 \times 3^2 \times 5^3$.

Example: (2007 International Tournament of the Towns- Junior A Level #2)

Pete and Basil choose three positive integers each. For each pair of his numbers, Pete writes their greatest common divisor on the blackboard. For each pair of his numbers, Basil writes their lowest common multiple on the blackboard. It turns out that the three numbers written by Pete are the same as those written by Basil (up to their order). Prove that all numbers written on the blackboard are equal.

Solution:

Consider a prime p

Suppose p^{a1}, p^{a2}, p^{a3} are the highest powers of p dividing Peter's numbers and p^{b1}, p^{b2}, p^{b3} are the highest powers of p dividing Basil's numbers where

$a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$

So the largest powers of p dividing the GCDs that Peter writes down are

p^{a1}, p^{a2}, p^{a1} where $a_1 = a_1 \leq a_2$

and the largest powers of p dividing the LCMs that Basil writes down are

p^{b2}, p^{b3}, p^{b3} where $b_2 \leq b_3 = b_3$

Now we need the following sets to be equal:

$\{p^{a1}, p^{a2}, p^{a1}\} = \{p^{b2}, p^{b3}, p^{b3}\}$

Therefore,

$a_1 = b_2, a_1 = b_3, a_2 = b_3$

-i.e. $a_1 = a_2 = b_2 = b_3$

So the numbers Peter and Basil write down are all the same since p is any prime.

University of Windsor Mathematics Contest Practice Problems
Number Theory

1. Roger becomes very bored one day and decides to write down a three digit number ABC, and the rearrangements of its digits. To his surprise, he finds that ABC is divisible by 2, ACB is divisible by 3, BAC is divisible by 4, BCA is divisible by 5, CAB is divisible by 6, and CBA is a divisor of 1995. Determine ABC.

2. Determine the smallest positive integer n with exactly 16 divisors.

3. Find the smallest positive integer, composed only of 0's and 1's which is divisible by 225.

4. Show that no integers n for which $n^4 - 20n^2 + 4$ is prime

5. Determine the smallest number p for which $p^3 + 2p^2 + p$ has exactly 42 divisors.

6. Paul has a collection of tickets. When the tickets are piled in groups of three, there is one left over. In piles of five, there are four left over. In piles of seven, there are none left over. How many tickets are in the collection? Is this answer unique?

7. What is the remainder when 3^{56} is divided by 7?

8. Two brothers sold a flock of sheep. For each sheep they received as many dollars as the number of sheep in the flock. The money was divided as follows: the older brother took 10 dollars, after which the younger brother took 10 dollars; next, the older brother took another 10 dollars, and so did the younger brother; and so on. At the end, the younger brother, whose turn it was, found that there was not a full 10 dollars left for him. He took what remained and, in order to even things up, the older brother game him his pen knife. What was the penknife worth? (Note: the penknife was worth an exact number of dollars.)