Chapter 4

Inclusion-Exclusion

4.1 The Principle of Inclusion-Exclusion

Let S be a set with $|S| = N$. Let $c_1, c_2, \ldots, c_t$ be a collection of conditional properties each of which may be satisfied by some of the elements of S.

Let $N(c_i), i = 1, \ldots, t$	$=$	the number of elements of S that satisfy c_i.
Let $N(c_i c_j), ij = 1, \ldots, t$	$=$	the number of elements of S that satisfy c_i and c_j.
Let $N(c_i c_j c_k), ijk = 1, \ldots, t$	$=$	the number of elements of S that satisfy c_i, c_j, and c_k.
Let $N(\bar{c}_i), i = 1, \ldots, t = N - N(c_i)$	$=$	the number of elements of S that do not satisfy c_i.
Let $N(\bar{c}_i \bar{c}_j), i = 1, \ldots, t = N - N(c_i c_j)$	$=$	the number of elements of S that do not satisfy c_i and do not satisfy c_j.

Theorem 4.1.1

$$\begin{aligned}
\overline{N} &= N(\bar{c}_1 \bar{c}_2 \ldots \bar{c}_t) \\
&= N - (N(c_i) + N(c_2) + \cdots + N(c_t)) + (N(c_1 c_2) + N(c_1 c_3) + \cdots + N(c_{t-1} c_t)) \\
&\quad - (N(c_1 c_2 c_3) + N(c_1 c_2 c_4) + \cdots) + \cdots + (-1)^t N(c_1 c_2 \cdots c_t)
\end{aligned}$$

Let $S_0 = N$
Let $S_1 = N(c_1) + N(c_2) + \cdots + N(c_t)$
Let $S_2 = N(c_1 c_2) + N(c_1 c_3) + \cdots + N(c_{t-1} c_t)$
Let $S_3 = N(c_1 c_2 c_3) + N(c_1 c_2 c_4) + \cdots$

then

$$\overline{N} = S_0 - S_1 + S - 2 - S_3 + \cdots + (-1)^t S_t$$

Proof:

- If x satisfies none of the conditions then x is counted once in $\overline{N}$ and once in N and not in any other term of S. Hence, it contributes 1 to both sides.

- If x satisfies exactly r of the conditions ($r < t$) then x does not contribute to $\overline{N}$, in the RHS x is counted.

  1. one time in N
  2. r times in $\sum N(c_i)$
  3. $\binom{r}{2}$ times in $\sum N(c_i c_j)$
  4. $\binom{r}{3}$ times in $\sum N(c_i c_j c_k)$

     $\vdots$

     $\binom{r}{r}$ times in $\sum N(c_1 c_2 \dots c_r)$

$\therefore$ in the RHS we have

$$1 - \binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = (1-1)^r = 0$$

$\square$

Example 4.1.1 *Determine the number of positive integers $n \leq 100$ and n is not divisible by 2, 3, or 5.*

Solution
$S = \{1, 2, \dots, 100\}, \quad N = 100$
$c_1 =$ the property that n is divisible by 2.
$c_2 =$ the property that n is divisible by 3.
$c_3 =$ the property that n is divisible by 5.

The answer is $N(\bar{c}_1 \bar{c}_2 \bar{c}_3)$

$$N(c_1) = \left\lfloor \frac{100}{2} \right\rfloor = 50, \qquad N(c_2) = \left\lfloor \frac{100}{3} \right\rfloor = 33, \qquad N(c_3) = \left\lfloor \frac{100}{5} \right\rfloor = 20$$
$$N(c_1 c_2) = \left\lfloor \frac{100}{6} \right\rfloor = 16, \qquad N(c_1 c_3) = \left\lfloor \frac{100}{10} \right\rfloor = 10, \qquad N(c_2 c_3) = \left\lfloor \frac{100}{15} \right\rfloor = 6$$
$$N(c_1 c_2 c_3) = \left\lfloor \frac{100}{30} \right\rfloor = 3$$

$\therefore$ the answer is $N(\bar{c}_1 \bar{c}_2 \bar{c}_3) = 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26$

The numbers are:
1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89, 91, 97