2007 International Tournament of the Towns
Senior O Level Paper

1. There are 100 pictures, and each of them includes images of an adult and of a child of smaller height (all 200 persons in the pictures are different). These pictures must be combined into a single large picture. We are allowed to change the scale of each picture by diminishing its size by some integer factor (independently for different pictures). Prove that we can do this so that on the combined picture all the adults are taller than all the children.

Solution: Here is a heuristic argument–I will leave the details to you. The pictures can be scaled so that the children are as close in height as you want. Each adult now appears to be taller than all of the children.

Why is it that by using scalings by integers, we can make the children as close to the same height as we want? Suppose we have two children, the first of height c_1 and another of height c_2. Can we find integers m_1 and m_2 so that $\frac{c_1}{m_1}$ and $\frac{c_2}{m_2}$ are as close to each other as we want? This is equivalent to asking if we can make $\frac{c_1}{m_1}/\frac{c_2}{m_2}$ as close to 1 as we want. Rewriting this one more time, can we make $\frac{c_1}{c_2}$ and $\frac{m_1}{m_2}$ as close to each other as we want? The answer is yes. $\frac{m_1}{m_2}$ is a rational number (a number of the form a/b where a and b are integers with $b \neq 0$) and given any real number, we can find a rational number as close to that real number as we want. For example, 3.1 is within .1 of π; 3.14 is within .01 of π; 3.141 is within .001 of π; etc.

Remaining details of this question are left to the reader.

2. The integer 1 and two positive numbers x and y are written on a sheet of paper. At each step we can write down the sum or the difference of some two numbers already written or write down the inverse of some number already written ($\frac{1}{a}$ is the inverse of a if $a \neq 0$). Is it possible to obtain at some step

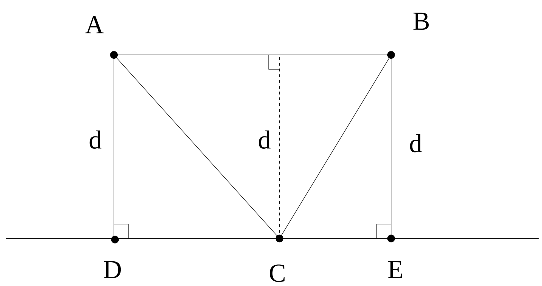
a) the number x^2?

b) the number xy?

Solution to a): Yes, it is possible to obtain x^2.
Add 1 and 1 and then take the inverse to get $\frac{1}{2}$.
Add 1 and x and then take the inverse to get $\frac{1}{1+x}$.
Subtract x from 1 and then take the inverse to get $\frac{1}{1-x}$.
$\frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}$.
Take the reciprocal: $\frac{1-x^2}{2}$.
$\frac{1}{2} - \frac{1-x^2}{2} = \frac{x^2}{2}$.
$\frac{x^2}{2} + \frac{x^2}{2} = x^2$.
Solution to b): Yes, it is possible to obtain xy. Make $x + y$.
From part a), we can make $x^2/2$, $y^2/2$, and $(x + y)^2/2$.
$(x + y)^2/2 - x^2/2 = (2xy + y^2)/2$
$(2xy + y^2)/2 - y^2/2 = 2xy/2 = xy$

3. A straight line and two points A and B on the same side of it and at the same distance from it are given. By compass and straightedge, find the point C on the line such that the product $AC \cdot BC$ is the minimal possible.

Solution by Anqi Zhang: Begin by finding points D and E on the line so that AD



and BE are perpendicular to the line. This can be done with straightedge and compass (`http://www.sonoma.edu/users/w/wilsonst/courses/math_150/c-s/default.html`). Let C be an arbitrary point on the line through DE. Note that the area of ΔABC is $\frac{1}{2}|AB| \times d$ which stays the same no matter where C is on the line (even if it isn't between D and E). On the other hand, the area of ΔABC is also equal to $\frac{1}{2}|AC| \cdot |BC| \sin \angle ACB$, and therefore to minimize $|AC| \cdot |BC|$, we must maximize $\sin \angle ACB$.

Maximizing $\sin \angle ACB$: $\sin \angle ACB$ achieves its maximum when $\angle ACB = 90°$. To construct such a point C:

1) bisect AB (this is possible with a compass and straightedge)

2) construct a circle with diameter AB (i.e. centered at the midpoint of AB)

3) take a point of intersection of the circle with the line to be C. Since AB is the diameter of the circle, we see that $\angle ACB = 90°$.

If the circle does not intersect the line (i.e. when $2d > |AB|$), you can show that $\sin \angle ACB$ is maximized when C is the midpoint of D. You can bisect DE with a compass and straightedge, so C can be constructed.

4. A magician is blindfolded and gives 29 cards with numbers from 1 to 29 to a spectator. The spectator hides two cards and returns the remaining cards to the magician's assistant. The assistant chooses two of them, and then the spectator communicates the numbers chosen by the assistant to the magician (in any order he prefers). After that, the magician guesses the numbers of the cards hidden by the spectator. How can the magician and the assistant arrange to succeed in performing this trick?

Solution by Frank Ban: Let the numbers hidden by the spectator be a and b where $a > b$. When the spectator returns the remaining 27 cards to the assistant, the assistant can clearly find the values of a and b by searching to see which two cards are missing. The assistant then considers the following two cases:

Case 1: $a - b \neq 1$ and $(a, b) \neq (29, 1)$: The assistant gives the magician the cards $a - 1$, $b - 1$, where card number 0 is identified with card number 29. Since $(a - 1) - (b - 1) = a - b \neq 1$, the magician can tell what case he has, and add 1 to both numbers to recover a and b.

Case 2: $a - b = 1$ or $(a, b) = (29, 1)$: The strategy from the previous case will not work here because $b = a - 1$ and both a and b are in the hands of the spectator. In this case, the assistant chooses cards with values $a - 2$, $b - 2$ (identifying -1 with 28 and 0 with 29). Since $(a-2)-(b-2) = a - b = 1$, if there has been no wrapping around, the magician can tell what case he has, and add 2 to both numbers to recover a and b. In the two instances where some wrapping around has occurred, $((1,2) \to (28,29)$ and $(29,1) \to (27,28))$, you can see that there is no conflict with Case 1, and the magician can tell he is in Case 2 and recover the spectator's numbers.

5. A square of side 1 cm on side is cut into three convex polygons. Is it possible that the diameter of each of them does not exceed

a) 1 cm;

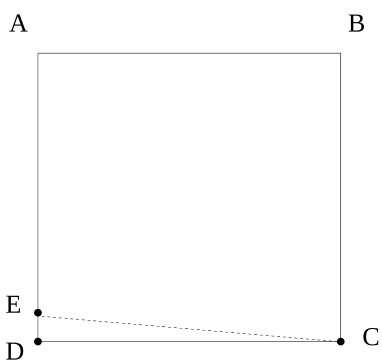
b) 1.01 cm;

c) 1.001 cm?

(The diameter of a polygon is the maximal distance between its vertices).

Solution by Jason Fan and Peter Wen:

a) By the Pigeon Hole Principle, one of the polygons must contain at least two vertices of the square. Furthermore, there is a polygon which contains at least two vertices of the square and at least one of the vertices does not belong to any of the other two polygons, or else there would be more than three polygons. The polygon therefore must contain three points C, D, and E as shown in the diagram. We see that $|CE| = \sqrt{|CD|^2 + |DE|^2} = \sqrt{1 + |DE|^2} > 1$. Therefore it is not possible that the diameter does not exceed 1cm.

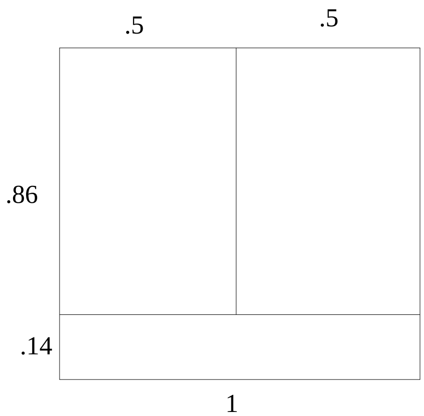


b) Yes. See the following diagram:

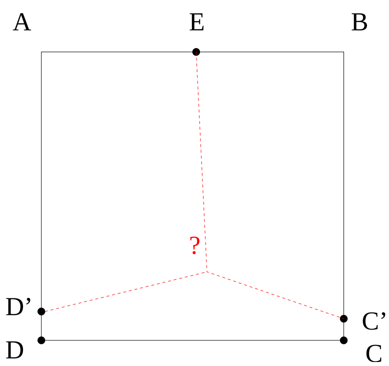
$\sqrt{(.5)^2 + (.86)^2} = .994\ldots < 1.01$

$\sqrt{1^2 + (.14)^2} = 1.00975\ldots < 1.01$

c) No. Refer to case a) and suppose C and D are vertices of one polygon (call it P) where D does not belong to the other two polygons. Let C' and D' be vertices of polygon P on AD and BC, respectively. (We allow $C' = C$.) Since the diameter of P cannot exceed 1.001, therefore $|CC'|, |DD'| \le \sqrt{1.001^2 - 1}$. The interior of AB must contain a vertex of a polygon. If not, then the top of the square resembles the bottom, and then A' and C' are two vertices of the remaining polygon and $|A'C'| > 1.001$. Thus



the interior of AB contains a polygon vertex E, and it must belong to the two polygons which aren't P (or else $|ED|$ or $|EC| > 1.001$). Either $|AE| \geq .5$ or $|BE| \geq .5$. Let's suppose the former is true. Then A, E, D' belong to the same polygon, and the diameter is $\geq |D'E|$. Now:



$$\begin{aligned} |D'E|^2 &= |AE|^2 + |AD'|^2 = |AE|^2 + (1 - |DD'|)^2 \\ &\geq (.5)^2 + (1 - \sqrt{1.001^2 - 1})^2 \\ &> 1.162 \end{aligned}$$

and therefore $|D'E| > \sqrt{1.162} > 1.001$. Similarly, if $|BE| \geq .5$, then E and C' are vertices of the same polygon and $|C'E| > 1.001$. Therefore it is impossible that the diameter of each polygon does not exceed 1.001cm.