2007 International Tournament of the Towns
Junior O Level Paper

1. What is the maximal number of black and white chips that can be placed on an 8×8 chess-board so that each horizontal and each vertical line contains twice as many white chips as black ones? (Each chip occupies a separate cell.)

Solution: The number of chips in a particular row must be a multiple of 3 since there are twice as many white chips as black chips, and thus any row contains at most 6 chips. Therefore the maximal number of black and white chips which may be placed on the chess board is 48: 32 white and 16 black. This maximal number is achieved, as shown in the diagram representing one quarter of the chess board:

	W	B	W
W		W	B
B	W		W
W	B	W	

Repeat this pattern in the remaining three quarters of the chess board.

2. The number 1 and of some non-integer number x are written on a sheet of paper. At each step we can write down the sum or the difference of some two numbers already written or write down the inverse of some number already written ($\frac{1}{a}$ is the inverse of a if $a \neq 0$). Is it possible to obtain x^2 at some step?

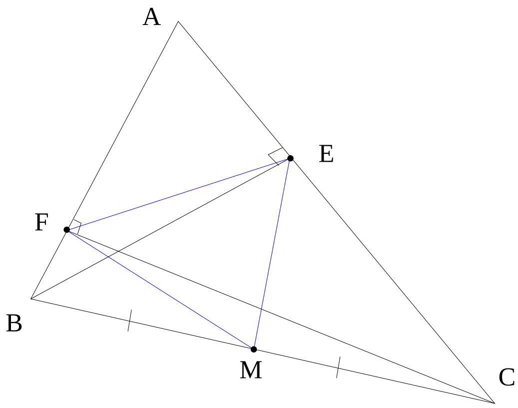
Solution: Yes, it is possible to obtain x^2.
Add 1 and 1 and then take the inverse to get $\frac{1}{2}$.
Add 1 and x and then take the inverse to get $\frac{1}{1+x}$.
Subtract x from 1 and then take the inverse to get $\frac{1}{1-x}$.
$\frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}$.
Take the reciprocal: $\frac{1-x^2}{2}$.
$\frac{1}{2} - \frac{1-x^2}{2} = \frac{x^2}{2}$.
$\frac{x^2}{2} + \frac{x^2}{2} = x^2$.

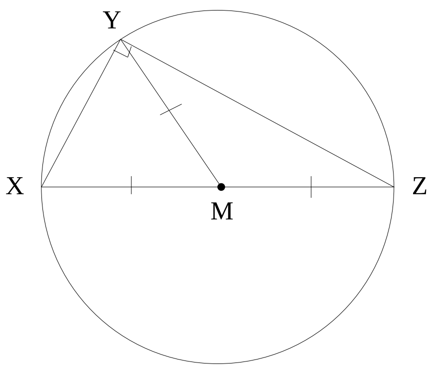
3. The midpoint of some side of a triangle and the bases of the altitudes drawn to two other sides form an equilateral triangle. Is it true that the original triangle is equilateral as well?

Solution: No. You can show that the original triangle could be a $30° - 60° - 90°$ triangle. More generally, if ΔABC is such that $\angle A = 60°$ and $\angle B$ and $\angle C$ are arbitrary, with the midpoint and altitudes labelled as in the first figure, ΔMEF is equilateral. We can prove this as follows:

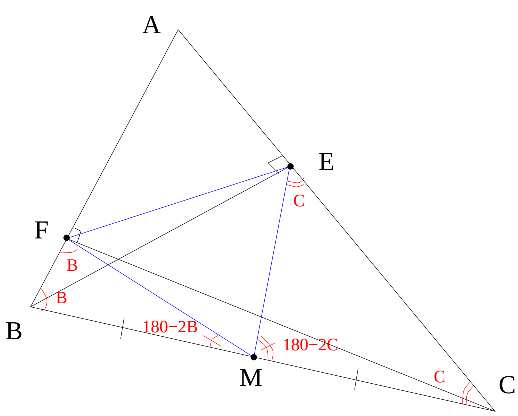
For any right angle triangle ΔXYZ where M is the midpoint of the hypotenuse as shown, $|XM| = |YM| = |ZM|$. We can prove that this is true by circumscribing the triangle with a circle and then noting that XZ is a diameter and therefore M is the centre of the circle. Therefore XM, YM, and ZM are all radii and therefore their lengths are equal.

Applying the above result to the right angle triangles ΔBFC and ΔBEC in the first diagram, we see that $|BM| = |FM| = |CM| = |EM|$. Therefore ΔBME and





ΔCME are isosceles triangles, and we label the angles as shown. Therefore $\angle FME = 180° - (180° - 2B) - (180° - 2C) = 2B + 2C - 180° = 2(120°) - 180° = 60°$. Now ΔMEF is such that $|EM| = |FM|$ and $\angle EMF = 60°$. Therefore ΔMEF is equilateral.



4. A 29×29 table contains the integers 1, 2, 3, ..., 29, each of them 29 times. It turns out that the sum of the numbers above the principal diagonal is three times greater than the sum of the numbers below this diagonal. Determine the number in the central cell of the table.

Solution by James Duyck: Let x be the sum of the numbers below the diagonal so that $3x$ is the sum of the numbers above the diagonal.

There are $1 + 2 + \cdots + 28 = 14 \times 29$ entries below the diagonal and the same number

above the diagonal.
Fill up the area below the diagonal with the smallest possible numbers to get the smallest possible value for x:

$$\begin{array}{rl} 1+1+\cdots+1 & \text{29 times} \\ 2+2+\cdots+2 & \text{29 times} \\ 3+3+\cdots+3 & \text{29 times} \\ \vdots & \vdots \\ 14+14+\cdots+14 & \text{29 times} \end{array}$$

Total sum: $(1+2+\cdots+14)\times 29 = 15\times 7\times 29$ so

$$x \geq 15\times 7\times 29.$$

Fill up the area above the diagonal with the largest possible numbers to get the largest possible value for $3x$:

$$\begin{array}{rl} 16+16+\cdots+16 & \text{29 times} \\ 17+17+\cdots+17 & \text{29 times} \\ 18+18+\cdots+18 & \text{29 times} \\ \vdots & \vdots \\ 29+29+\cdots+29 & \text{29 times} \end{array}$$

Total sum: $(16+17+\cdots+29)\times 29 = 45\times 7\times 29$ so

$$3x \leq 45\times 7\times 29.$$

The only way both inequalities are satisfied is if $x = 15\times 7\times 29$. Thus there are only the numbers $1, 2, \ldots 14$ below the diagonal and the numbers $16, 17, \ldots, 29$ above the diagonal, leaving only 15's on the diagonal. Therefore the centre cell is 15.

5. A magician is blindfolded and gives five cards with numbers from 1 to 5 to a spectator. The spectator hides two cards and returns the remaining three to the magician's assistant. The assistant chooses two of them, and then the spectator communicates the numbers chosen by the assistant to the magician (in any order he prefers). After that, the magician guesses the numbers of the cards hidden by the spectator. How can the magician and the assistant arrange to succeed in performing this trick?
Solution: We can record the strategy in the following table:

Spectator's cards	Assistant's cards
1, 2	3, 4
1, 3	2, 4
1, 4	2, 5
1, 5	2, 3
2, 3	4, 5
2, 4	3, 5
2, 5	1, 3
3, 4	1, 5
3, 5	1, 4
4, 5	1, 2

See Question 4 from the Senior Competition.