International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2007.**

1. Let K be a point on the side CD of the rhombus $ABCD$ such that $AD = BK$. Let P be the point of intersection of BD with the perpendicular bisector of BC. Prove that A, K and P are collinear.

2. (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down on the blackboard the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down on the blackboard the least common multiple of the two numbers. If the three numbers written down by Peter are the same three numbers, in some order, written down by Basil, prove that all six numbers are the same.

   (b) If each of Peter and Basil thinks of four positive integers instead, will the twelve numbers on the blackboard be the same?

3. Michael is at the centre of a circle of radius 100 metres. Each minute, he will announce the direction in which he will be moving. Catherine can leave it as is, or change it to the opposite direction. Then Michael moves exactly 1 metre in the direction determined by Catherine. Does Michael have a strategy which guarantees that he can get out of the circle, even though Catherine will try to stop him?

4. Two players take turns entering a symbol in an empty cell of a $1 \times n$ chessboard, where n is an integer greater than 1. The first player always enters the symbol X and the second player O. Two identical symbols may not occupy adjacent cells. A player without a move loses the game. Which player has a winning strategy?

5. Attached to each of a number of objects is a tag which states its correct mass. The tags have fallen off and got mixed up. We wish to restore all of them to the correct objects by using exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side. The lever either stays horizontal or tilts to one side. Is this task always possible?

6. The audience arranges n coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience.

   (a) Prove that if this is possible for some n, then it is also possible for $2n$.

   (b) Determine all n for which this is possible.

7. For each letter in the English alphabet, William assigns an English word which contains that letter. His first document consists only of the word assigned to the letter A. In each subsequent document, he replaces each letter of the preceding document by its assigned word. The foriteth document begins with "Till whatsoever star that guides my moving ". Prove that this sentence reappears later in this document.

Note: The problems are worth 5, 3+3, 6, 7, 8, 4+5 and 9 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2007.**

1. (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down on the blackboard the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down on the blackboard the least common multiple of the two numbers. If the three numbers written down by Peter are the same three numbers, in some order, written down by Basil, prove that all six numbers are the same.
   (b) If each of Peter and Basil thinks of four positive integers instead, will the twelve numbers on the blackboard be the same?

2. Let k, L, M and N be the midpoints of the sides AB, BC, CD and DA of a cyclic quadrilateral $ABCD$. Let P be the point of intersection of AC and BD. Prove that the circumradii of triangles PKL, PLM, PMN and PNK are equal to one another.

3. Determine all finite arithmetic progressions in which each term is the reciprocal of a positive integer and the sum of all the terms is 1.

4. Attached to each of a number of objects is a tag which states its correct mass. The tags have fallen off and got mixed up. We wish to restore all of them to the correct objects by using exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side. The lever either stays horizontal or tilts to one side. Is this task always possible?

5. The audience arranges n coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience.
   (a) Prove that if this is possible for some n_1 and n_2, then it is also possible for $2n_1n_2$.
   (b) Determine all n for which this is possible.

6. Let P and Q be two convex polygons. Let h be the length of the projection of Q onto a line perpendicular to a side of P which is of length ℓ. Define $f(P,Q)$ to be the sum of the products $h\ell$ over all sides of P. Prove that $f(P,Q) = f(Q,P)$.

7. There are 100 boxes, each containing either a red cube or a blue cube. Alex has a sum of money initially, and places bets on each box in turn. The bet can be anywhere from 0 up to everything he has at the time. After the bet has been placed, the box is opened. If Alex loses, his bet will be taken away. If he wins, he will get his bet back, plus a sum equal to the bet. Then he moves onto the next box, until he has betted on the last one, or until he runs out of money. What is the maximum amount he can guarantee for himself if he knows that the exactly number of blue cubes is
   (a) 1;
   (b) some integer $n > 1$.

Note: The problems are worth 2+2, 6, 6, 6, 4+4, 8 and 3+5 points respectively.