L semis, $\quad L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \quad \leftarrow$ eigenspaces L_α called <u>root spaces</u>

<u>root space decomposition</u>

$\alpha \in \Phi$ called roots.

$K_L(\ ,\ ) \rightsquigarrow$ non-deg. symmetric bilinear form on H.

Use $(\ ,\ )$ to denote $K_L|_{H \times H}$, K_L.

$\lambda \in H^* \rightsquigarrow t_\lambda \in H$ so that

$$\lambda(h) = (t_\lambda, h) \quad \forall h \in H.$$

<u>Proposition:</u> a) Φ spans H^*

Pf: If not, $\exists\, h \in H$, $h \neq 0$ so that $\alpha(h) = 0 \quad \forall \alpha \in \Phi$

$\Rightarrow [L_\alpha, h] = \alpha(h) L_\alpha = 0 \quad \forall \alpha \in \Phi$

$\Rightarrow h \in Z(L) = \{0\}$ but $h \neq 0$ – contradiction.

b) $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$

Pf: $L_\alpha \perp L_\beta$ if $\alpha \neq -\beta$ and Killing form non-degenerate for L semis.

c) $x \in L_\alpha$, $y \in L_{-\alpha} \Rightarrow [x,y] = (x,y)\, t_\alpha$.

Pf: $([x,y], h) = (x, [y,h]) = \alpha(h)(x,y)$

$= (x,y)(t_\alpha, h) \quad \forall h \in H$

$\Rightarrow [x,y] = (x,y)\, t_\alpha$ –

d) $[L_\alpha, L_{-\alpha}]$ is one-dimensional

<u>Pf:</u> non-deg. of Killing form + c)

e) $\alpha(t_\alpha) = (t_\alpha, t_\alpha) \neq 0 \quad \forall \alpha \in \Phi$

<u>Pf:</u> $\underset{\substack{\uparrow \\ L_\alpha}}{x_\alpha}$, $\underset{\substack{\uparrow \\ L_{-\alpha}}}{y_\alpha}$, t_α generate a subalgebra by c) and definition of L_α

If $\alpha(t_\alpha) = 0$, then:

$$[t_\alpha, x] = \alpha(t_\alpha)x = 0$$
$$[t_\alpha, y] = -\alpha(t_\alpha)y = 0$$

$\langle x, y, t_\alpha \rangle \cong$ Heisenberg Lie alg $\Rightarrow$ solvable

so by Lie's Theorem: $x = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$, $y = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$,

$t_\alpha = \begin{pmatrix} 0 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ 0 & & & 0 \end{pmatrix}$ nilpotent — contradiction —

So $\alpha(t_\alpha) \neq 0 \Rightarrow \langle x, y, t_\alpha \rangle \cong \mathfrak{sl}_2$

f) If $x_\alpha \in L_\alpha$, then $\exists\ y_\alpha \in L_{-\alpha}$ s.t.

$S_\alpha := \langle x_\alpha, y_\alpha, [x_\alpha, y_\alpha] = h_\alpha \rangle \cong \langle x, y, h \rangle$ $\mathfrak{sl}_2$

$x_\alpha \mapsto x$ etc. in the isomorphism

Pf: $h_\alpha = c\, t_\alpha$ for some constant c.

$[c t_\alpha, x] = c\,\alpha(t_\alpha)x$ Choose $c\,\alpha(t_\alpha) = 2$

so $\boxed{h_\alpha = \dfrac{2t_\alpha}{\alpha(t_\alpha)}} = \dfrac{2t_\alpha}{(t_\alpha, t_\alpha)}$ Note that $\boxed{\alpha(h_\alpha) = 2}$

Each root α gives rise to a subalgebra $S_\alpha \cong \mathfrak{sl}_2$.

Consider $M := H \oplus \bigoplus_{c \in \mathbb{Z}} L_{c\alpha}$

M is an S_α-submodule of L w.r.t. adjoint action

M: h_α acts by 0. $\dim H$ 0 eigenvalues

$L_{c\alpha}$: h_α acts by $c\,\alpha(h_\alpha) = 2c$

$\mathfrak{sl}_2$-theory requires $2c \in \mathbb{Z}$ i.e. $c \in \frac{1}{2}\mathbb{Z}$

Show only possible values for c are ± 1:

$\underbrace{\{\alpha : \alpha(h) = 0\}}_{\dim = \dim H - 1} \subset H$ Subrep of M

0 eigenvalues

$S_\alpha \subset M$ is an S_α-subrep. of M via adjoint action giving one more eigenvalue 0.

This exhausts the 0 eigenvalues.

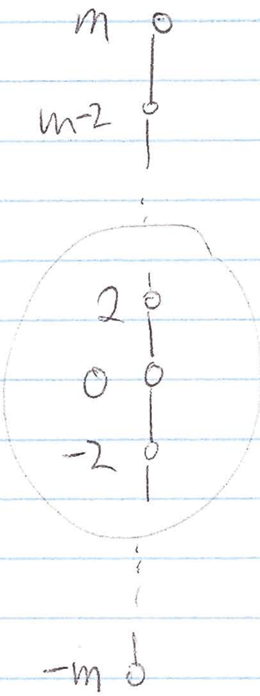
$\therefore$ only even weights occurring in M are $0, \pm 2$.

$\Downarrow$

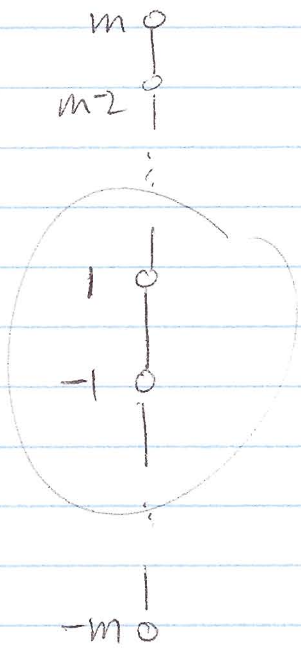
$$\boxed{\dim L_\alpha = 1}$$

$\shortparallel$ dim of 2 weight space

m even



m odd



General statement: α a root $\Rightarrow 2\alpha$ not a root

Conclude: α a root $\Rightarrow \frac{1}{2}\alpha$ not a root either.

α only c giving $L_{c\alpha}$ weight 1

Weight 1 does not appear $\Rightarrow$ no odd weights.

$\therefore M = H + S_\alpha = \ker\alpha \oplus S_\alpha$.

$$\boxed{\alpha \in \Phi,\ c\alpha \in \Phi \Rightarrow c = \pm 1}$$

$(,)$ on $H \times H$ vs $(,)$ on $H^* \times H^*$

$(\alpha, \beta) = (t_\alpha, t_\beta)$.

$\beta(t_\alpha) = (t_\beta, t_\alpha) = (\alpha, \beta)$.

$h_\alpha = \frac{t_\alpha}{(t_\alpha, t_\alpha)} = \frac{t_\alpha}{(\alpha,\alpha)}$.

Now consider L_β where $\beta \neq c\alpha$:

$\bigoplus_{i \in \mathbb{Z}} L_{\beta + i\alpha}$ is an S_α-submodule

h_α acts on $L_{\beta+i\alpha}$ by multiplication by $(\beta + i\alpha)(h_\alpha) = 2i + \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$

So: $\boxed{\alpha, \beta \in \Phi \Rightarrow \underbrace{\frac{2(\alpha,\beta)}{(\alpha,\alpha)}}_{\text{Cartan integer}} \in \mathbb{Z}}$

$$[L_{\beta+i\alpha}, L_\alpha] \subset L_{\beta+(i+1)\alpha}$$
$$[L_{\beta+i\alpha}, L_{-\alpha}] \subset L_{\beta+(i-1)\alpha}$$
$$\Rightarrow \bigoplus_{i \in \mathbb{Z}} \overset{\text{1-dim'l}}{L_{\beta+i\alpha}} \quad \text{irred. } S_\alpha\text{-module}$$

weights: $-k, -k+2, \ldots, k-2, k$

$\boxed{\beta + i\alpha \in \Phi \text{ for some interval } i \in [-r, p]}$

$$k = -\left(-2r + \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\right) = 2p + \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$$

$$\boxed{\Rightarrow r-p = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}}$$

$r, p \geq 0$ so $0 \leq r-p \leq r$, $0 \leq p-r \leq p$

So $\boxed{\alpha, \beta \in \Phi \Rightarrow \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Phi}$

$\boxed{\alpha, \beta \in \Phi \text{ and } \alpha+\beta \in \Phi \Rightarrow [L_\alpha, L_\beta] = L_{\alpha+\beta}}$

Span $\Phi = H^*$.

Choose $\{\alpha_1, \ldots, \alpha_n\} \subset \Phi$ basis of H^*.

Proposition: if $\alpha \in \Phi$, then $\alpha = \sum c_i \alpha_i$ for $c_i \in \mathbb{Q}$.

Pf:
$$\underbrace{\frac{(\beta,\alpha_j)}{(\alpha_j,\alpha_j)}}_{\in \mathbb{Q}} = \sum_i c_i \underbrace{\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}}_{\in \mathbb{Q}}$$

for each j

dim L equ's (indexed by j)

in dim L unknowns (indexed by i)

Cramer's Rule $\Rightarrow$ $c_i \in \mathbb{Q}$

Need invertibility of corresponding matrix
i.e. the c_i's uniquely determined
to use Cramer's Rule.
Killing form non-deg. $\longleftrightarrow$ matrix with entries (α_i, α_j) invertible

Proposition: $\lambda, \mu \in H^*$. Then

$$\boxed{(\lambda, \mu) = \sum_{\alpha \in \Phi} (\alpha, \lambda)(\alpha, \mu)}$$

Pf: $(\lambda, \mu) = (t_\lambda, t_\mu) = \mathrm{Tr}_L(\mathrm{ad}(t_\lambda)\,\mathrm{ad}(t_\mu))$

$\mathrm{ad}(t_\lambda)\mathrm{ad}(t_\mu)\, x_\alpha$
$= \mathrm{ad}(t_\lambda)\,\alpha(t_\mu)\, x_\alpha$
$= \alpha(t_\lambda)\,\alpha(t_\mu)\, x_\alpha$
so trace is

$$= \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu)$$

$$= \sum_{\alpha \in \Phi} (\alpha, \lambda)(\alpha, \mu)$$

Corollary: $(\ ,\ )$ restricted to $\mathbb{Q}$-span Φ is positive definite.

Pf: $(\beta, \beta) = \sum_{\alpha \in \Phi} (\alpha, \beta)^2 > 0$ if $\beta \neq 0$.

real, not complex

& actually need $(\alpha, \beta) \in \mathbb{R}$ to draw this conclusion. In fact,

Corollary: In fact, $(\alpha, \beta) \in \mathbb{Q}$ also.

Pf: $$\frac{1}{(\beta, \beta)} = \sum_{\alpha \in \Phi} \Bigg(\underbrace{\frac{(\alpha, \beta)}{(\beta, \beta)}}_{\in \mathbb{Q}} \Bigg)^2 \in \mathbb{Q}$$

$\frac{1}{(\beta,\beta)} \in \mathbb{Q}$ along with $\frac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Q} \implies (\alpha, \beta) \in \mathbb{Q}$

Φ - root system

- had certain properties

Define abstract concept of root system:

Def'n: E - real vector space with positive-def'n inner-product

$\Phi \subset E$ is a root system if

1) $\alpha \in \Phi$, $c\alpha \in \Phi \Rightarrow c = \pm 1$
2) Φ is finite, $\mathbb{R}$-Span $\Phi = E$, $0 \notin \Phi$
3) $\alpha, \beta \in \Phi \Rightarrow \langle \alpha, \beta \rangle := \dfrac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$
4) $\alpha, \beta \in \Phi \Rightarrow \alpha - \langle \alpha, \beta \rangle \beta \in \Phi$

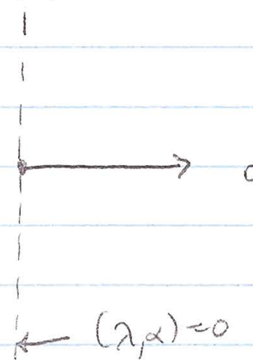
$\dim E$ = rank of Φ

For $\alpha \in E$: define $S_\alpha : E \to E$ linear map

$$\lambda \longmapsto \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$$

$S_\alpha(\alpha) = -\alpha$

$S_\alpha(\lambda) = \lambda$ if $(\lambda,\alpha) = 0$ ← This condition defines a hyperplane in E



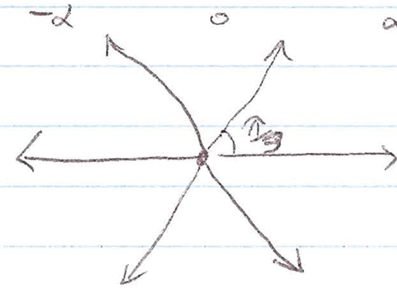
S_α = refl. through the hyperplane orthogonal to α (dotted line)

$\boxed{\text{Geometric picture: } \Phi \text{ preserved by reflection } S_\alpha \quad \alpha \in \Phi}$

eg. 1) $L = \mathfrak{sl}_2$ dim $H = 1$

A_1

2) $L = \mathfrak{sl}_3$, dim $H = 2$.



A_2

Let's study geometry of an abstract root system $\Phi \subset E$:

Two roots $\alpha, \beta \in \Phi$.

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)}, \quad \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$$

$$\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} \in \mathbb{Z}$$

Recall: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where $\cos\theta$ = angle between $\vec{a}$ and $\vec{b}$

$$\frac{(\vec{a}\cdot\vec{b})^2}{\|\vec{a}\|^2\|\vec{b}\|^2}$$

$$4\cos^2\theta \in \mathbb{Z}$$

Can be $0, 1, 2, 3, 4$.

0: $(\alpha,\beta)=0$ $\qquad \theta = \frac{\pi}{2}$

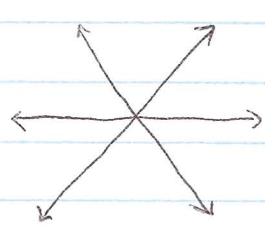
1: $\cos^2\theta = \frac{1}{4}$ $\qquad \cos\theta = \pm\frac{1}{2}$ $\qquad \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2(\alpha,\beta)}{(\beta,\beta)} = \pm 1$$

since they are integers whose product is 1.

$\Rightarrow (\alpha,\alpha) = (\beta,\beta)$ or $\frac{(\alpha,\alpha)}{(\beta,\beta)} = 1$ $\qquad \alpha, \beta$ have same length.

eg.



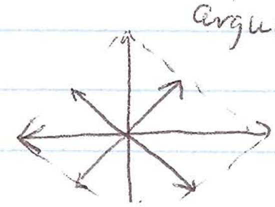
A_2

2: $\cos^2\theta = \frac{1}{2}$ $\qquad \cos\theta = \frac{\pm 1}{\sqrt{2}}$ $\qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Cartan integers are $\{1, 2\}$ or $\{-1, -2\}$ (like previous argument)

$$\Rightarrow \frac{(\alpha,\alpha)}{(\beta,\beta)} = 2 \text{ or } \frac{1}{2}$$

eg.

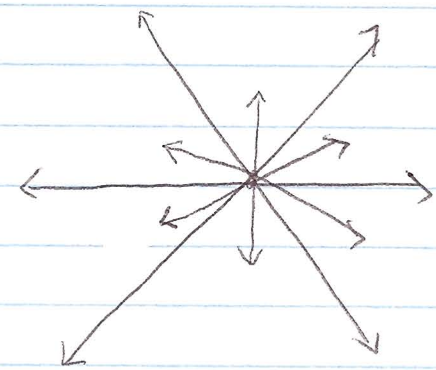


B_2

$$3: \quad \cos^2\theta = \frac{3}{4} \qquad \cos\theta = \frac{\pm\sqrt{3}}{2} \qquad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Cartan integers are $\{1, 3\}$ or $\{-1, -3\}$

$$\Rightarrow \quad \frac{(\alpha,\alpha)}{(\beta,\beta)} = 3 \text{ or } \frac{1}{3}$$



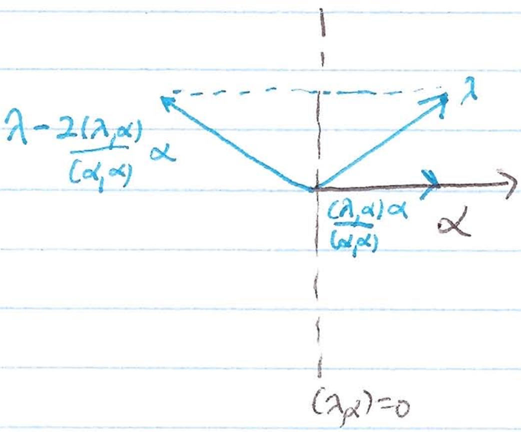
G_2

Connect geometry to algebra:

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$$

Reflection through the hyperplane orthogonal to α

$$\{\lambda : (\lambda,\alpha) = 0\}.$$



Yesterday:

$\alpha, \beta \in \Phi$

$$\Rightarrow \alpha - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \Phi$$

i.e. $\alpha, \beta \in \Phi \Rightarrow s_\alpha(\beta) \in \Phi$

Reflections $s_\alpha : \Phi \circlearrowleft$ $\qquad$ Φ preserved.

$\{s_\alpha\}_{\alpha \in \Phi}$ generate a group called the Weyl group