Back to Weyl's Theorem:

Prove: For any submodule $W \subseteq V$ if a split exact sequence $0 \to W \to V \to W' \to 0$.

First: Enough consider $W$ of codimension $1$, i.e. $\dim V/W = 1$.

$0 \to W \to V \to \frac{V}{W} \to 0$ is exact.

$L$ is semi-simple, so this must be trivial module by Lemma 1.

Why? Consider $L$-module $\text{Hom}_L(V, W)$

$L$-submodules

$U := \{ f \in \text{Hom}_L(V, W) \mid f|_W = 0 \} \subseteq \text{Hom}_L(V, W)$.

For $x \in L$, $w \in W$, $f \in U$:

$(x \cdot f)(w) = x \cdot (f(w)) = f(x \cdot w) = x \cdot (f(w)) = 0$.

so $L \cdot U \subseteq U$ so certainly $U$ and $W$ are $L$-modules.

$\dim \frac{U}{W} = 1$ since $f$ maps to $0 \in \frac{W}{W}$.

If $0 \to W \to V \to \frac{V}{W} \to 0$ splits then

Let $f \in U = \text{Hom}_L(V, W)$ such that $\text{Span} \{ f \} = W'$. Can assume we chose $f$ so that $f|_W = 1 \cdot \text{Id}$

know that $L \cdot f = 0$, so $f \in \text{Hom}_L(V, W) = \text{Hom}_L(V|_W)$.

$0 \to W \to V \to \frac{V}{W} \to 0$

So $c = \text{Id}$ so this splits.

So can assume $\text{codim} W = 1$.

Show: can assume $W$ irreducible:

induction on $\dim W$. 
W not imed. \implies W'

\[ 0 \to W \xrightarrow{w'} V \xrightarrow{v} \mathfrak{l} \to 0 \quad \text{splits by induction} \]

\[ \text{dim } W'/W' < \text{dim } W \]

then \( W'/W' \) contains one-dim lile \( V'/W' \) = complement of \( W'/W' \)

\[ 0 \to W' \xrightarrow{v'} V' \xrightarrow{v} \mathfrak{l} \to 0 \quad \text{splits by induction} \]

\[ \text{dim } W' < \text{dim } W \]

so \( V' \) contains L-invariant line

\[ V' \leq V \] so \( V \) contains L-invariant line

so

\[ 0 \to W \xrightarrow{w} V \xrightarrow{v} \mathfrak{l} \to 0 \quad \text{splits} \]

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New topic: \( \mathfrak{sl}_2 (\mathfrak{k}) \)

A fundamental object in rep theory

First: more on constructing representations.

Last week: tensor product \( V_1 \otimes V_2 \)

2nd symmetric power: consider \( V \otimes V \).

\[ \text{Sym}^2 V = S^2 V = \frac{V \otimes V}{\sim} \]

where \( V \otimes V \sim w \otimes v \).

\( \{ \mathfrak{v}_1, \ldots, \mathfrak{v}_3 \} \) basis for \( V \). Then basis for \( S^2 V \):

\[ \mathfrak{v}_i \cdot \mathfrak{v}_j \quad i \leq j \]

use \( \otimes \) in place of \( \otimes \) for elements of \( S^2 V \).

Has module structure arising from module structure on \( V \otimes V \).

Could write \( S^2 V = V \otimes V / I \) instead.
I is subspace generated by \( \{ v \otimes w - w \otimes v \} \)

\[
X(v \otimes w - w \otimes v) = (X \cdot v) \otimes w + v \otimes (X \cdot w) - (X \cdot w) \otimes v - w \otimes (X \cdot v)
\]

So I closed under \( L \)-action.

I is a submodule of \( V \otimes V \) in \( V \otimes V \) with \( L \)-module.

2nd exterior power: (alternating power)

\[
\Lambda^2 V = \Lambda^2 V = \overline{V \otimes V}
\]

where \( v \otimes v = 0 \)

\[
\Rightarrow v \otimes w \sim -w \otimes v \quad \text{if char } k \neq 2
\]

\( \{ v_1, \ldots, v_n \} \) basis for \( V \). Then basis for \( \Lambda^2 V \):

\[
v_i \wedge v_j \quad i < j
\]

Use \( \wedge \) in place of \( \otimes \) for elements of \( \Lambda^2 V \).

Again, has \( L \)-module structure from \( L \)-module structure on \( V \otimes V \):

\[
X \cdot (v_i \wedge v_j) = (X \cdot v_i) \wedge v_j + v_i \wedge (X \cdot v_j)
\]

HW. Exercise: Show \( V \otimes V \cong S^2 V \oplus \Lambda^2 V \).

Tensor powers:

\[
V^{\otimes 2} = V \otimes V
\]

\[
V^{\otimes 3} = V \otimes V \otimes V
\]

\[
V^{\otimes n} \quad n^{th} \text{ tensor power}
\]

Can generalize \( S^2 V \) to \( S^n V \):

\[
S^n V = \frac{V^{\otimes n}}{\sim}
\]

where \( v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \sim v_{i+1} \otimes \cdots \otimes v_i \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \).
\[ V_1 \otimes \cdots \otimes V_n \cong V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)} \]

for every permutation \( \sigma \in S_n \) of \( \{1, \ldots, n\} \).

Similarly, generalize \( \Lambda^2 V \):

\[ \Lambda^n V = V^{\otimes n} / \sim \]

where

\[ V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_n \cong V_1 \otimes \cdots \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_n \]

Swap \( V_i, V_{i+1} \).

(Recall: \( \det(\sigma) = \pm 1 \)) so have group homomorphism from \( S_n \to \mathbb{Z}_2 \) (cyclic group of order 2)

\[ \Sigma(\sigma) = \text{Sgn}(\sigma) \]

\[ \Sigma(\sigma) = \text{Cycl}(\sigma) \]

\[ \Sigma(\sigma) = (-1)^{d(\sigma)} \]

\[ d(\sigma) = \text{min. # of permutations of the form } (i \ i+1) \text{ required to write } \sigma = \text{product of these} \]

\[ \text{Now, } \mathfrak{s}l_2(\mathbb{C}) \text{ Take & alg. closed, characteristic } 0 \]

we will use \( \mathbb{C} = \mathbb{C} \) in our discussion.

\[ \mathfrak{s}l_2(\mathbb{C}) : \text{ basis } H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

Sometimes \( E \), sometimes \( F \)

\[ [H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \]

Consider an irreducible representation \( V \) of \( \mathfrak{s}l_2(\mathbb{C}) \).

Lie's Theorem applied to \( \text{Span} \{H^k\} = \mathbb{C} v \) \( v \in V \) an eigenvalue of \( H \) (really \( \alpha(H) \) say) but we'll leave about \( \alpha \)'s.

i.e.

\[ H \cdot v = \alpha v \]

We call \( v \) a vector of weight \( \alpha \).
\[ HXV = XHV + 2Xv = 2Xv + 2Xv = (\lambda + 2)Xv \]

\( Xv \) is a vector of weight \( \lambda + 2 \)

\[ HYV = YHV - 2Yv = 2Yv - 2Yv = (\lambda - 2)Yv \]

\( Yv \) is a vector of weight \( \lambda - 2 \)

\( V \) is finitely dimed so can only have finitely many weights (eigenvectors)

So there is some \( k \) so that \( X^{k-1}v \neq 0 \) but \( X^k v = 0 \).
So if some non-zero weight vector \( v \) s.t. \( XV = 0 \).
\( v \) is called a singular vector.

From this \( V \), from the vector space generated by
\[ V, \; YV, \; Y^2V, \ldots \]

Lemma: \( \langle V, \; YV, \; Y^2V, \ldots \rangle \) is an \( \mathbb{K}_2 \) submodule of \( V \).

\( hV = 2v \) for some \( 2 \) since \( v \) is a weight vector

\[ hY^k v = (\lambda - 2k)Y^k v \]

\( h \)-invariant \( \checkmark \)

\( Y \)-invariant: clear.

\( X \)-invariant: induction

Can show \( XY^k v = C_k Y^{k-1} v \) by induction on \( k \).

\( k = 1 \):
\[ XYv = YXv + Hv = 0 + 2v - 2v \]

\( v \) is singular.

General \( k \):
\[ XY^k v = YXY^{k-1} v + HY^{k-1} v \]
\[ = YC_{k-1} Y^{k-2} v + (\lambda - 2(k-1))Y^{k-1} v \]
\[ = C_k Y^{k-1} v \]

\( C_k = C_{k-1} + \lambda - 2(k-1) = k (\lambda - k + 1) \).

So \( \langle v, \; Yv, \; Y^2v, \ldots \rangle \)

For some \( m \), \( Y^m v \neq 0 \) but \( Y^{m+1} v = 0 \).

Basis for \( V \):
\[ V, \; Yv, \; Y^2v, \ldots, \; Y^m v \]

different weights

\[ \dim V = m + 1 \].
Last class: saw irreducible finite dim'l reps. of the form
\[ x^r A x^s B y \]
\[ x^r x^s y \]
\[ x^r y \]
\[ x^r y \]
\[ x^r y \]
\[ x^r y \]
\[ x^r y \]
\[ x^r y \]
\[ x^r y \]
\[ x^r y \]
\[ x^r y \]
\[ x^r y \]

\[ H_v = \lambda_v \]
\[ H Y^k v = (\lambda - 2k) Y^k v \]
\[ X_v = 0 \]

**Theorem:** For any \( m \in \mathbb{Z}^+ \), there exists a unique irreducible \( \mathfrak{sl}_2 \) rep. \( V(m) \) of dimension \( m+1 \).

**Proof:**
- \( m = 1 \):
  \[ V(1) = \mathbb{C}^2 \]
  canonical rep. of \( \mathfrak{sl}_2 (\mathbb{C}) \)
- General \( m \):
  \[ V(m) = S^m V(1) \]

**Basis of \( S^m V(1) \):**
\[ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
\[ j \text{ times} \]
\[ m-j \text{ times} \]

\[ \dim S^m V(1) = m+1 \]

Just need to show irreducibility.

\[ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ Y_{(w \circ v \circ w)} v_1 = v_2, \quad Y v_2 = 0 \]

\[ Y_{(w \circ v \circ w)} \begin{pmatrix} 1 \\ j+1 \\ \vdots \end{pmatrix} = \begin{pmatrix} W_1 \\ Y v_1 \\ \vdots \end{pmatrix} \]

\[ Y_{(w \circ v \circ w)} \begin{pmatrix} 1 \\ j+1 \\ \vdots \end{pmatrix} = \begin{pmatrix} W_1 \\ Y v_1 \\ \vdots \end{pmatrix} + \begin{pmatrix} 1 \\ j+1 \\ \vdots \end{pmatrix} + \begin{pmatrix} 1 \\ j+1 \\ \vdots \end{pmatrix} + \cdots \]

etc.
Apply $Y$ to $v_i^m$ repeatedly. Get all basis vectors
along with $x_i (v_i^j, v_2^{(m-j)}) = (m-j) v_i^j, v_2^{(m-j)}$

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

$H \cdot v_1 = v_1$
$H \cdot v_2 = -v_2$

So $H \cdot (v_i^j, v_2^{(m-j)}) = j v_i^j, v_2^{(m-j)} - (m-j) v_i^j, v_2^{(m-j)}$

\[= (2j - m) v_i^j, v_2^{(m-j)}\]

Weights: $m, m-2, m-4, \ldots, -m$

Singular vector: $v = v_i^m$
$\lambda = m$

We have shown existence.

Uniqueness: Need to show that $\lambda$ in diagram on first page is always $\lambda = m$.

Last class:

$XY^k v = \kappa (\lambda-k+1) Y^{k-1} v$

Applying repeatedly:

$XY^k v = \kappa (\lambda-k+1) (\lambda-k+2) Y^{k-2} v$

$XY^k v = \kappa ! \lambda (\lambda-1) (\lambda-2) \cdots (\lambda-k+1) v$

Now if $\kappa = m+1$, then $Y^{m+1} v = 0$.

So $\kappa (\lambda-1) (\lambda-2) \cdots (\lambda-m) = 0$.

Each $XY^k v$ for $k = 1, \ldots, m$ is non-zero.

Or else you would $Y^k v$

in a singular vector

which generates subrep:

$\left< Y^k v, Y^{k+1} v, \ldots \right>$

So $\lambda, \lambda-1, \ldots, \lambda-m+1$ are non-zero. (See *)

Forces $\lambda = m$ in (*)