Back to Weyl's Theorem:

Prove: For any submodule $W \subset V$ $\exists$ a split exact sequence $0 \to W \hookrightarrow V \twoheadrightarrow W' \to 0$.

First: Enough to Consider W of codimension 1

i.e. $\dim V/W = 1$.

$0 \to W \to V \to (V/W) \to 0$ is exact.

V/W is a L semis so

this must be trivial module by Lemma

Why? Consider L-module $\mathrm{Hom}_{\mathbb{k}}(V, W)$

$\mathcal{V} := \{ f \in \mathrm{Hom}_{\mathbb{k}}(V,W) \mid f|_W = \lambda \mathrm{Id} \}$

$\cup$

$\mathcal{W} := \{ f \in \mathrm{Hom}_{\mathbb{k}}(V,W) \mid f|_W = 0 \}$ $\leftarrow \lambda = 0$.

L-Submodules

For $x \in L$, $w \in W$, $f \in \mathcal{V}$:

$(x \cdot f)(w) = x \cdot (f(w)) - f(x \cdot w) = x \cdot (\lambda w) - \lambda (x \cdot w) = 0$

so $L \cdot \mathcal{V} \subset \mathcal{W}$ so certainly $\mathcal{V}$ and $\mathcal{W}$ are L-modules.

$\dim \mathcal{V}/\mathcal{W} = 1$ since f modulo $\mathcal{W}$ $\longleftrightarrow$ $\lambda \in \mathbb{k}$.

If $0 \to \mathcal{W} \to \mathcal{V} \to \mathbb{k} \to 0$ splits then

$\mathbb{k} \cong W'$

Let $f \in \mathcal{V} \subset \mathrm{Hom}_{\mathbb{k}}(V, W)$ such that

$\mathrm{Span}\{f\} = W'$. Can assume we chose f so that

$f|_W = 1 \cdot \mathrm{Id}$

Know that $L \cdot f = 0$, so $f \in \mathrm{Hom}_L(V, W) = \mathrm{Hom}_{\mathbb{k}}(V,W)^L$.

$0 \to W \xrightarrow[i]{} V \xrightarrow[p]{} V/W \to 0$, with $f : V \to W$

$f \circ i = \mathrm{Id}$ so this splits.

So can assume $\mathrm{codim}\, W = 1$.

Show: can assume W irreducible:

Induction on $\dim W$.

W not irred. $\supset W'$

$$0 \to W/W' \to V/W' \to k \to 0$$ splits (by induction) since $\dim W/W' < \dim W$

then V/W' contains one-dim'l L-invariant line V'/W' = complement of W/W'

$$0 \to W' \to V' \to k \to 0$$ splits by induction since $\dim W' < \dim W$

so V' contains L-invariant line

$V' \subset V$ so V contains L-invariant line

so $0 \to W \to V \to k \to 0$ splits.

New topic: $sl_2(k)$

← fundamental object in rep. theory

First: more on constructing representations.

Last week: tensor product $V_1 \otimes V_2$

2nd symmetric power: Consider $V \otimes V$.

$$\mathrm{Sym}^2 V = S^2 V = \frac{V \otimes V}{\sim}$$

where $v \otimes w \sim w \otimes v$.

$\{v_1, \dots, v_n\}$ basis for V. Then basis for $S^2 V$:

$v_i \cdot v_j$ $\quad i \le j$

↑ use $\cdot$ in place of $\otimes$ for elements of $S^2 V$.

Has module structure arising from module structure on $V \otimes V$:

Could write $S^2 V = \frac{V \otimes V}{I}$ instead

I = subspace generated by $\{ v \otimes w - w \otimes v \}$

$$X \cdot (v \otimes w - w \otimes v) = (X \cdot v) \otimes w + v \otimes (X \cdot w) - (X \cdot w) \otimes v - w \otimes (X \cdot v)$$

So I closed under L-action.

I is a submodule of $V \otimes V$ $\rightsquigarrow$ $\frac{V \otimes V}{I}$ is an L-module

2nd exterior power: (alternating power)

$$Alt^2 V = \Lambda^2 V = \frac{V \otimes V}{\sim}$$

where $v \otimes v \sim 0$

$\Rightarrow v \otimes w \sim -w \otimes v$ if char $k \neq 2$

$\{v_1, \ldots, v_n\}$ basis for V. Then basis for $\Lambda^2 V$:

$v_i \wedge v_j$ $\quad i < j$

↑ use $\wedge$ in place of $\otimes$ for elements of $\Lambda^2 V$.

Again, has L-module structure from L-module structure on $V \otimes V$:

$$X \cdot (v_i \wedge v_j) = (X \cdot v_i) \wedge v_j + v_i \wedge (X \cdot v_j)$$

HW. Exercise: Show $V \otimes V \cong S^2 V \oplus \Lambda^2 V$.

Tensor powers: $V^{\otimes 2} = V \otimes V$ 2nd tensor power

$V^{\otimes 3} = V \otimes V \otimes V$ 3rd —//—

$\vdots$ $\vdots$

$V^{\otimes n}$ n^{th} tensor power

Can generalize $S^2 V$ to $S^n V$:

$S^n V = V^{\otimes n} / \sim$ where $v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n$

Swap v_i, v_j $\rightarrow$ $\sim v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n$

$\rightsquigarrow$ $v_1 \otimes \cdots \otimes v_n \sim v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$

for every permutation $\sigma \in S_n$ of $\{1, \ldots, n\}$.

Similarly, generalize $\Lambda^2 V$:

$$\Lambda^n V = V^{\otimes n} / \sim$$

where $v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \sim - v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n$

(swap v_i, v_{i+1})

$\rightsquigarrow$ $v_1 \otimes \cdots \otimes v_n \sim \operatorname{sgn}(\sigma)\, v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$.

(recall: $\det(\sigma) = \pm 1$ — matrix corresp. to permutation

so have gp. homo. from $S_n \longrightarrow \mathbb{Z}_2$ — cyclic gp. of order 2

$\operatorname{sgn}(\sigma) = \det(\sigma)$

$= (-1)^{\ell(\sigma)}$

$\ell(\sigma)$ = min. # of permutations of the form $(i\ i+1)$ required to write σ = product of these ↑)

Now: $\mathfrak{sl}_2(k)$ Take k alg. closed, characteristic 0. We will use $k = \mathbb{C}$ in our discussion.

$\mathfrak{sl}_2(k)$: basis $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (sometimes E), $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (sometimes F)

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Consider an irreducible fin. dim'l representation V of $\mathfrak{sl}_2(\mathbb{C})$.

Lie's Theorem applied to $\operatorname{Span}\{H\}$ $\Rightarrow \exists\ v \in V$ an eigenvector of H (really $\pi(H)$ say, but we'll leave out π's)

i.e. $H \cdot v = \lambda v$

We call v a weight vector of weight λ.

$HXV = XHv + 2Xv = \lambda Xv + 2Xv = (\lambda+2)Xv$

Xv is a weight vector of weight $\lambda+2$

$HYV = YHv - 2Yv = \lambda Yv - 2Yv = (\lambda-2)Yv$

Yv is a weight vector of weight $\lambda-2$

V is fin. dim'l so can only have finitely many weights (eigenvals)

So there is some k so that $X^{k-1}v \neq 0$ but $X^k v = 0$.

So $\exists$ some non-zero weight vector v s.t. $Xv = 0$.

v is called a <u>singular vector</u>.

From this v, form the vector space generated by

$v, Yv, Y^2v, \ldots$

Lemma: $\langle v, Yv, Y^2v, \ldots \rangle$ is an $\mathfrak{sl}_2$- submodule of V.

$hv = \lambda v$ for some λ since v is a weight vector

$hY^k v = (\lambda - 2k)Y^k v$

h- invariant ✓

Y-invariant: clear.

X- invariant: induction.

Can show $XY^k v = C_k Y^{k-1} v$ by induction on k.

$k=1$: $XYv = (YX)v + Hv = 0 + \lambda v = \lambda v$

(v is singular.)

General k:

$$\begin{aligned} XY^k v &= YXY^{k-1}v + HY^{k-1}v \\ &= Y C_{k-1} Y^{k-2} v + (\lambda - 2(k-1))Y^{k-1}v \\ &= C_k Y^{k-1} v. \end{aligned}$$

$C_k = C_{k-1} + \lambda - 2(k-1) = k(\lambda - k + 1)$.

So $V = \langle v, Yv, Y^2v, \ldots \rangle$

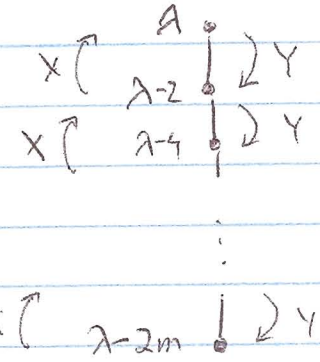
For some m, $Y^m v \neq 0$ but $Y^{m+1} v = 0$

Basis for V: $v, Yv, Y^2v, \ldots, Y^m v$

(different weights)

$\dim V = m+1$.

Last class: saw irreducible finite dim'l rep. are of the form



$$\begin{array}{ccc} & \lambda & \\ X \uparrow & \downarrow Y & \\ & \lambda-2 & \\ X \uparrow & \downarrow Y & \\ & \lambda-4 & \\ & \vdots & \\ X \uparrow & \downarrow Y & \\ & \lambda-2m & \end{array}$$

$\leftarrow$ $m+1$ dim'l rep.

$Hv = \lambda v$ $\qquad$ $HY^k v = (\lambda - 2k) Y^k v$. $\qquad$ weight vectors.

$Xv = 0$.

<u>Theorem:</u> For any $m \in \mathbb{Z}^+$, there exists a unique irred. sl_2-rep. $V(m)$ of dimension $m+1$.

<u>Proof:</u> $m=1$: $\quad V(1) = \mathbb{C}^2$ canonical rep'n of $sl_2(\mathbb{C})$

General m: $\quad V(m) = S^m V(1)$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad$ basis of $V(1)$.

Basis of $S^m V(1)$: $\quad \underbrace{v_1 \cdot v_1 \cdot \cdots \cdot v_1}_{j \text{ times}} \cdot \underbrace{v_2 \cdot \cdots \cdot v_2}_{m-j \text{ times}}$

$\dim S^m V(1) = m+1$.

Just need to show irreducibility.

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$Y v_1 = v_2$. $\qquad$ $Y v_2 = 0$.

$Y \cdot (u \otimes v \otimes w)$
$= (Y \cdot u) \otimes v \otimes w$
$+ u \otimes Y \cdot v \otimes w$
$+ u \otimes v \otimes Y \cdot w$
etc.

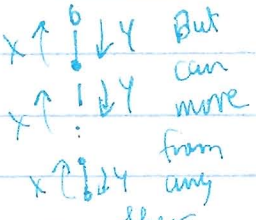
$$Y \cdot \left(v_1^{\cdot j} \cdot v_2^{\cdot (m-j)} \right) = \overset{Yv_1}{v_2} \cdot v_1 \cdot \cdots \cdot v_1 \cdot v_2 \cdots v_2 + \overset{Yv_1}{v_1} \cdot v_2 \cdot \cdots \cdot v_1 \cdot v_2 \cdots v_2$$
$$+ \cdots + v_1 \cdot v_1 \cdot \cdots \cdot \underset{Yv_1}{v_2} \cdot v_2 \cdots v_2 + 0 + \cdots + 0$$
$$= j \, v_1^{\cdot j+1} \cdot v_2^{\cdot (m-j-1)}$$

← Can move from $v_1^{\cdot m}$ to any other node.

Apply Y to $v_1^{\cdot m}$ repeatedly. Get all basis vectors. ∴ irreducible. along with $X \cdot (v_1^{\cdot j} \cdot v_2^{\cdot (m-j)}) = (m-j)\, v_1^{\cdot (j+1)} \cdot v_2^{\cdot (m-j-1)}$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$H \cdot v_1 = v_1$ $\qquad$ $H \cdot v_2 = -v_2$



But can move from any other node to $v_1^{\cdot m}$ too.

So $H \cdot (v_1^{\cdot j} \cdot v_2^{\cdot (m-j)}) = j\, v_1^{\cdot j} \cdot v_2^{\cdot (m-j)} - (m-j)\, v_1^{\cdot j} \cdot v_2^{\cdot (m-j)}$

$= (2j - m)\, v_1^{\cdot j} \cdot v_2^{\cdot (m-j)}$

Weights: $\boxed{m, m-2, m-4, \ldots, -m}$

Singular vector: $v = v_1^{\cdot m}$

$\lambda = m$.

We have shown existence.

Uniqueness: Need to show that λ in diagram on first page is always $\lambda = m$.

Last class: $XY^k v = k(\lambda - k + 1) Y^{k-1} v$

Applying repeatedly: $X^2 Y^k v = k(k-1)(\lambda-k+2)(\lambda-k+1) Y^{k-2} v$

$X^k Y^k v = k!\, \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-k+1)\, v$.

Now if $k = m+1$, then $Y^{m+1} v = 0$

so $\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-m) = 0$. (*)

Each $XY^k v$ for $k = 1, \ldots, m$ is non-zero

$\| $

(*) $k(\lambda-k+1) Y^{k-1} v$

or else you would $Y^k v$ is a singular vector which generates subrep. $\langle Y^k v, Y^{k+1} v, \ldots, \rangle$

So $\lambda, \lambda-1, \ldots, \lambda-m+1$ are non-zero. (see (*))

Forces $\lambda = m$ in (*)