Defn:
V is a representation of L (or \( L \)-module) if we have \( \Phi : L \rightarrow \text{gl}(V) \) Lie algebra homomorphism
\[ \Phi[x,y] = [\Phi(x), \Phi(y)] = \Phi(x)\Phi(y) - \Phi(y)\Phi(x) \]

Notation: \( \Phi(x)v \) or \( xv \) & Above statement in this notation: \( \Phi[x,y]v = (xy - yx)v \)

Defn: \( V \) \( L \)-module, \( U \subset V \) subspace
\( U \) is a submodule if \( xu \in U \) \( \forall x \in L, u \in U \)

Examples:
1) trivial rep \( \Phi = 0 \): every subspace is a submodule
\[ \Phi(x)v = 0 \] \( \forall x \in L, v \in V \)
2) adjoint representation of \( L \) on \( L \): submodules are ideals
3) representation of \( L = \text{gl}(V) \) on \( V \): only submodules are trivial \( \Phi \) and \( V \) itself

\( V \) is irreducible if its only submodules are \( \Phi \) and itself

Constructing representations:
\( V \) \( L \)-module, \( V = U \) submodule
\( U \) is also a \( L \)-module
Quotient $V/U$ is naturally an $L$-module

$x(v + U) = xv + U$ well defined

Can check that $[x, y]$ acts as $xy - yx$

Here we mean thinking of $x, y$ as elements of $\text{gl}(V/U)$ so really $\phi(x) \phi(y) - \phi(y) \phi(x)$

where $\phi: L \rightarrow \text{gl}(V/U)$

$i: U \rightarrow V$ inclusion is a homomorphism of $L$-modules

$\pi: V \rightarrow V/U$ projection is a homomorphism of $L$-modules

\[ \pi(x + U) = \pi(x) + U \]

i.e. $\pi([x, y]) = [\pi(x), \pi(y)]$

$\phi : L \rightarrow \text{gl}(V)$

Kernel of representation $\phi = \text{Ker} \phi = \{x \in L \mid \phi(x) = 0\}$

Ker$\phi \triangleleft L$ ideal

$\phi$ is faithful if Ker$\phi = 0$

$\Rightarrow L \cong$ subalgebra of $\text{gl}(V)$

Theorem (Ado–Iwasawa) If $L$ is a finite dim' Lie algebra, then $L$ admits a faithful finite dim' rep., i.e. $L$ is a subalgebra of some $\text{gl}_n(k)$

We'll apply the above material to the study of nilpotent Lie algebras
L is a Lie algebra.

I, J ideals of L.

Then \([I, J]\) is an ideal of \(L\):

\[
[x, [i, j]] = -\sum_{I} \frac{[i, [x, j]]}{e_i} e_j \quad \text{any element of } L
\]

In particular, \([L, L]\) is an ideal of \(L\).

\([L, [L, L]]\) is an ideal of \(L\).

\[
\begin{align*}
L &> [L, L] > [L, [L, L]] > \cdots \\
&\text{finite dim} L
\end{align*}
\]

so that stabilizes after finitely many steps: consider \(\dim L\).

Notation: \(C^1 L = L\), \(C^2 L = [L, C^{1-1} L]\)

\(\text{so } [L, L] = C^2 L, \quad [L, [L, L]] = C^3 L, \quad \text{etc.}\)

\[
C^1 L \supset C^2 L \supset C^3 L \supset \cdots
\]

Descending central series of \(L\)

Defn: \(L\) is a nilpotent Lie algebra if

\(C^m L = 0\) for some \(m\).

i.e. descending central series stabilizes at \(0\).

\(\text{eg. 1) } \mathfrak{sl}_2(\mathbb{C})\) - only ideals are \(0\)s, \(\mathfrak{sl}_2(\mathbb{C})\)

\(\text{Saw that } [\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C})] = \mathfrak{sl}_2(\mathbb{C})\)

Descending central series is

\[
L > L > 0
\]

stabilizes at \(L\).
$\mathfrak{sl}_2(\mathbb{C})$ is not nilpotent

2) $L$ abelian: $[L, L] = \mathfrak{so}_3$ so descending central series is $L = \mathfrak{so}_3 \supset \mathfrak{so}_2 \supset \cdots$

Abelian $\Rightarrow$ nilpotent

3) 2 dim'l non-abelian Lie algebra:

$\mathfrak{L} = \text{span} \{ e_1, e_2 \}$

$[e_1, e_2] = e_2$

$C^1 L = \text{span} \{ e_1, e_3 \}$

$C^2 L = \text{span} \{ e_2, e_3 \}$

$C^3 L = \text{span} \{ e_3 \}$ from $\mathfrak{L}$ not nilpotent

4) Heisenberg Lie algebra:

$L = \text{span} \{ e_1, e_2, e_3 \}$

$[e_1, e_3] = [e_2, e_3] = 0$

$[e_1, e_2] = e_3$

\[
\begin{pmatrix}
0 & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]


$\begin{pmatrix}
0 & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$

Note: $\begin{pmatrix}
0 & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$

Can also define ascending central series:

$C_0 L = \mathfrak{so}_3$

$\Pi_{i-1} : L \rightarrow L / C_i(L)$ projection

$x \mapsto x + C_i(L)$

preimage

$C_i L = \Pi_{i-1} \left( \mathbb{Z} \left( L / C_i(L) \right) \right)$

Here $\mathbb{Z}( \cdot )$ denotes the centre

Note: preimage of an ideal $A$ in an ideal $B$ is $\{ z \in L \mid \Pi_{i-1}(z) \in \mathbb{Z}( L / C_i(L) ) \}$
Note: \( C_1 L = Z(L) \)
\[
\begin{align*}
& C_0 L \subset C_1 L \subset C_2 L \subset \cdots \quad \text{ascending central series}
\end{align*}
\]

E.g., Heisenberg Lie algebra:
\[
C_1(L) = Z(L) = \text{span} \{ e_3 \}
\]
\[
L/C_1(L) = \text{span} \{ e_1 + C_1(L), e_2 + C_1(L) \}
\]
\[
Z(L/C_1(L)) = \text{span} \{ e_1 + C_1(L), e_2 + C_1(L) \} = \text{span} \{ e_3 + C_1(L) \}
\]
\[
[e_1 + C_1(L), e_2 + C_1(L)] = [e_1, e_2] + C_1(L)
\]
\[
= e_3 + C_1(L) = 0 + C_1(L)
\]

\[
\Pi^{-1} (Z(L/C_1(L))) = L
\]

Ascending central series:
\[
\emptyset \subset \text{span} \{ e_3 \} \subset C_1(L) \subset C_2(L) \subset \cdots
\]

Proposition: The following are equivalent:
1) \( C^m L = \emptyset \) for some \( m \) (i.e. \( L \) nilpotent, by definition)
2) \( C_m L = L \) for some \( m \)
3) For any \( x_1, x_2, \ldots, x_m \in L \),
\[
[x_1, [x_2, [x_3, \ldots, x_m]]] = \text{ad}(x_1) \text{ad}(x_2) \cdots \text{ad}(x_{m-1}) x_m = 0
\]
4) There is a chain of ideals
$L = I_1 \supset I_2 \supset I_3 \supset \ldots \supset I_m = 0$

Such that $[L, I_k] \subset I_{k+1}$ for all $k$.

or equivalently $I_k / I_{k+1} \subset \mathbb{R} (L / I_{k+1})$.

Proof:

1) $\Rightarrow$ 4):

Let $I_k = C^k (L)$.

$[L, C^k (L)] = C^{k+1} (L) \subset C^{k+1} (L)$

$[L, I_k] = I_{k+1} \subset I_{k+1}$ (true).

4) $\Rightarrow$ 3):

$I_1 = L$ so $x_m \in I_1$.

Then $[x_{m-1}, x_m] \in [L, I_2] \subset I_2$.

Then $[x_{m-2}, [x_{m-1}, x_m]] \in [L, I_3] \subset I_3$.

$[x_1, [x_2, [x_3, \ldots, x_m]], \ldots] \in I_m = \{0\}$.

3) $\Rightarrow$ 1):

Can see that $C^k (L) \subset I_k$:

$C^k (L) = L = I_1$.

If $C^k (L) \subset I_k$ then $[L, C^k (L)] \subset [L, I_k] \subset I_{k+1}$.

$C^m (L) \subset I_m = \{0\}$.

1) $\Leftarrow$ 3)

$\Rightarrow$ 4)

1), 3), 4) are equivalent.

4) $\Rightarrow$ 2):

$C_k (L) \supset I_{m-k}$ analogously.

2) $\Rightarrow$ 3):

Similar to above arguments $x \in C_k (L)$ by definition means $[x, L] \subset C_{k+1} (L)$.

Thus $x_m \in C_m (L)$, $[x_{m-1}, x_m] \in C_{m-1} (L)$, $\ldots$,

$[x_0, [x_1, [x_2, [x_3, \ldots, x_{m-1}], \ldots]]] \in C_0 (L) = \{0\}$. 
Exercise: \( L \) nilpotent \( \Rightarrow \) any subalgebra, quotient algebra, is also nilpotent.

But converse is NOT true:
\[ L \triangleright I \text{ ideal}, \quad I, \ L/I \text{ nilpotent} \]
does NOT imply that \( L \) is nilpotent.

\[ [L, L] = \text{span} \{ X^2 \} \]
\[ L = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \text{span} \{ H, X^2 \} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
\[ x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

- Subalgebra of \( \mathfrak{sl}_2(\mathbb{C}) \)
  - but NOT an ideal of \( \mathfrak{sl}_2(\mathbb{C}) \).

\[ [H, x] = 2x \]
so \( [L, L] = \text{span} \{ X^2 \} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \) & 1-dim.\( \Rightarrow \) so \( [L, L] \)
  - is abelian.

\[ [L, [L, L]] = [L, \text{span} \{ X^2 \}] = \text{span} \{ X \} \]
  - nilpotent.

\( L \) is NOT nilpotent.
\( C^1 L = L \supset C^2 L = \text{span} \{ X^2 \} \supset C^3 L = \text{span} \{ X^3 \} \supset \)
  - stabilizes at \( \text{span} \{ X^2 \} \).

\( I = [L, L] \) nilpotent ideal
\( L/I \triangleright 1 \)-dim.\( \Rightarrow \) abelian \( \Rightarrow \) nilpotent.

But \( L \) is NOT abelian nilpotent.

Also:
\( L \) nilpotent, \( \varphi : L \to \text{gl}(V) \) map.
  - then image \( \varphi (L) \) is nilpotent too.
Other facts:

a) \( L \neq 0 \) nilpotent \( \Rightarrow Z(L) \neq 0 \) (i.e., has non-trivial central term of descending central series)

\[ P_f: \text{Central in abelian last non-zero term of descending central series} \]

\[ \text{since } [L, C^{m-1}L] = C^mL = \{0\} \]

b) \( L/Z(L) \) nilpotent \( \Rightarrow L \) nilpotent.

\[ C^mL/Z(L) = \{0\} \text{ i.e. } [L, [L, \ldots, L]] \subset Z(L) \text{ in m+1 times} \]

\[ \Rightarrow [L, [L, \ldots, L]] \subset \{0\} \text{ m+1 times} \]

Flags:

\( V \) - finite dim'ed vector space.

\( F_i \) - flag in \( V \): \( \{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V \)

\[ \dim V_i = i \]

\[ n(F) := \{ x \in \text{gl}(V) : xV_i \subset V_{i-1} \text{ for all } i \} \]

\( n(F) \) is a Lie subalgebra of \( \text{gl}(V) \).

See e.g. on pg. 10
Choose a basis compatible with the flag 
\( \{ e_i \} \) with \( e_i \in V_i \setminus V_{i-1} \).

Then with respect to this basis 
\( x \in n(F) \) is strictly upper triangular

\[
\begin{pmatrix}
0 & * & & & \\
0 & 0 & * & & \\
0 & 0 & 0 & * & \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\( n(F) \) is nilpotent.

See:

\[ [n(F), n(F)] \subset \{ x \in \text{gl}(V) \mid xV_i \subset V_{i-2} \} \]
\[ [n(F), C^2 n(F)] \subset \{ x \in \text{gl}(V) \mid xV_i \subset V_{i-3} \} \]
\[ \vdots \]
\[ C^n n(F) = 0 \]

Engel's Theorem: Let \( L \) be a finite dim'ed Lie alg. \( V \) fin. dim'ed v. s. p.

\[ \pi : L \to \text{gl}(V) \text{ rep. } \pi(\pi(x)) = \text{End}_V(\pi(x)) \text{ nilpotent as endomorphism} \]

If \( \pi(x) \) is nilpotent for any \( x \in L \) then there exists a flag \( F \) of \( V \) s. t. \( \pi(L) \subset n(F) \).

For \( x \).
\( \pi(x) \) nilpotent so \( \exists \) flag \( F_x \) s. t.

\[
\pi(x) V^x_i \subset V^x_{i-1} \quad \text{Jordan basis where } \pi(x) \text{ is triangular}
\]

Theorem says have single flag working for all \( x \).
**Corollary:** (Box's version of Engel's Theorem)

Finite dim'l Lie algebra \( L \) is nilpotent iff. \( \text{ad}(x) \) is nilpotent for any \( x \).

**Proof:** Adjoint representation:

\[
\text{ad} : L \to \text{gl}(L)
\]

**Fi:** \[ 0 = L_0 \subset L_1 \subset \ldots \subset L_n = L \]

\( \text{ad}(x)L_i \subset L_{i-1} \quad \forall x \in L \)

---

**eg. of a flag for an infinite dimensional algebra:**

\( V = \text{polynomials} \)

**Fi:** \( V_0 = \mathcal{S}o(f), \quad V_1 = \text{Span} \{ x^3 \}, \quad V_2 = \text{Span} \{ x, x^2 \}, \quad V_3 = \text{Span} \{ 1, x, x^2 \}, \ldots \)

\( V_0 \subset V_1 \subset V_2 \subset V_3 \subset \ldots \)

\[
\frac{d}{dx} : V_i \to V_{i-1}
\]

\[
\frac{d}{dx} \in \mathcal{N}(\text{Fi})
\]