Lecture #3

07/01/23

- representations
- nilpotent Lie algebras
- Engel's Theorem

Def'n:

V is a <u>representation</u> of L (or <u>L-module</u>) if we have $\varphi : L \to \mathfrak{gl}(V)$ Lie algebra homomorphism.

$$\varphi[x,y] = [\varphi(x), \varphi(y)] = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$$

Notation: $\varphi(x)v$ or xv ← Above statement in this notation: $[x,y]v = (xy - yx)v$

Def'n: V L-module, $U \subset V$ subspace

U is a <u>submodule</u> if $xu \in U$ $\forall x \in L, u \in U$

Examples:

1) trivial rep on any V: every subspace is a submodule

$\varphi(x) \cdot v = 0$ $\forall x \in L, v \in V$

2) adjoint representation of L on L: submodules are ideals

3) representation of $L = \mathfrak{gl}(V)$ on V: only submodules are trivial $\{0\}$ and V itself ← can map from any nonzero vector to another using some linear map

V is <u>irreducible</u> if its only submodules are $\{0\}$ and itself.

Constructing representations:

V L-module, $V \supset U$ submodule

U is also an L-module

Quotient V/U is naturally an L-module

$x(v+U) = xv + U$ & well defined

Can check that $[x,y]$ acts as $\underbrace{xy - yx}$

Here we mean thinking of x, y as elements of $\mathfrak{gl}(V/U)$ so really $\varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ where $\varphi : L \to \mathfrak{gl}(V/U)$

$i : U \hookrightarrow V$ inclusion is a homomorphism of L-modules

$\pi : V \longrightarrow V/U$ projection is a homomorphism of L-modules
$x \longmapsto x+U$

i.e. $\pi([x,y]) = [\pi(x), \pi(y)]$.

<u>Def'n</u>: $\varphi : L \to \mathfrak{gl}(V)$

<u>Kernel</u> of representation φ $= \operatorname{Ker}\varphi := \{x \in L \mid \varphi(x) = 0\}$

$\operatorname{Ker}\varphi \subset L$ ideal

φ is <u>faithful</u> if $\ker\varphi = \{0\}$.

$\Longrightarrow$ $L \cong$ subalgebra of $\mathfrak{gl}(V)$

<u>Theorem</u> (Ado – Iwasawa) If L is a finite dim'l Lie algebra, then L admits a faithful finite dim'l rep. i.e. L is $\cong$ to a subalgebra of some $\mathfrak{gl}_n(k)$

We'll apply the above material to the study of nilpotent Lie algebras.

L Lie algebra

I, J ideals of L.

Then $[I, J]$ is an ideal of L:

$$[x, [i,j]] = -[\underset{\in I}{i}, \underbrace{[j,x]}_{\in J}] - [\underset{\in J}{j}, \underbrace{[x,i]}_{\in I}] \in [I, J]$$

x any element of L

In particular, $[L,L]$ is an ideal of L

$[L,[L,L]]$ is an ideal of L

$\vdots$

$$\boxed{\underbrace{L \supset [L,L]}_{\text{finite dim'l}} \supset [L,[L,L]] \supset \cdots}$$

so this stabilizes after finitely many steps: consider dim.

Notation: $C^1 L = L$, $C^r L = [L, C^{r-1} L]$

(so $[L,L] = C^2 L$, $[L,[L,L]] = C^3 L$, etc.)

$$\boxed{C^1 L \supset C^2 L \supset C^3 L \supset \cdots}$$

<u>Descending central series of L</u>

<u>Def'n</u>: L is a <u>nilpotent</u> Lie algebra if $C^m L = 0$ for some m.

i.e. descending central series stabilizes at $\{0\}$.

eg. 1) $sl_2(\mathbb{C})$ – only ideals are $\{0\}$, $sl_2(\mathbb{C})$

Saw that $[sl_2(\mathbb{C}), sl_2(\mathbb{C})] = sl_2(\mathbb{C})$

Descending central series is

$L \supset L \supset L \supset \cdots$ stabilizes at L

$sl_2(\mathbb{C})$ is NOT nilpotent

2) L abelian. $[L,L] = \{0\}$ so descending central series is $L \supset \{0\} \supset \{0\} \supset \cdots$

Abelian $\Rightarrow$ nilpotent

3) 2 dim'l non-abelian Lie algebra:

$L = \text{span}\{e_1, e_2\}$ $\qquad [e_1, e_2] = e_2$

$C^1 L = \text{span}\{e_1, e_2\}$

$C^2 L = \text{span}\{e_2\}$

$C^3 L = \text{span}\{e_2\}$ from

not nilpotent ✓

4) Heisenberg Lie algebra:

$L = \text{span}\{e_1, e_2, e_3\}$

$[e_1, e_3] = [e_2, e_3] = 0$

$[e_1, e_2] = e_3$

$$\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$L \qquad [L,L] \qquad [L,[L,L]]$ Nilpotent.

Note: $\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ so

Can also define <u>ascending central series</u>:

$C_0 L = \{0\}$ $\qquad \pi_{i-1} : L \longrightarrow L/C_{i-1}(L)$ projection

$x \longmapsto x + C_{i-1}(L)$

$$C_i L = \overset{\text{preimage}}{\pi_{i-1}^{-1}}\left(Z\left(L/C_{i-1}(L) \right) \right)$$ ← Here $Z(\ )$ denotes the centre

Note: preimage of an ideal is an ideal

$$:= \left\{ z \in L \;\middle|\; \pi_{i-1}(z) \in Z\left(L/C_{i-1}(L) \right) \right\}$$

Note: $C_1 L = Z(L)$.

$$\boxed{C_0 L \subset C_1 L \subset C_2 L \subset \cdots}$$ ascending central series

eg. Heisenberg Lie algebra.

$C_1(L) = Z(L) = \text{span}\{e_3\}$.

$L/C_1(L)$: $\text{span}\{e_1 + C_1(L), e_2 + C_1(L)\}$.

$Z(L/C_1(L)) = \cancel{\text{span}\{e}\ L/C_1(L)$ since

$[e_1 + C_1(L), e_2 + C_1(L)] = [e_1, e_2] + C_1(L)$

$= e_3 + C_1(L) = 0 + C_1(L)$

$\pi_1^{-1}(Z(L/C_1(L))) = L$.

Ascending central series:

$$\underset{C_0(L)}{\{0\}} \subset \underset{C_1(L)}{\text{span}\{e_3\}} \subset \underset{C_2(L)}{L} \subset L \subset \cdots$$

<u>Proposition</u>: The following are equivalent:

1) $C^m L = \{0\}$ for some m (i.e. L nilpotent, by definition)

2) $C_m L = L$ for some m

3) For any $x_1, x_2, \ldots, x_m \in L$,

$[x_1, [x_2, [x_1 \cdots x_m]] \cdots] = ad(x_1) ad(x_2) \cdots ad(x_{m-1}) x_m = 0$

4) There is a chain of ideals

$L = I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_m = 0$

Such that $[L, I_k] \subset I_{k+1} \quad \forall k$

or equivalently $I_k / I_{k+1} \subset Z(L/I_{k+1})$.

<u>Proof</u>: 1) $\Rightarrow$ 4): Let $I_k = C^k(L)$.

$[L, C^k(L)] = C^{k+1}(L) \subset C^{k+1}(L)$

$[L, I_k] = I_{k+1} \subset I_{k+1}$ true.

4) $\Rightarrow$ 3): $I_1 = L$ so $x_m \in I_1$.

Then $[x_{m-1}, x_m] \in [L, I_1] \subset I_2$

Then $[x_{m-2}, [x_{m-1}, x_m]] \in [L, I_2] \subset I_3$

$\vdots$

$[x_1, [x_2, [x_3, \cdots x_m] \cdots]] \in I_m = \{0\}$.

3) $\Rightarrow$ 1): Can see that $C^k(L) \subset I_k$:

$C^1(L) = L = I_1$ ✓

If $C^k(L) \subset I_k$ then $\underbrace{[L, C^k(L)]}_{C^{k+1}(L)} \subset [L, I_k] \subset I_{k+1}$ ✓

$C^m(L) \subset I_m = \{0\}$.

$$\begin{tikzcd} 1) \arrow[dr, Rightarrow] & & 3) \arrow[ll, Rightarrow] \\ & 4) \arrow[ur, Rightarrow] & \end{tikzcd}$$

1), 3), 4) are equivalent.

4) $\Rightarrow$ 2): $C_k(L) \supset I_{m-k}$ analogously

2) $\Rightarrow$ 3) Similar to above arguments

$x \in C_k(L)$ by definition means $[x, L] \subset C_{k-1}(L)$.

Thus $x_m \in C_m(L)$, $[x_{m-1}, x_m] \in C_{m-1}(L)$, $\cdots$,

$[x_0, [x_1, [x_2, [x_3, \cdots x_m] \cdots]]] \in C_0(L) = \{0\}$.

<u>Exercise</u>: L nilpotent $\Longrightarrow$ any subalgebra, quotient algebra is also nilpotent.

But converse is <u>NOT</u> true:

$L \supset I$ ideal. I, L/I nilpotent does NOT imply that L is nilpotent,

eg. $L = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \text{span}\{H, X\}$ $\quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

- subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ but NOT an ideal of $\mathfrak{sl}_2(\mathbb{C})$.

$[H, X] = 2X$

so $[L, L] = \text{span}\{X\} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ $\leftarrow$ 1-dim'l so $[L, L]$ is abelian.

$[L, [L, L]] = [L, \text{span}\{X\}] = \text{span}\{X\}$ $\therefore$ nilpotent.

L is <u>NOT</u> nilpotent.

$C^1 L = L \supset C^2 L = \text{span}\{X\} \supset C^3 L = \text{span}\{X\} \supset \cdots$

stabilizes at $\text{span}\{X\}$.

$I = [L, L]$ nilpotent ideal

L/I is 1-dim'l, $\therefore$ abelian $\Rightarrow$ nilpotent.

But L is NOT nilpotent.

Also: L nilpotent, $\varphi : L \to \mathfrak{gl}(V)$ rep.

then image $\varphi(L)$ is <u>nilpotent</u> too.

Other facts:

a) $L \neq 0$ nilpotent $\Rightarrow Z(L) \neq 0$ (i.e. has non-trivial centre)

Pf: $C^{m-1} L \neq 0$ is ~~abelian~~ central

since $[L, C^{m-1} L] = C^m L = \{0\}$.

last non-zero term of descending central series

b) $L/Z(L)$ nilpotent $\Rightarrow L$ nilpotent.

$\Uparrow$

$C^m L/Z(L) = \{0\}$ i.e. $\underbrace{[L,[L, \cdots L]]}_{m \text{ times}} \subset Z(L)$

$\Rightarrow [L, \underbrace{[L, \cdots, L] \cdots]}_{m+1 \text{ times}} \cancel{\subset} \{0\}$.

Flags:

V - finite dim'l vector space

$\mathcal{F}$ - <u>flag in V</u>: $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$

$\dim V_i = i$

$\mathfrak{n}(\mathcal{F}) := \{ x \in \mathfrak{gl}(V) : x V_i \subset V_{i-1} \quad \forall i \}$

$\mathfrak{n}(\mathcal{F})$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

See eg. on pg. 10

Choose a basis compatible with the flag

i.e. $\{e_1, \dots, e_n\}$ with $e_i \in V_i \setminus V_{i-1}$.

Then with respect to this basis

$x \in n(\mathcal{F})$ is strictly upper triangular

$$\begin{pmatrix} 0 & * & \cdots & * \\ & 0 & * & \vdots \\ & & 0 & * \\ 0 & & & 0 \end{pmatrix}$$

nilpotent matrix

$n(\mathcal{F})$ is <u>nilpotent</u>.

See: $[n(\mathcal{F}), n(\mathcal{F})] \subset \{x \in \mathfrak{gl}(V) \mid xV_i \subset V_{i-2}\}$

$[n(\mathcal{F}), C^2 n(\mathcal{F})] \subset \{x \in \mathfrak{gl}(V) \mid xV_i \subset V_{i-3}\}$

$\vdots$

$C^n n(\mathcal{F}) = \{0\}$.

<u>Engel's Theorem</u>: Let L be a finite dim'l Lie alg.
V fin. dim'l v.sp.

$\pi : L \to \mathfrak{gl}(V)$ rep.

$\pi(x) \in \mathfrak{gl}(V) = (\mathrm{End}\, V, [,])$ endomorphism

We mean $\pi(x)$ is nilpotent as ~~antilinear map~~ $(\pi(x))^m = 0$ for some m

If $\pi(x)$ is nilpotent for any $x \in L$ then there exists a flag $\mathcal{F}$ of V s.t. $\pi(L) \subset n(\mathcal{F})$.

(Fix x. $\pi(x)$ nilpotent so $\exists$ flag $\mathcal{F}_x$ s.t.
$\pi(x) V_i^x \subset V_{i-1}^x$ · Jordan basis where $\pi(x)$ is triangular

Theorem says have single flag ~~working~~ that works for all x.)

<u>Corollary</u>: (Book's version of Engel's Theorem)
Finite dim'l Lie algebra L is nilpotent iff. $ad(x)$ is nilpotent for any x.

<u>Proof</u>: Adjoint representation:

$$ad : L \to \mathfrak{gl}(L)$$

$$\mathcal{F}: \quad 0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$$

$$ad(x) L_i \subset L_{i-1} \qquad \forall x \in L$$

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eg. of a flag for an infinite dimensional algebra:

V = polynomials

$\mathcal{F}$: $V_0 = \{0\}$, $V_1 = \mathrm{Span}\{1\}$, $V_2 = \mathrm{Span}\{1, x\}$, $V_3 = \mathrm{Span}\{1, x, x^2\}$, ---

$V_0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots$

$$\frac{d}{dx} : V_i \longrightarrow V_{i-1}$$

$$\frac{d}{dx} \in n(\mathcal{F}).$$