eg. G_2 $\underset{1}{\circ} \Rrightarrow \underset{2}{\circ}$

$$\langle \alpha_1, \alpha_2 \rangle \langle \alpha_2, \alpha_1 \rangle = 3$$

$$\| $$

$$\frac{2(\alpha_1,\alpha_2)}{(\alpha_2,\alpha_2)} \times \frac{2(\alpha_2,\alpha_1)}{(\alpha_1,\alpha_1)}$$

$$\cap \qquad\qquad \cap$$

$$\mathbb{Z} \qquad\qquad \mathbb{Z}$$

3	1
-3	-1
$\boxed{+1}$	$\boxed{3}$
$\boxed{-1}$	$\boxed{-3}$

α_1 short α_2 long.

α_1, α_2 simple so $(\alpha_1, \alpha_2) < 0$.

$$\frac{2(\alpha_1,\alpha_2)}{(\alpha_2,\alpha_2)} = -1 \qquad \frac{2(\alpha_2,\alpha_1)}{(\alpha_1,\alpha_1)} = -3.$$

$$s_2(\alpha_1) = \alpha_1 - \overbrace{\frac{2(\alpha_1,\alpha_2)}{(\alpha_2,\alpha_2)}}^{-1} \alpha_2 = \alpha_1 + \alpha_2$$

$$s_1(\alpha_2) = \alpha_2 - \overbrace{\frac{2(\alpha_2,\alpha_1)}{(\alpha_1,\alpha_1)}}^{+3} \alpha_1 = \alpha_2 + 3\alpha_1$$

etc.

Topic ~~Rest~~ today: finite dim'l reps of semis. L.

Mimic sl_2 - theory

Preliminary discussion:

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \qquad L_\alpha = \{ x \in L \mid [h, x] = \alpha(h) x \ \forall h \in H \}$$

$$= \bigoplus_{\alpha \in \Phi^+} L_\alpha \oplus H \oplus \bigoplus_{\alpha \in \Phi^-} L_\alpha$$

triangular decomposition.

Let $\mathcal{N} = \bigoplus_{\alpha \in \Phi^+} L_\alpha$ $\qquad n^- = \bigoplus_{\alpha \in \Phi^-} L_\alpha$.

e.g. $\mathfrak{sl}_n(\mathbb{C})$

$$\begin{pmatrix} & H & \mathcal{N} \\ n^- & & \end{pmatrix}$$

Let $\mathcal{B} := H \oplus \mathcal{N}$ $\rightsquigarrow$ $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

This is a <u>Borel subalgebra</u>

i.e. maximal solvable subalgebra

Solvable: $[H \oplus N, H \oplus N] \subset \cancel{[H,H]}^{0} \oplus \underbrace{[N,H]}_{\subset N} \oplus \underbrace{[N,N]}_{\subset N}$

$\subseteq N$ nilpotent.

Fact: $\mathcal{N}$ is nilpotent. $\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$

$[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ $\rightsquigarrow$ $[L_{\alpha_1}, [L_{\alpha_2}, [L_{\alpha_3}, \ldots [L_{\alpha_{N-1}}, L_{\alpha_N}]] \ldots]]$

Φ^+ is finite $\qquad \subset L_{\alpha_1+\alpha_2+\cdots+\alpha_N}$

$\Rightarrow$ for N big enough,

$\alpha_1 + \cdots + \alpha_N \notin \Phi^+$ (or in Φ)

Work w/ $L = N \oplus H \oplus N^-$

Let V be a fin. dim'l L-module.

Prop: $\exists$ a singular vector $v \in V$ i.e. $v \neq 0$ s.t.

$\mathcal{N} v = 0$

Pf: B solvable so by Lie's Thm

$\exists\, v \in V$ s.t. $bv = \lambda(b) v \quad \forall\, b \in B$.

$\lambda \in B^*$

$n \in N \Rightarrow \lambda(n) = 0$ $\leftarrow$ Why? Lie's Theorem.

$$\begin{pmatrix} 0 & * & * \\ & \ddots & * \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow nv = 0 \quad \forall\, n \in N$ ✓

So really $\boxed{\lambda \in H^*}$.

V irreducible $\Rightarrow$ V generated by v.

$\langle v, Y_{\alpha_1} v, Y_{\alpha_1}^2 v, \ldots \quad Y_{\alpha_2} v, Y_{\alpha_2}^2 v, \ldots \quad Y_{\alpha_1} Y_{\alpha_2} v \quad \ldots$ etc. $\rangle$

[λ is called the <u>highest weight</u> of V

Recall: $\quad \alpha \in \Phi^+ \rightsquigarrow X_\alpha, Y_\alpha, H_\alpha \leftarrow \mathfrak{sl}_2$.

$$\underset{L_\alpha}{X_\alpha}, \ \underset{L_{-\alpha}}{Y_\alpha}, \ \underset{H}{H_\alpha}$$

$$[X_\alpha, Y_\alpha] = H_\alpha \qquad [H_\alpha, X_\alpha] = 2X_\alpha$$
$$[H_\alpha, Y_\alpha] = -2Y_\alpha.$$

$\Delta = \{\alpha_1, \ldots, \alpha_n\}$ simple roots

$$(\mathfrak{sl}_2)_i := \langle \underset{X_i}{X_{\alpha_i}}, \underset{Y_i}{Y_{\alpha_i}}, \underset{H_i}{H_{\alpha_i}} \rangle$$

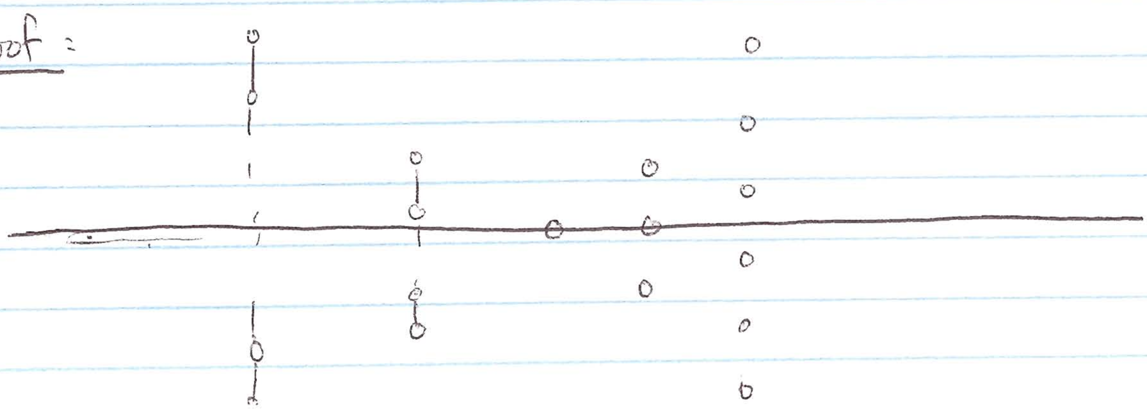
<u>Proposition</u>: λ is highest weight $\Rightarrow$ $\lambda(H_i) \in \mathbb{Z}_{\geq 0}$.

<u>Proof</u>: $L \supset (\mathfrak{sl}_2)_i$ so V is an $(\mathfrak{sl}_2)_i$-module.

$$X_i v = 0 \qquad H_i v = \underbrace{\lambda(H_i)}_{\in \mathbb{Z}_{\geq 0}} v$$

<u>Corollary</u>: If u is a vector in V of weight μ, then
$$\mu = \lambda - \sum_{i=1}^n k_i \alpha_i \qquad k_i \in \mathbb{Z}_{\geq 0}$$

<u>Proof</u>:



$\leftarrow V$ decomposed into $(\mathfrak{sl}_2)_i$ reps.

symmetric about weight 0

V irred. $\Rightarrow$ V generated by singular vector v
$\langle v, Y_{\alpha_1} v, Y_{\alpha_1}^2 v, \dots, Y_2 v, Y_2^2 v, \dots Y_1 Y_2 \rangle$

Proof: If $\mu \neq \lambda$, then there is some X_α s.t.

$X_\alpha u \neq 0$.

$$h(X_\alpha u) = [h, X_\alpha]u + X_\alpha h u$$

$$= \alpha(h) X_\alpha u + \mu(h) X_\alpha u$$

$$= (\mu+\alpha)(h)\ (X_\alpha u)$$

$X_\alpha u$ is a vector of weight $\mu+\alpha$

Repeat. Eventually have $\mu + \alpha_{i_1} + \dots + \alpha_{i_k}$ weight vector annihilated by all X_α's

$\Rightarrow \mu + \alpha_{i_1} + \dots + \alpha_{i_k} = \lambda$ $\square$

Corollary: If μ is a weight of V, then $\mu(H_i) \in \mathbb{Z}$.

Can define partial order on weights:

$$\lambda \geq \mu \iff \lambda - \mu = \sum_i k_i \alpha_i \quad k_i \in \mathbb{Z}_{\geq 0}$$

Result says if μ is a weight of V, then $\mu \leq \lambda$.

Corollary: highest weight vector is unique up to scaling.

Theorem: For each $\lambda \in H^*$, $\lambda(H_i) \in \mathbb{Z}_{\geq 0}$ $i = 1, \dots, n$
$\exists$ a unique L-module $V(\lambda)$ with highest wt. λ.

Won't prove this but we'll discuss the structure of fin. dim'l modules of highest weight λ somewhat.

$$L \rightsquigarrow H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

Joint eigenspace decomposition.

$$V \rightsquigarrow \bigoplus_{\mu \in H^*} V_\mu$$

fin. dim'l L-module

$$V_\mu = \{ u \in V : h \cdot u = \mu(h) u \ \ \forall h \in H \}$$

__Proposition:__ $w \in W$, $\mu \in H^*$

$$\dim V_\mu = \dim V_{w\mu}.$$

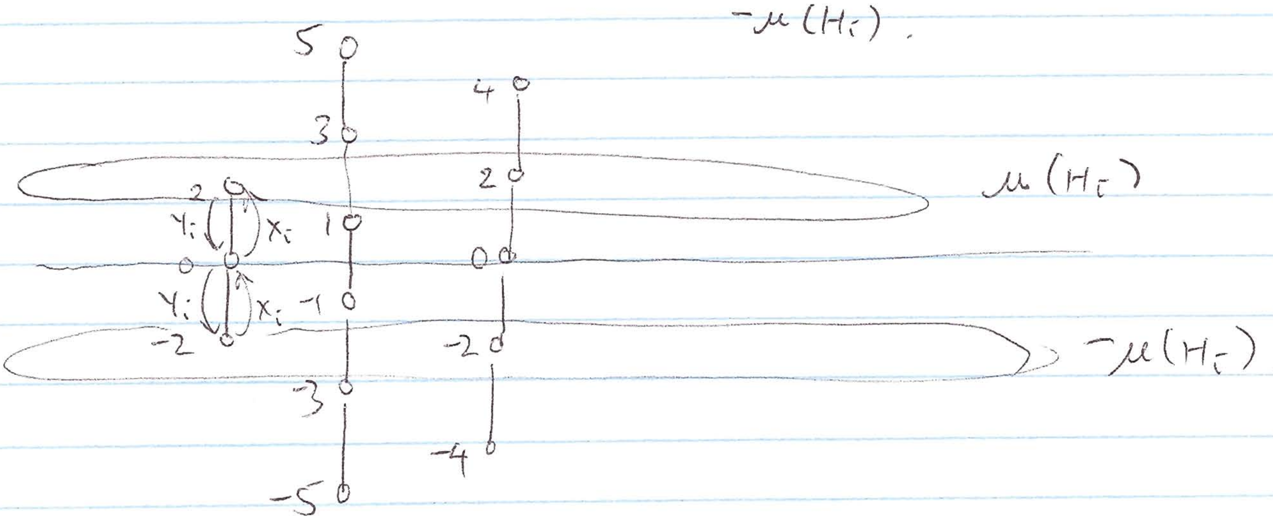
__Proof:__ Enough to prove for $w = s_{\alpha_i} = s_i$

$$s_i \mu = \mu - \frac{2(\mu, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i = \mu - \mu(H_i) \alpha_i$$

Consider V as an $(\mathfrak{sl}_2)_i$ rep.

$V_\mu \subset V_{\mu(H_i)}$ H_i-eigenspace

$V_{s_i \mu} \subset V_{\underbrace{\mu(H_i) - \mu(H_i) \overset{2}{\alpha_i(H_i)}}_{-\mu(H_i).}}$ H_i-eigenspace



Decompose V as $(\mathfrak{sl}_2)_i$-rep.

Symmetric about weight 0.

$\Delta\ \lambda(H_i) = n_i \in \mathbb{Z}_{\geq 0}$. $\quad v \in V_\lambda$ hw vector.

$$\boxed{\begin{array}{lll} \text{Formulas:} & Y_i^{n_i} v & \neq 0 \\ & Y_i^{n_i+1} v & = 0. \\ \text{If } j \neq i, & X_j Y_i^{n_i+1} v & = 0 \end{array}}$$

Why? $\quad X_j Y_i = Y_i X_j + [X_j, Y_i] \in L_{\alpha_j - \alpha_i} = 0$

Not a root $\quad$ Since α_i, α_j are simple.

So $X_j Y_i = Y_i X_j$.

$$X_j Y_i^{n_i+1} v = Y_i^{n_i+1} (X_j v) = 0.$$

Now $\quad X_i Y_i^{n_i+1} v = (n_i+1)(\lambda(H_i) - n_i) Y_i^{n_i} v$

$= 0 \quad$ from $\mathfrak{sl}_2$ theory.

Another way to think of this: $(\mathfrak{sl}_2)_i$ picture.

$$(\lambda - k\alpha_i)(H_i) = n_i - 2k.$$

weight λ — v ← weight n_i

weight $\lambda - \alpha_i$ — $Y_i v$ ← weight $n_i - 2$

$\lambda - 2\alpha_i$ — $Y_i^2 v$ — $n_i - 4$

$\vdots$

$\lambda - n_i\alpha_i$ — $Y^{n_i} v$ — $-n_i$