Theorem: Let $\Delta$ be a base of $\Phi$.

a) $\gamma \in E$ regular $\implies \exists w \in W$ s.t. $w(\gamma) \succ 0 \quad \forall \alpha \in \Delta$ (W acts transitively on Weyl chambers)

b) $\Delta'$ another base $\implies \exists w \in W : w(\Delta') = \Delta$

c) $\alpha \in \Delta = \exists w \in W : w(\alpha) \subset \Delta$

d) $W$ is generated by simple reflections

e) $w(\Delta) = \Delta \implies w = 1$ (W acts simply transitively on Weyl chambers)

Proof: Let $W' \subset W$ be the group generated by simple reflections.

a) Choose $w \in W'$ so that $(w(\gamma), \rho)$ is smallest possible. For $\alpha \in \Delta$:

$$ (s_\alpha w(\gamma), \rho) = (w(\gamma), s_\alpha \rho) = (w(\gamma), \rho - \alpha) $$

Will show on final exam:

$$ (s_\alpha, \rho) = (\gamma, \rho) $$

$w$ is invariant under simple reflections.

$$ \implies (w(\gamma), \alpha) > 0 \quad \forall \alpha \in \Delta $$

$\gamma$ regular $\implies (w(\gamma), \alpha) > 0 \quad \forall \alpha \in \Delta$.

b) This implies $w(\gamma)$ is a fundamental Weyl chamber for $\Delta$. Then if $\Delta' = \Delta(w(\gamma))$, then

$\Delta = \Delta(w(\gamma))$, proving (b).

c) $H_\alpha = H_\beta \iff \beta = \pm \alpha$.

So pick $\alpha$ regular so that $\alpha$ is very close to $H_\alpha$ but far away from $H_\beta$. Do so in such a way that

$$ 0 < (\gamma, \alpha) < \varepsilon $$

but $| (\gamma, \beta) | > \varepsilon$ for some $\varepsilon > 0$. 

\begin{center}
\includegraphics[width=0.5\textwidth]{diagram.png}
\end{center}
Then \( \alpha \in \Delta(\Phi) \) and you can choose \( w \) using (2).
\[
(\Phi, x) > 0 \implies \alpha \in \Phi^+(\Phi)
\]
\[
(\Phi, \beta) > (\Phi, \alpha) \quad \text{for any other } \beta \in \Phi^+(\Phi) \quad \beta \neq \alpha
\]
\[
\implies \alpha \text{ must be indecomposable } \implies \alpha \in \Delta(\Phi).
\]

d) \( W = \langle s_{\alpha} : \alpha \in \Phi \rangle \) so it suffices to show that every reflection \( s_{\alpha} \) \( \alpha \in \Phi \in W' \).

Given \( \alpha \in \Phi \), let \( w = W' \) so that \( w(\alpha) = \beta \in \Delta \setminus \{ \alpha \} \).

Then \( s_{\alpha w} = w s_{\alpha} w^{-1} \)
\[
= s_{\beta} \quad \text{so } \quad s_{\alpha} = w^{-1}s_{\beta}w \in W'.
\]

e) Choose \( w = s_{\alpha} \cdots s_{\alpha} \) so that \( t \) is minimal.

If \( t > 0 \), then the string is non-empty, then

\( W(\alpha_t) < 0 \) - contradiction.

So \( w = 1 \).

We've already seen that root systems have very restrictive properties:

- finite list of possible angles between two roots
- finite # of possible root length ratios.

Let's try to classify all root systems.

**Defn.** \( \Phi \) is **irreducible** if \( \Phi \) cannot be written as

a union of two root systems \( \Phi_1, \Phi_2 \), \( \Phi_1 \perp \Phi_2 \).

eg. \( A_1 \times A_1 \) is reducible, \( A_2 \) is irreducible.

**Proposition.** \( \Phi \) is irreducible \( \iff \Delta \) cannot be decomposed
\[
\Delta = \Delta_1 \cup \Delta_2
\]
\[
\Delta_1 \perp \Delta_2
\]

**Proof:** \( \iff \Delta_1 = \Phi_1 \cap \Delta \quad \Delta_2 = \Phi_2 \cap \Delta \).
\[ \Rightarrow: \quad \Delta = \Delta_1 \cup \Delta_2. \]

Define \( \Delta_1 = \{ x \in \Delta \mid \exists w \in \Delta, \, w \cdot x \in \Delta \} \)
\[ \Delta_2 = \{ x \in \Delta \mid \forall w \in \Delta, \, w \cdot x \in \Delta \}. \]

Claim: \( \Delta_1 \subset \text{Span} \Delta \).

Consider: \( s_x(\Delta) = \Delta - 2 (\Delta, x) x \).

\[ \alpha \in \Delta_1 \Rightarrow s_x(\Delta_1) \subset \text{Span} \Delta_1. \]
\[ \alpha \in \Delta_2 \Rightarrow s_x(\Delta_1) \subset \text{Span} \Delta_1 \quad \text{also since} \quad \Delta_2 \perp \Delta_1. \]

Insert into prev. page:

\[ w = s_{i_1} s_{i_2} \ldots s_{i_t} \quad \text{product of simple reflections} \]

called reduced expression when \( t \) is minimal. Can then write \( l(w) = t \) for the length of \( w \) relative to \( \Delta \).

Another characterization of length:

\[ n(w) = \# \{ \alpha \in \pm \Delta^+: \, w \cdot \alpha < 0 \}. \]

Proposition: \( l(w) = n(w) \)

E.g. For \( \alpha \) simple,
\[ \{ \beta \in \pm \Delta^+: s_x \beta < 0 \} = \{ \pm s_x \}. \]
\[ l(s_x) = 1 = n(s_x). \]

Proof of Proposition: Induction on \( l(w) \).

\[ l(w) = 0, \quad 1 \quad \text{clear.} \]

Suppose the proposition holds for all \( v \) s.t. \( l(v) < l(w) \).

Let \( w = s_{i_1} \ldots s_{i_t} \) be a reduced expression for \( w \). Then
\[ w \cdot \alpha < 0. \]

Now \( w \Delta^+ = s_{i_1} \ldots s_{i_t} \Delta^+ = s_{i_1} s_{i_t}(\Delta^+ \cap s_{i_1}^{-1} \Delta^+ \cup \Delta^+ \setminus s_{i_1}^{-1} \Delta^+) \]
\[ = w s_{i_t} (\Delta^+ \cap s_{i_1}^{-1} \Delta^+ \cup \Delta^+ \setminus s_{i_1}^{-1} \Delta^+) \]
\[ \text{and } s_{i_1} \ldots s_{i_{t-1}} \alpha > 0 \quad (\text{Corollary, from last class}) \]
\[ n(w) = n(ws_{x_t}) + 1 \]

Now \( l(ws_{x_t}) = l(s_{x_t}) = t-1 \implies l(w) = t \)

So by induction, \( l(ws_{x_t}) - n(ws_{x_t}) = t-1 \).

Then \( \diamondsuit \) \( n(w) = t = l(w) \)

End insert

**Lemma:** If \( \Phi \) is irreducible:

1. \( E \) is any fixed subspace of \( N \)
2. \( w\)-orbit of any not \( \alpha \) spans \( E \)

**Proof:** Span \( w\alpha \) is a \( W \)-invariant subspace of \( E \) so (a) \( \implies \) (b).

To prove (a): let \( E' \subset E \) be a non-zero \( W \)-invariant subspace of \( E \), \( E'' \) - orthogonal complement.

For \( \beta \in E \), \( s_{\beta} = 1 \implies s_{\beta} E' = E' \)

and \( W \)-invariance of \( E' \)

\[ \exists \beta \in E \text{ or } E' \subset H_0 \implies \beta \in E'' \]

\[ E = E' \oplus E'' \text{ - decomposed.} \]

partitions \( E \) into two \( \perp \) subspaces

\( \Phi \)-irred. \( E' \neq 0 \implies E'' = 0 \).

\[ \implies E' = E. \]

**Lemma:** If \( \Phi \) is irreducible, then at most 3

- \( W \)-orbits of root lengths occur in \( \Phi \) and
- all roots of a given length are conjugate under \( W \).

**Proof:** \( \forall \alpha, \beta \in \Phi \). Not all \( Wx \perp \beta \) since the \( Wx \) span \( E \) by prev. lemma. So assume \( (\alpha, \beta) \neq 0 \).

Possible squares of root length ratios are \( 1, 2, 3, 4, 1/2, 1/3 \).

If there were 3 or more root lengths, get ratios \( 6, 3, 2, 3/2, 2/3, \text{ or } 1 \)

- contradiction.

If \( \Phi \) is irreducible with two distinct root lengths, roots are
called *short roots* or *long roots*

\[ \Delta \rightarrow \text{Dynkin diagram} \]

\[ \Delta = \{ \alpha_1, \ldots, \alpha_n \} \]

\( n \) vertices labelled 1, \ldots, \( n \)

\( i \text{th} \) and \( j \text{th} \) vertices joined by

\( \langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle \) edges

\[
\frac{\langle \alpha_i, \alpha_j \rangle}{\| \alpha_i \|^2 \| \alpha_j \|^2} = 0, 1, 2, 3
\]

Need more info. to determine which of \( \alpha_1, \alpha_2 \) is long, which \( \alpha_1, \alpha_2 \) is short

Add arrow pointing to shorter root on double and triple edges

Theorem: If \( \Phi \) is an irreducible root system of rank \( n \), (i.e. \( |\Delta| = n \)) then its Dynkin diagram is one of the following:

- \( A_n \):
  \[
  \begin{array}{ccccccc}
  0 & - & - & - & - & - & 0 \\
  1 & 2 & \cdots & n
  \end{array}
  \]

- \( B_n \):
  \[
  \begin{array}{ccccccc}
  0 & - & - & - & - & - & 0 \\
  1 & 2 & \cdots & n-2 & n-1 & n
  \end{array}
  \]
Type $A_n$, $B_n$, $C_n$, $D_n$ are infinite families of classical root systems.

$E_6$:

```
  0  1  2  3  4  5  6
```

$E_7$:

```
  0  1  2  3  4  5  6  7
```

$E_8$:

```
  0  1  2  3  4  5  6  7  8
```

$F_4$:

```
  0  1  2  3  4
```

$G_2$:

```
  0  1  2
```

Next class: Finite dim'l reps of semi-simple Lie algebras.