Recall:

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

Φ = root system of (L, H) $\rightsquigarrow$ abstract root system Φ

$\alpha \in \Phi \quad \rightsquigarrow \quad s_\alpha(\lambda) = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$

reflection through hyperplane $\perp$ to α.

$\alpha, \beta \in \Phi \Rightarrow s_\alpha(\beta) \in \Phi$

Reflections $s_\alpha : \Phi \circlearrowleft$ $\quad$ Φ preserved

$\{s_\alpha\}_{\alpha \in \Phi}$ generate a group called the <u>Weyl group</u> W

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\pi/2$	?
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1.
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

<u>Corollary</u>: Let α, β be nonproportional roots i.e. $\beta \neq c\alpha$.

① If $(\alpha,\beta) > 0$ then $\alpha - \beta$ is a root.

② If $(\alpha,\beta) < 0$ then $\alpha + \beta$ is a root. (Follows from ①)

<u>Pf</u> of ①: Either $\langle\alpha,\beta\rangle = 1$ or $\langle\beta,\alpha\rangle = 1$ from table.

$\Downarrow$

$s_\beta(\alpha) = \alpha - \langle\alpha,\beta\rangle\beta = \alpha - \beta \in \Phi$

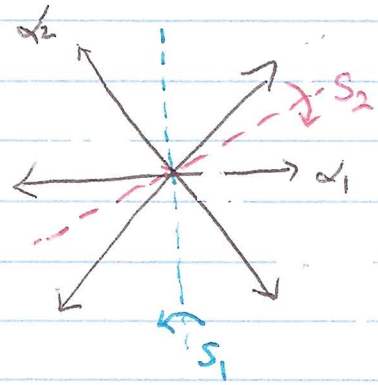
$\rightsquigarrow s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha = \beta - \alpha \in \Phi$

$\Rightarrow \alpha - \beta \in \Phi$.

<u>Properties</u>: - W is finite

- $(\alpha, \beta) = 0 \Rightarrow s_\alpha s_\beta = s_\beta s_\alpha$
- $w s_\alpha w^{-1} = s_{w\alpha}$

eg- A_2



s_1 short for s_{α_1}

Elements of W:

e, s_1, s_2.

$s_1^2 = e \quad s_2^2 = e$.

$s_2 s_1$ = rotate by $\frac{2\pi}{3}$ clockwise

$s_1 s_2$ = rotate by $\frac{2\pi}{3}$ counterclockwise

$(s_2 s_1)^{-1}$

use $s_i^{-1} = s_i$ from $s_i^2 = 1$

it is its own inverse

Can see $s_1 s_2 s_1 = s_2 s_1 s_2$: $\alpha_1 \mapsto -\alpha_2$, $\alpha_2 \mapsto -\alpha_1$

Note: this is the same as refl. s_3 through remaining root hyperplane. See line 3 of this page.

From $\triangle$: $s_1 s_2 s_1 s_2 s_1 s_2 = e$

Strings in s_1, s_2 of length ≥ 4 can be rewritten to have length ≤ 3.

$\therefore W = \{ e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1 \}$

A concrete realization of this:

Think of root system associated to sl_3 $\quad E_{ij} \leftrightarrow e_i - e_j$

$E = \mathrm{Span}\{e_1 - e_2, e_2 - e_3\} \subset \mathbb{R}^3$

$(\ ,\ )$ = usual dot product

$t_{\alpha_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \leftarrow t_{\alpha_2}$

$e_i \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = h_i$

$\alpha_1 = e_1 - e_2$

$\alpha_2 = e_2 - e_3$

$\alpha_3 = e_1 - e_3$

See picture on next page

α_1

$$s_1\begin{pmatrix}1\\-1\\0\end{pmatrix} = -\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$ ← from $s_\alpha(\alpha) = -\alpha$

$$\boxed{s_1(e_1 - e_2) = e_2 - e_1}$$

$$s_1\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix} - \frac{2\begin{pmatrix}0\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\underbrace{\cancel{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}}_{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\boxed{s_1(e_2 - e_3) = e_1 - e_3}$$

$$s_2\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix} - 2\frac{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}}{\underbrace{\cancel{\begin{pmatrix}0\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}}}_{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\boxed{s_2(e_1 - e_2) = e_1 - e_3}$$

$$s_2\underbrace{\begin{pmatrix}0\\+1\\-1\end{pmatrix}}_{\alpha_2} = -\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\boxed{s_2(e_2 - e_3) = e_3 - e_2}$$

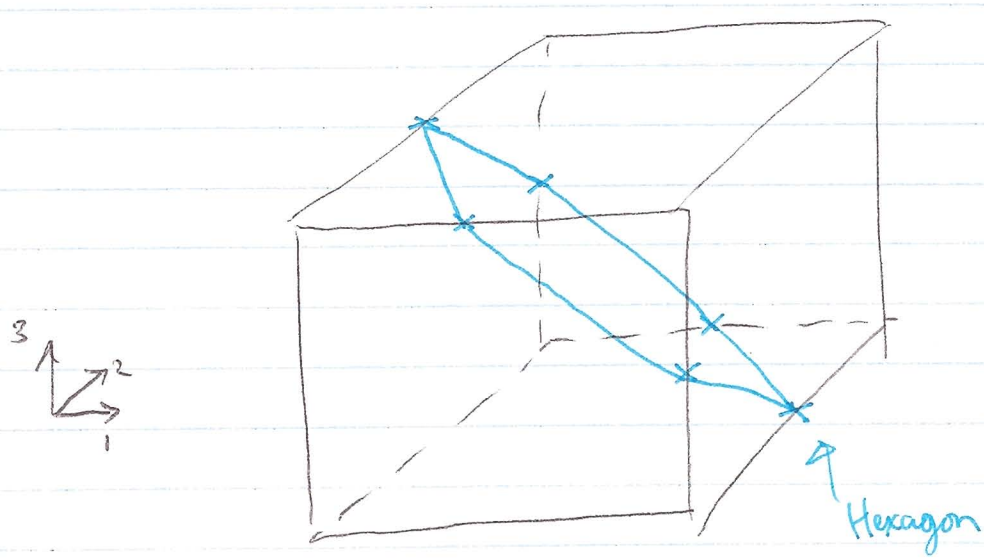
s_1 : swap indices 1, 2

s_2 = swap indices 2, 3.

s_1, s_2 : generate group of permutations of $\{1,2,3\} = S_3$

$s_1 \longleftrightarrow (12)$

$s_2 \longleftrightarrow (23)$



Note: $W = \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2 s_1)^2 = (s_2 s_1 s_2)^2 = 1 \rangle$

generator s_3 is redundant.

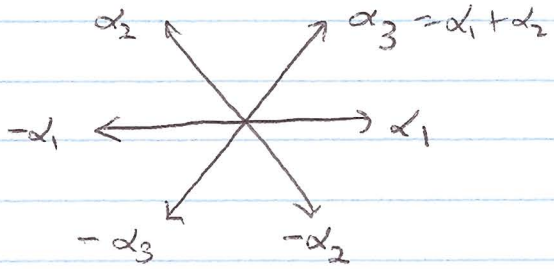
Related to idea of a <u>base</u>:

<u>Def'n</u>: $\Delta \subset \Phi$ is a <u>base</u> if

B1) Δ is a basis of E

B2) $\forall \beta \in \Phi$, $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ Either: all $c_\alpha \geq 0$ OR all $c_\alpha \leq 0$.

eg.

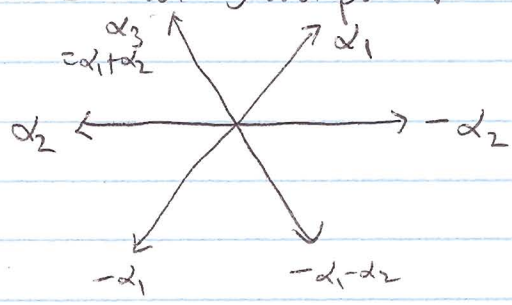


$\Delta = \{\alpha_1, \alpha_2\}$ is a base.

$\alpha_3 = \alpha_1 + \alpha_2$.

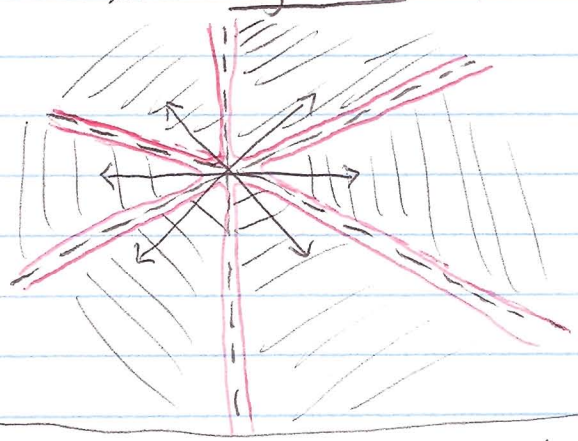
Bases are not unique.

eg.



$\gamma \in E$ is <u>regular</u> if $(\gamma, \alpha) \neq 0 \quad \forall \alpha \in \Phi$

eg.



← anywhere but hyperplanes = regular elements.

$E \setminus \bigcup_\alpha H_\alpha$ = regular elements

$H_\alpha = \{x \in E : (x, \alpha) = 0\}$

Connected components are called <u>Weyl chambers</u>

If γ regular, then either $(\gamma, \alpha) > 0$ or $(\gamma, \alpha) < 0$.

$\Phi = \Phi_+(\gamma) \cup \Phi_-(\gamma)$ $\qquad$ $\Phi_+(\gamma) = \{\alpha \in \Phi : (\alpha, \gamma) > 0\}$

$\Phi_-(\gamma) = \{ -\!\!\shortparallel\!\!- \quad (\alpha, \gamma) < 0\}$

$\alpha \in \Phi_+(\gamma)$ then $-\alpha \in \Phi_-(\gamma)$.

$\Delta(\gamma)$ = set of $\underbrace{\text{indecomposable}}_{\text{cannot be expressed as sum of two roots of } \Phi_+(\gamma)}$ roots in $\Phi_+(\gamma)$

<u>Theorem</u>: $\Delta(\gamma)$ is a base and any base is of this form.

<u>Proof</u>: in steps.

1) Every $\alpha \in \Phi^+(\gamma)$ is $\mathbb{Z}$-linear comb. of elems. of $\Delta(\gamma)$.
Suppose not, and suppose (γ, α) minimal w.r.t. this property.

$$\alpha = \beta_1 + \cdots + \beta_k \qquad \beta_i \in \Phi^+(\gamma)$$

$$(\gamma, \alpha) = \underbrace{(\gamma, \beta_1) + \cdots + (\gamma, \beta_k)}_{\text{smaller than } (\gamma, \alpha)}$$

so β_i's must be in $\mathbb{Z}$-span $\Delta(\gamma)$

But then $\alpha = \beta_1 + \cdots + \beta_k \in$ —$\shortparallel$— too

— contradiction.

2) $\alpha, \beta \in \Delta(\gamma)$ $\quad \alpha \neq \beta \Rightarrow (\alpha, \beta) \leq 0$.
(i.e. angle between two roots in base $\geq 90°$)

If $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Phi$ (see Corollary on first page)

If $\alpha - \beta \in \Phi_+(\gamma)$ then

$\alpha = (\alpha - \beta) + \beta$ — but α not decomposable.

If $\beta - \alpha \in \Phi_+(\gamma)$ then

$\beta = (\beta - \alpha) + \alpha$ — but β not decomposable

– contradiction both cases. so $(\alpha,\beta) \leq 0$.

3) $\Delta(\gamma)$ linearly independent.

Suppose not. Then $\sum_{\alpha \in \Delta(\gamma)} r_\alpha \alpha = 0$ for some constants r_α.

Split r_α's into >0 and <0. $s_\alpha > 0$, $t_\beta > 0$.

Then $\sum s_\alpha \alpha = \sum t_\beta \beta =: \varepsilon$

$$(\varepsilon,\varepsilon) = \sum_{\alpha,\beta} s_\alpha t_\beta (\alpha,\beta) \leq 0$$

$(x,x) \geq 0$ always with equality only when $x=0$

so: $\varepsilon = 0 \Rightarrow (\gamma,\varepsilon) = \sum s_\alpha \underbrace{(\gamma,\alpha)}_{>0} = 0$

so <u>all</u> $s_\alpha = 0$.

Similarly, all $t_\beta = 0$, so all $r_\alpha = 0$.

We've now shown $\Delta(\gamma)$ is a basis (B1) and B2) follows from 1)

4) All bases of Φ are of the form $\Delta(\gamma)$ for some regular γ.

For a base Δ of Φ, choose γ s.t.

$(\gamma,\alpha) > 0 \quad \forall \alpha \in \Delta$ (possible – Exercise).

$\Delta = \Delta(\gamma)$.

$\Delta(\gamma)$ therefore only depends on which connected component of $E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ γ belongs to.

Conclusion: bases $\xleftrightarrow{1-1}$ Weyl chambers.

If $(\gamma, \alpha) > 0 \quad \forall \alpha \in \Delta$ (base)
then the Weyl chamber containing γ is called the fundamental Weyl chamber wrt. Δ.

Elements of Δ are called <u>simple roots</u>
$\alpha \in \Delta$ $\quad s_\alpha$ called <u>simple reflection</u>

$\boxed{W \text{ is generated by simple reflections}}$

Show later. First some lemmas:

<u>Lemma A:</u>
If $\alpha \in \Phi^+$ is not simple, then $\alpha - \beta \in \Phi^+$ for some $\beta \in \Delta$.

<u>Proof:</u> Recall that $(\alpha, \beta) > 0 \Rightarrow \alpha - \beta \in \Phi$.
Enough to prove $(\alpha, \beta) > 0$ for some $\beta \in \Delta$.
If $(\alpha, \beta) \leq 0$ for every $\beta \in \Delta$, then α is linearly independent to Δ by step 3) of our previous proof where we showed that the properties:
1) $\Delta(\gamma)$ consists of positive roots by def'n
2) (β_1, β_2) for any $\beta_1 \neq \beta_2 \in \Delta(\gamma)$
$\rightsquigarrow \Delta(\gamma)$ linearly indep.

This shows $\alpha - \beta \in \Phi$.
$\alpha - \beta \in \Phi^+$ since $\beta - \alpha \notin \Phi^+$
or $\beta = \beta - \alpha + \alpha$ is indecomposable.

<u>Corollary:</u> Any $\beta \in \Phi^+$ can be written as a sum of simple roots (not necessarily uniquely)
$$\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_k, \qquad \alpha_i \in \Delta$$
such that $\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_k \in \Phi^+$

<u>Lemma B:</u> If α is simple, then s_α permutes $\Phi^+ \setminus \{\alpha\}$.

<u>Proof:</u> Write $\beta \in \Phi^+ \setminus \{\alpha\}$ as $\beta = \sum_{\gamma \in \Delta} r_\gamma \gamma \quad r_\gamma \in \mathbb{Z}_{\geq 0}$.

For some γ; $r_\gamma > 0$. Now

$$s_\alpha \beta = \beta - \langle \beta, \alpha \rangle \alpha$$ only changes coefficient in front of α

so r_γ for $s_\alpha \beta$ doesn't change - still >0.

One coefficient >0 $\Rightarrow$ all coefficients ≥ 0

so $s_\alpha \beta \in \Phi^+$.

$s_\alpha(\alpha) = -\alpha$ while $s_\alpha(s_\alpha \beta) = \beta \neq -\alpha$

so $s_\alpha \beta \in \Phi^+ \setminus \{\alpha\}$.

<u>Corollary</u>: Let $\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$ ← very important element of H^*.

Then $s_\alpha \rho = \rho - \alpha$.

<u>Lemma C</u>: Given $\alpha_1, \ldots, \alpha_t \in \Delta$ (not necessarily distinct)

If $s_1 s_2 \cdots s_{t-1}(\alpha_t) < 0$, then for some index $1 \leq j < t$,

$$s_1 s_2 \cdots s_t = s_1 \cdots s_{j-1} s_{j+1} \cdots s_{t-1}.$$

<u>Proof</u>: Let $\beta_i = s_{i+1} \cdots s_{t-1}(\alpha_t)$, $0 \leq i \leq t-2$.

Let $\beta_{t-1} = \alpha_t$.

$\beta_0 = s_1 \cdots s_{t-1}(\alpha_t) < 0$ but $\beta_{t-1} = \alpha_t > 0$

so there is a smallest index j such that

$\beta_j > 0$.

$s_j \beta_j = s_j s_{j+1} \cdots s_{t-1}(\alpha_t) = \beta_{j-1} < 0$

so by Lemma B, $\beta_j = \alpha_j$ simple

Recall that for $w \in W$, $s_{w\alpha} = w s_\alpha w^{-1}$.

$$\alpha_j = \beta_j = \underbrace{s_{j+1} \cdots s_{t-1}}_{w} \underbrace{(\alpha_t)}_{\alpha}$$

$$s_j = s_{j+1} \cdots s_{t-1} \, s_t \, s_{t-1} \, s_{t-2} \cdots s_{j+1}$$

Thus
$$s_1 \cdots s_j \cdots s_t = s_1 \cdots s_{j-1} \left(s_{j+1} \cdots \underbrace{s_t \, s_{t-1} \cdots s_{j+1}} \right) \underbrace{s_{j+1} \cdots s_t}$$

$$= s_1 \cdots s_{j-1} \, s_{j+1} \cdots s_{t-1}$$

<u>Corollary</u>: If $w = s_1 \cdots s_t$ is such that t is as small as possible, then $w(\alpha_t) < 0$.

[Note: leads to definition for <u>length</u> of elements of Weyl group. $s_1 \cdots s_t$ is called a <u>reduced expression</u> form]

<u>Theorem</u>: Let Δ be a base of Φ.

a) $\gamma \in E$ regular $\Rightarrow \exists\, w \in W$ s.t. $(w(\gamma), \alpha) > 0 \quad \forall \alpha \in \Delta$.

b) Δ' another base $\Rightarrow \exists\, w \in W : w(\Delta') = \Delta$.

c) $\alpha \in \Phi \Rightarrow \exists\, w \in W \quad w(\alpha) \in \Delta$

d) W is generated by simple reflections

e) $w(\Delta) = \Delta \Rightarrow w = 1$

(i.e. Weyl group acts simply transitively on bases)

<u>Proof</u>: next week.