Recall:

\[ L = H \oplus \bigoplus_{\alpha \in \Phi} L_{\alpha} \]

\[ \Phi = \text{root system of } (L, H) \Rightarrow \text{abstract root system } \Phi \]

\[ \alpha \in \Phi \quad \Rightarrow \quad S_{\alpha}(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \alpha \]

Reflection through hyperplane \( \perp \alpha \).

\[ \alpha, \beta \in \Phi \quad \Rightarrow \quad S_{\alpha}(\beta) \in \Phi \]

Reflections \( S_{\alpha} \) of \( \Phi \) preserved.

\[ \{ S_{\alpha} \}_{\alpha \in \Phi} \] generate a group called the Weyl group \( W \).

<table>
<thead>
<tr>
<th>( \langle \alpha, \beta \rangle )</th>
<th>( \langle \beta, \alpha \rangle )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{2\pi}{3} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{3\pi}{4} )</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{5\pi}{6} )</th>
<th>( \frac{|\beta|^2}{|\alpha|^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>\frac{\pi}{2}</td>
<td>?</td>
<td>1</td>
<td>1.</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>\frac{2\pi}{3}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>\frac{\pi}{4}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
<td>\frac{3\pi}{4}</td>
<td>1</td>
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<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
<td>\frac{\pi}{6}</td>
<td>1</td>
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<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>-3</td>
<td>\frac{5\pi}{6}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Corollary: Let \( \alpha, \beta \) be nonproportional roots i.e. \( \beta \neq \alpha \).

1. If \( \langle \alpha, \beta \rangle > 0 \) then \( \alpha - \beta \) is a root.
2. If \( \langle \alpha, \beta \rangle < 0 \) then \( \alpha + \beta \) is a root. (Follows from 1)

Proof of 1: Either \( \langle \alpha, \beta \rangle = 1 \) or \( \langle \beta, \alpha \rangle = 1 \) from table.

\[ S_{\alpha}(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta - \alpha \in \Phi \]

\[ S_{\alpha}(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta = \alpha - \beta \in \Phi \]
Properties:
- $W$ is finite
- $(c_1, c_2) = 0 \Rightarrow s_{c_1}s_{c_2} = s_{c_2}s_{c_1}$
- $w_s w^{-1} = s_w$

Elements of $W$:
- $e, s_1, s_2, s_3$
- $s_1^2 = e, s_2^2 = e$
- $s_3^2 = e$

$s_2 s_1 = \text{rotate by } \frac{2\pi}{3}$
- clockwise

$s_1 s_2 = \text{rotate by } \frac{2\pi}{3}$
- counter-clockwise

$(s_2 s_1)^{-1}$

It is its own inverse.

Can see $s_1 s_2 s_1 = s_2 s_1 s_2 \Rightarrow \alpha_1 \mapsto -\alpha_2$
- $\alpha_2 \mapsto -\alpha_1$

Note: this is the same as refl. $s_3$ through remaining rot hyperplane. See lines of this page.

From $A_2$: $s_1 s_2 s_1 s_2 s_1 s_2 = e$

Strings in $s_1 s_2$ of length $\geq 4$ can be rewritten to have length $\leq 3$.

$W = \{ e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1 s_2 \}$

A concrete realization of this:

$E = \text{Span}\{e_1 - e_2, e_2 - e_3\} \subset \mathbb{R}^3$

$
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix}
= e_2 - e_1$

$E_i(h_i, h_2) = h_3$

$\alpha_1 = e_1 - e_2$

$\alpha_2 = e_2 - e_3$

$\alpha_3 = e_1 - e_3$

Thick of root system associated to $\alpha_3$ $\Rightarrow e_1 - e_3$

$E_{ij} = e_i - e_j$

See picture on next page
$S_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$S_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - 2 \left( \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$S_1 \begin{pmatrix} e_1 - e_2 \end{pmatrix} = e_2 - e_1$

$S_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 2 \left( \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$S_2 \begin{pmatrix} e_2 - e_3 \end{pmatrix} = e_1 - e_3$

$S_2 \begin{pmatrix} e_2 - e_3 \end{pmatrix} = e_2 - e_3$

$S_2 \begin{pmatrix} e_2 - e_3 \end{pmatrix} = e_3 - e_2$

$S_1$: Swap indices 1, 2.

$S_2$: Swap indices 2, 3.

$S_1, S_2$: generate group of permutations of $\{1, 2, 3\} = S_3$

$S_1 \leftarrow (12)$

$S_2 \leftarrow (23)$

Hexagon
Note: \( W = \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2 s_1)^2 = (s_2 s_1 s_2)^2 = 1 \rangle \)

Related to idea of a base:

**Def'n:** \( \Delta \subset \Phi \) is a base if

\( \beta_1 \) \( \Delta \) is a basis of \( E \)

\( \beta_2 \) \( \forall \beta \in \Phi, \beta = \sum_{\alpha \in \Delta} c_{\alpha} \alpha \)

Either: all \( c_{\alpha} \geq 0 \)

or all \( c_{\alpha} \leq 0 \).

**eg.** \( \alpha_2 \)

\( -\alpha_1 \)

\( -\alpha_3 \)

\( \alpha_3 = \alpha_1 + \alpha_2 \)

\( \Delta = \{ \alpha_1, \alpha_2 \} \) is a base.

\( \alpha_3 = \alpha_1 + \alpha_2 \).

Bases are not unique.

**eg.** \( \alpha_2 \)

\( \alpha_1 \)

\( -\alpha_1 \)

\( -\alpha_2 \)

\( -\alpha_3 \)

\( \alpha_3 = \alpha_1 + \alpha_2 \)

\( \Delta = \{ \alpha_1, \alpha_2, \alpha_3 \} \) is a base.

\( \alpha_3 = \alpha_1 + \alpha_2 \).

\( \gamma \in E \) in regular if \( (\gamma, \alpha) \neq 0 \) \( \forall \alpha \in \Phi \)

**eg.**

\( \Delta \) anywhere but hyperplanes

\( = \) regular elements

\( E \setminus \bigcup_{\alpha} H_\alpha = \) regular elements

\( H_\alpha = \{ x \in E : (x, \alpha) = 0 \} \)

Connected components are called Weyl chambers.
If $\gamma$ regular, then either $(\gamma, x) > 0$ or $(\gamma, x) < 0$.

$\Phi = \Phi_+(x) \cup \Phi_-(x)$

$\Phi_+(x) = \{ x \in \Phi : (x, x) > 0 \}$

$\Phi_-(x) = \{ x \in \Phi : (x, x) < 0 \}$

$\alpha \in \Phi_+(x)$ then $-\alpha \in \Phi_-(x)$.

$\Delta(x) =$ set of indecomposable roots in $\Phi_+(x)$

cannot be expressed as sum of two roots of $\Phi_+(x)$.

**Theorem:** $\Delta(x)$ is a base and any base is

of this form.

**Proof:** In steps.

1) Every $\alpha \in \Phi^+(x)$ is $\mathbb{Z}$-linear comb of elements of $\Delta(x)$.

Suppose not, and suppose $(\gamma, x)$ minimal with this property.

$\alpha = \beta_1 + \ldots + \beta_k$

$\beta_i \in \Phi^+(x)$

$(\gamma, \alpha) = (\gamma, \beta_1) + \ldots + (\gamma, \beta_k)$

smaller than $(\gamma, x)$

so $\beta_i$ s must be in $\mathbb{Z}$-span $\Delta(x)$

But then $\alpha = \beta_1 + \ldots + \beta_k \in \Phi^+(x)$ too

-- contradiction --

2) $x, y \in \Delta(x)$ $x \neq y \Rightarrow (x, y) \leq 0$

(i.e. angle between two roots in base $\geq 90^\circ$)

If $(x, y) > 0$ then $x - y \in \Phi$ (Corollary) 

If $x - y \in \Phi^+(x)$ then

$x = (x - y) + y$ -- but $x$ not decomposable.

If $y - x \in \Phi^+(x)$ then

$y = (y - x) + x$ -- but $x$ not decomposable.
Contradiction both cases. So \( \Delta(x) \leq 0 \).

3) \( \Delta(x) \) linearly independent.

Suppose not. Then \( \sum_{\alpha \in \Delta(x)} r_{\alpha} \alpha = 0 \) for some constants \( r_{\alpha} \).

Split \( r_{\alpha} \)'s into \( > 0 \) and \( < 0 \).

Then \( \sum_{\alpha} a_{\alpha} \alpha = \sum_{\beta} t_{\beta} \beta \) = \( \varepsilon \)

\((\varepsilon, \varepsilon) = \sum_{\alpha, \beta} a_{\alpha} t_{\beta} (\alpha, \beta) \leq 0\)

\((\alpha, \alpha) \geq 0 \) always with equality only when \( \alpha = 0 \).

So: \( \varepsilon = 0 \) \( \Rightarrow \) \( (\varepsilon, \varepsilon) = \sum_{\alpha} a_{\alpha} (\alpha, \alpha) = 0 \)

\( \Rightarrow \) all \( a_{\alpha} = 0 \).

Similarly, all \( t_{\beta} = 0 \), so all \( r_{\alpha} = 0 \).

We've now shown \( \Delta(x) \) is a basis (B1) and (B2) follows from 1).

4) All bases of \( \mathfrak{A} \) are of the form \( \Delta(x) \) for some regular \( x \).

For a base \( \Delta \in \mathfrak{A} \), choose \( x \) s.t.

\( (x, x) > 0 \) \( \forall \alpha \in \Delta \) (possible - Exercise).

\( \Delta = \Delta(x) \).

\( \Delta(x) \) therefore only depends on which connected component of \( E \setminus \bigcup_{x \in E} H_{-} \) \( x \) belongs to.

Conclusion: bases \( \xrightarrow{1-1} \) Weyl chambers.
If \((\gamma, \gamma) > 0\) \(\forall \gamma \in \Delta\),
then the Weyl chamber containing \(\gamma\) is called the fundamental Weyl chamber w.r.t. \(\Delta\).

Elements of \(\Delta\) are called simple roots \(\alpha \in \Delta\) & called simple reflection \(W\) is generated by simple reflections.

Show later. First some lemmas:

Lemma A:
If \(\alpha \in \Phi^+\) is not simple, then \(\alpha - \beta \in \Phi^+\) for some \(\beta \in \Delta\).

Proof: Recall that \((\alpha, \beta) > 0 \Rightarrow \alpha - \beta \in \Phi^+\).

Enough to prove \((\alpha, \beta) > 0\) for some \(\beta \in \Delta\).

If \((\alpha, \beta) \leq 0\) for every \(\beta \in \Delta\), then \(\alpha\) is linearly independent to \(\Delta\) by step 3 of our previous proof where we showed that the properties:
1) \(\Delta(\beta)\) consists of positive roots by def'n.
2) \((\beta, \beta) > 0\) for every \(\beta \in \Delta\).
\(\Rightarrow\) \(\Delta(\beta)\) linearly indep.

Corollary: Any \(\beta \in \Phi^+\) can be written as \(\alpha\) a sum of simple roots (not necessarily uniquely):
\[\beta = \alpha_1 + \alpha_2 + \ldots + \alpha_k, \quad \alpha_i \in \Delta\]

such that \(\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \ldots + \alpha_k \in \Phi^+\).

Lemma B: If \(\alpha\) is simple, then so permutes \(\Phi^+ \setminus \{\alpha\}\).

Proof: Write \(\beta \in \Phi^+ \setminus \{\alpha\}\) as \(\beta = \sum_{\gamma \in \Delta} \gamma \in \mathbb{Z}^\alpha\).
For some \( \delta \), \( r_\delta > 0 \). Now
\[
S_\delta \beta = \beta - \langle \beta, \alpha \rangle \alpha
\]
only changes coefficient in front of \( \alpha \)

so \( r_\delta \) for \( S_\delta \beta \) doesn't change - still \( > 0 \).

One coefficient \( > 0 \) \( \Rightarrow \) all coefficients \( \geq 0 \)

so \( S_\delta \beta \in \overline{\mathbb{H}}^+ \).

\( S_\delta (\alpha) = -\alpha \) while \( S_\delta (S_\delta \beta) = \beta \neq \alpha \)

so \( S_\delta \beta \in \overline{\mathbb{H}}^+ \setminus 2\overline{\mathbb{H}}. \)

**Corollary:** Let \( \rho = \frac{1}{2} \sum \beta \) \( \in \) element of \( \mathbb{H}^+ \).

Then \( S_{\rho} \beta = \rho - \alpha \).

**Lemma C:** Given \( \alpha_1, \ldots, \alpha_t \in \Delta \) (not necessarily distinct)

If \( S_1 S_2 \cdots S_{t-1} (\alpha_t) < 0 \), then for some index \( 1 \leq j < t \),

\( S_1 S_2 \cdots S_t = S_1 \cdots S_{j-1} S_{j+1} \cdots S_{t-1} \).

**Proof:** Let \( \beta_i = S_{i+1} \cdots S_{t-1} (\alpha_t) \), \( 0 \leq i \leq t-2 \).

Let \( \beta_{t-1} = \alpha_t \).

\( \beta_0 = S_1 \cdots S_{t-1} (\alpha_t) < 0 \) but \( \beta_{t-1} = \alpha_t > 0 \)

so there is a smallest index \( j \) such that \( \beta_j > 0 \).

\( S_j \beta_j = S_j S_{j+1} \cdots S_{t-1} (\alpha_t) = \beta_{j-1} < 0 \)

so by Lemma \( \beta_j \), \( \beta_j = \alpha_j \) simple...
Recall that for \( w \in W \), \( s_{w x} = w s_x w^{-1} \).

\[
\alpha_j - \beta_j = \underbrace{s_{j+1} \cdots s_{t-1}}_{W}(\alpha_t) \alpha
\]

\[
S_j = s_{j+1} \cdots s_{t-1} s_t s_{t-1} s_{t-2} \cdots s_{j+1}
\]

Thus
\[
S_i \cdots S_j \cdots S_t = S_i \cdots S_{j-1} \underbrace{(S_{j+1} \cdots S_t s_{t-1} \cdots s_{j+1})}_{S_{j+1} \cdots S_t}
\]

\[
= S_i \cdots S_{j-1} S_{j+1} \cdots S_{t-1}
\]

**Corollary:** If \( w = s_i \cdots s_t \) is such that \( t \) is as small as possible, then \( W(\alpha_t) < 0 \).

(Note: leads to definition for length of elements of Weyl group. \( s_i \cdots s_t \) is called a reduced expression for \( w \).

**Theorem:** Let \( \Delta \) be a base of \( \Phi \).

a) \( \sigma \in E \) regular \( \Rightarrow \exists w \in W \) s.t. 
\[
(w(\sigma), \alpha) > 0 \quad \forall \alpha \in \Delta.
\]

b) \( \Delta' \) another base \( \Rightarrow \exists w \in W : w(\Delta') = \Delta. \)

c) \( \alpha \in \Phi \Rightarrow \exists w \in W \quad w(\alpha) \in \Delta \)

d) \( W \) is generated by simple reflections

e) \( w(\Delta) = \Delta \Rightarrow w = 1 \)

(i.e. Weyl group acts simply transitively on bases)

**Proof:** next week.