Lecture #2 b) 07/01/19

Saw last class : L 1-dim'l $\Rightarrow$ L abelian

What are the 2-dim'l Lie algebras up to isomorphism?

(1) L abelian , $\dim L = 2$

(2) L is NOT abelian:

$L = \text{span}\{e_1, e_2\}$

Necessarily $[e_1, e_1] = [e_2, e_2] = 0$

$[e_1, e_2] = ae_1 + be_2 \neq 0$ WLOG, $b \neq 0$.

Set $\underbrace{e_1' = \frac{1}{b} e_1 , \quad e_2' = [e_1, e_2]}_{\text{still linearly ind.}} = ae_1 + be_2$.

$[e_1', e_2'] = \frac{1}{b}[e_1, ae_1 + be_2] = \frac{1}{b} \cdot a \cancel{[e_1, e_1]}^{0} + [e_1, e_2]$

$= e_2'$

Up to isomorphism, 2-dim'l Lie algebras are : abelian or have some basis $\{e_1, e_2\}$ such that $[e_1, e_2] = e_2$.

Lecture #2

Some constructions of Lie algebras:

$(A, *)$ an algebra, I a subspace of A

I is a <u>(left) ideal</u> of A if and only if

Define right, two-sided ideal analogously

$x * y \in I \quad \forall x \in A, \ y \in I$

$B \subset A$ is a <u>subalgebra</u> if and only if
$B * B \subset B$.

Remark: an ideal is also a subalgebra.

Note: ideals in a Lie algebra must be two-sided.

Pf: Let $I \subset L$ be a left ideal. Then

$$\underset{-[y,x]}{\overset{\|}{[x,y]}} \in I \quad \forall x \in I, \ y \in L$$

Since I is a subspace of A, therefore

$+[y,x] \in I \quad \forall x \in I, \ y \in L$

$\Rightarrow I$ is a right ideal also.

Similarly, if I is a right ideal, then it must be a left ideal also. $\square$

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L Lie algebra, I an ideal.

(1) I is a subalgebra of L.

s, l for special linear

sum of diagonal entries

eg. $sl_2(\mathbb{C})$: 2x2 matrices of trace 0

Basis: $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$[X,Y] \overset{XY-YX=}{=} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H$$

$$[H,X] \overset{HX-XH=}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 2X$$

Since $X^t = Y$, by taking transposes,
$[H,Y] = -2Y$ from

Fact: $sl_2(\mathbb{C})$ has no non-trivial ideals.

Proof: If $H \in I$ a non-zero ideal of $sl_2(\mathbb{C})$ then:

$[H,X] = 2X \in I \quad \Rightarrow X \in I$

$[H,Y] = -2Y \in I \quad \Rightarrow Y \in I$

Basis vectors $X, Y, H \in I \Rightarrow I = sl_2(\mathbb{C})$.

Now assume $0 \neq v \in I$ an ideal of $sl_2(\mathbb{C})$.

$v = aX + bY + cH$ for some scalars a, b, c.

At least one scalar not zero.

If $a \neq 0$: $[H, v] = 2aX - 2bY \in I$

$\Rightarrow [2aX - 2bY, Y] = 2aH \in I \Rightarrow H \in I$

$\Rightarrow I = sl_2(\mathbb{C})$.

If $b \neq 0$: again by taking transposes, $\Rightarrow I = sl_2(\mathbb{C})$.

If $c \neq 0$: $[v, X] = [aX + bY + cH, X] = -2bH + \underbrace{2c}_{\neq 0}X \in I$

So by the case $a \neq 0$, $\Rightarrow I = sl_2(\mathbb{C})$.

Only ideals of $sl_2(\mathbb{C})$ are $\{0\}$, $sl_2(\mathbb{C})$.

(2) The quotient of L by I $\quad L/I$
is a Lie algebra.
What is the quotient of L by I?
Sets of the form $\quad x + I \quad\quad x \in L$.
Note that $\quad x+I = x+i+I \quad$ for every $i \in I$.
i.e. $\quad x+I = y+I \quad$ for $x, y \in L$
is equivalent to $x - y \in I$.
Another way of stating this:
L/I is L modulo the relation $\sim$ $\quad (L/\sim)$
where $x \sim y \iff x - y \in I$.

Can check that by defining
$$[x+I, y+I] = [x,y] + I$$
we get a Lie algebra structure on L/I.

(3) Direct sums:
$$\underbrace{W = U \oplus V}_{U,\,V\text{ subspaces of }W} \qquad \text{direct sum of vector spaces}$$

means for every $w \in W$, there exist unique $u \in U, v \in V$
so that $w = u + v$.
In other words, we can think of W as
the set of pairs $\{(u,v) \mid u \in U, v \in V\}$

$L_1 \oplus L_2$ for Lie algebras $(L_1, [\,,\,]_1)$, $(L_2, [\,,\,]_2)$
with
$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1]_1, [x_2, y_2]_2)$$
$$x_i, y_i \in L_i \qquad i = 1, 2$$
is a Lie algebra.
L_1 and L_2 can be thought of as subalgebras

(and ideals) of $L_1 \oplus L_2$.

L_1 : $\{ (x,0) \mid x \in L_1 \}$

$$[(x,0),(y,0)] = ([x,y],0) \quad \forall x,y \in L_1$$

so this is clearly a subalgebra.

$$[(x,0),(y_1,y_2)] = ([x,y_1],[0,y_2]) = ([x,y_1],0)$$

$\forall\ x, y_1 \in L_1, \quad y_2 \in L_2$

So it is an ideal.

What if:

L Lie algebra and

$L = I \oplus A$ as vector spaces, I ideal, A subalgebra.

If we know Lie alg. structures of I, A, how much do we need to know to understand fully the Lie algebra structure of L?

Know: $[I,I]$, $[A,A]$.

How to take Lie brackets of an element of I with an element of I, an element of A with an element of A.

Need to know: $[A,I]$ (which will tell us $[I,A]$ also by anticommutativity).

For every $a \in A$, have map

$$\underbrace{ad(a)\big|_I}_{\text{Recall this is a derivation}} : I \longrightarrow I$$

$$i \longmapsto [a,i]$$

$\theta = \mathrm{ad}|_A : A \longrightarrow \mathrm{Der}(I)$

Lie algebra homomorphism

describes Lie bracket of elements of A with elements of I.

So in summary:

We knew I, A's Lie algebra structures.

Only need to understand $\theta : A \to \mathrm{Der}(I)$ Lie alg homo.

~~that~~ $\rightsquigarrow$ Lie alg structure of L.

Vice versa:

I, A are Lie algebras.
$\theta : A \longrightarrow \mathrm{Der}(I)$ Lie alg homo.

$\rightsquigarrow$

$L = I \oplus A$ as vector ~~[illegible]~~ spaces

a Lie algebra with ideal I, subalgebra A.

(Know how to define $[\,,]$ on I and $[\,,]$ on A.
$[a, i] = \theta(a)\, i$ for $a \in A$, $i \in I$
takes care of $[A, I]$, $[I, A]$.)

~~more and $[e_i, e_j] = -[e_j, e_i]$ for $i,j = 1, \dots, n$~~

~~i.e. $C_{ij}^k = -C_{ji}^k$ for every i,j,k~~

Definition:

$$C(L) = \{ x \in L \mid [x,y] = 0 \ \forall y \in L \}$$

is the <u>centre of L</u>.

Note: $C(L)$ is the kernel of $\mathrm{ad}: L \to L$.

$$\mathrm{ad}(x)(y) = 0 \quad \forall y \in L$$
$$\iff [x,y] = 0 \quad \forall y \in L.$$

Why <u>centre</u>? Suppose L comes from an associative algebra $(A, *)$.

$[x,y] = x*y - y*x$ ← commutator of x and y

so $[x,y] = 0$ means $x*y = y*x$.

(Kernels of Lie algebra homomorphisms are ideals.)

Often, if you take the quotient by an ideal defined by some property, then the quotient doesn't have that property.

Does $L/C(L)$ have trivial centre $\{0\}$?

Not necessarily ...

Consider the 3-dim'l Heisenberg algebra

basis $\{e_1, e_2, e_3\}$

$$[e_1, e_3] = [e_1, e_2] = 0$$

$$[e_1, e_2] = e_3$$

$$I := C(L) = \text{Span}\{e_3\}$$

$$[e_1 + I, e_2 + I] = [e_1, e_2] + I = e_3 + I = I.$$

$L/C(L)$ is <u>abelian</u> so its centre isn't trivial.

Heisenberg algebra $\subset gl_3(\mathbb{C})$.

$$\underset{e_1}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \quad \underset{e_2}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} \quad \underset{e_3}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$