Lecture #2a 1

Describing / Finding Finite Dimensional Lie algebras

L = Lie algebra (these are vector spaces too) with basis $\{e_1, e_2, \dots, e_n\}$ & $\dim L = n$

$$[e_i, e_j] = \sum_k C_{ij}^k \, e_k$$

The constants C_{ij}^k completely determine the Lie algebra structure on L since $[,]$ is linear in each variable.

i.e. $x = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$

$y = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n$.

We know exactly what $[x,y]$ should be:

$$[x,y] = [a_1 e_1 + a_2 e_2 + \cdots + a_n e_n, \; b_1 e_1 + \cdots + b_n e_n]$$

$$= \left[\sum_{i=1}^n a_i e_i, \sum_{j=1}^n b_j e_j\right]$$

$$= \sum_{1 \le i,j \le n} a_i b_j [e_i, e_j]$$

$$= \sum_{1 \le i,j,k \le n} a_i b_j C_{ij}^k \, e_k$$

Because $[,]$ is bilinear, to check if $[x,x] = 0$ and if Jacobi identity holds for all of L, enough to check certain (almost) identical properties for the basis elements.

~~2) Jacobi identity is linear in each variable, so~~

i.e. The constants C_{ij}^k which describe $[,]$ define a Lie alg.

$\Leftrightarrow$

1) $[e_i, e_i] = 0$ for $i = 1, \dots, n$

and $[e_i, e_j] = -[e_j, e_i]$ for $i, j = 1, \dots, n$

[i.e. $C_{ii}^{k} = 0$ for $i, k = 1, \dots, n$

$C_{ij}^{k} = -C_{ji}^{k}$ for $i, j, k = 1, \dots, n$]

2) $[e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 0$

for all indices i, j, k

[i.e. $\sum_{l=1}^{n} \left(C_{il}^{m} C_{jk}^{l} + C_{jl}^{m} C_{ki}^{l} + C_{kl}^{m} C_{ij}^{l} \right) = 0$]

Comments: 2) $[x,[y,z]]$ is linear in each of the variables x, y, z so enough to check Jacobi identity on basis elements.

1) $[x,x]$ is linear in x

[It does not follow from bilinearity of $[,]$ that] so ~~we need to check~~ not enough to check $[x,x] = 0$ for basis elements.

eg. The 3-dimensional Heisenberg algebra

$L = \text{span}\{e_1, e_2, e_3\}$

$[e_1, e_3] = [e_1, e_2] = 0$

$[e_1, e_2] = e_3$

$e_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad e_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

embedding of Heisenberg algebra into $\mathfrak{gl}_3(\mathbb{C})$

Can check:

$$[e_1, e_2] = e_1 e_2 - e_2 e_1 = \begin{pmatrix} 0 1 0 \\ 0 0 0 \\ 0 0 0 \end{pmatrix}\begin{pmatrix} 0 0 0 \\ 0 0 1 \\ 0 0 0 \end{pmatrix} - \begin{pmatrix} 0 0 0 \\ 0 0 1 \\ 0 0 0 \end{pmatrix}\begin{pmatrix} 0 1 0 \\ 0 0 0 \\ 0 0 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 0 1 \\ 0 0 0 \\ 0 0 0 \end{pmatrix} = e_3 \qquad \text{etc.}$$

eg. $L = \text{span}\{e\}$ 1-dim'l Lie algebra.

Forced to have $[e, e] = 0$.

$[ae, be] = ab\,[e, e] = 0 \qquad \forall\, a, b \in k$

1-dim'l Lie algebra is <u>abelian</u>

Abelian: $[x, y] = 0 \qquad \forall\, x, y \in L$.

Why abelian? If L comes from an associative algebra, $[x, y] = \underbrace{x*y - y*x}$

$= 0$ if x, y commute w.r.t. multiplication in the associative algebra