MATH 410/510 SOLVED PROBLEMS 3

1. Show that if $f \geq 0$ is improper Riemann integrable on $[0, \infty)$ then f is λ-integrable there and $\int_{[0,\infty)} f\, d\lambda = \int_0^\infty f(x)\, dx$.

Soln. Suppose f is improper Riemann integrable on $[0, \infty)$. Then, by definition, for all $b > 0$, f is Riemann intrable on $[0, b]$ and $\lim_b \int_0^b f\, dx$ exists in $\mathbb{R}$. This limit is what is meant by $\int_0^\infty f(x)\, dx$.

By the connection between Riemann and Lebesgue integrability, we know therefore that for each $b > 0$, f is Lebesgue integrable on $[0, b]$, with $\int_{[0,b]} f\, d\boldsymbol{\lambda} = \int_0^b f(x)\, dx$. Now, since $\int_0^b f(x)\, dx$ is increasing as a function of b, we can just use limits over natural numbers: $\int_0^\infty f(x)\, dx = \lim_n \int_0^n f(x)\, dx$.

Put $f_n = f\mathbf{1}_{[0,n]}$. Then $0 \leq f_n(x) \nearrow f(x)$, for all $x \in [0, \infty)$, so by the Monotone Convergence Theorem $\int_0^\infty f(x)\, dx = \lim_n \int_0^n f(x)\, dx = \lim_n \int_{[0,n]} f\, d\boldsymbol{\lambda} = \lim_n \int f_n\, d\boldsymbol{\lambda} = \int f\, d\boldsymbol{\lambda}$, as required □

2. If $|f|$ is improper Riemann integrable on $[0, +\infty)$, then f is λ-integrable on $[0, \infty)$, but not necessarily improper Riemann integrable. On the other hand, f can be improper Riemann integrable but not Lebesgue integrable.

Soln. Consider $f : [0, +\infty) \to \mathbb{R}$, defined by $f(x) = \begin{cases} e^{-x}, & \text{if } x \text{ is rational} \\ -e^{-x}, & \text{if } x \text{ is irrational} \end{cases}$
Then, f is not Riemann integrable on $[0, b]$, for any choice of $b > 0$. But $|f|(x) = e^{-x}$, for all x.
Thus, $\int_0^b |f|(x) = 1 - e^{-b} \xrightarrow[b\to\infty]{} 1$, so $\int |f|\, d\boldsymbol{\lambda} = \int_0^\infty |f(x)|, dx = 1$.

On the other hand, the function $f(x) = \sum_{n=1}^\infty (-1)^n \frac{1}{n^2} \mathbf{1}_{(n-1,n]}$ has improper Riemann integral $\sum_{n=1}^\infty (-1)^n 1/n$, which converges by the alternating series test, but cannot be Lebesgue integrable, because then $|f|$ would be also, and this has integral $\sum_{n=1}^\infty \frac{1}{n}$ which diverges.

3. For a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, if f is monotone, then f is Borel measurable.

Proof. We assume, without loss of generality, that f is increasing.

We show that for all $c \in \mathbb{R}$, $(f < c) = f^{-1}((-\infty, c)) \in \mathcal{B}(\mathbb{R})$, for then f will be Borel measurable, since the family of sets $(-\infty, c)$ for $c \in \mathbb{R}$ generates the Borel sets.

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Let $c \in \mathbb{R}$. If $x \in (f < c)$, then $f(x) < c$ and for all $t < x$, $f(t) < f(x) < c$, so $t \in (f < c)$ also.
This shows that $(f < c)$ is an infinite interval.
More precisely, put $b = \sup(f < c)$. If $f(b) < c$, we have $(f < c) = (-\infty, b]$.
If $f(b) \geq c$, then $(f < c) = (-\infty, b)$.
Thus, in any case, for $c \in \mathbb{R}$, $(f < c)$ is a Borel set, so f is $\mathcal{B}(\mathbb{R})$-measurable. □

4. Let $\mathcal{S}$ be a σ-algebra in a space S. Let μ be a σ-finite measure on $\mathcal{S}$. (Thus, there exists a sequence (K_n) of elements of $\mathcal{S}$ with $\mu(K_n)$ finite for all n, such that $\bigcup_{n \in \mathbb{N}} K_n = S$). If $\mathcal{H}$ is a disjoint family of sets with $\mu(A) > 0$, for all $A \in \mathcal{H}$, then $\mathcal{H}$ is countable.

Proof.

Let $\mathcal{H}$ be as stated. Choose a sequence (K_n) as stated. For each $n \in \mathbb{N}$, let $\mathcal{H}_n = \{A \in \mathcal{H} : \mu(A \cap K_n) > 0\}$. Then $\mathcal{H} = \bigcup_{n \in \mathcal{H}} \mathcal{H}_n$. So it is enough to prove $\mathcal{H}_n$ is countable. For a fixed $\varepsilon > 0$, if $A_1, A_2, \ldots, A_N$, with $\mu(A_i \cap K_n) > \varepsilon$, for each i, we have $\mu(K_n) \geq \sum_{i=1}^{N} \mu(A_i \cap K_n) > N\varepsilon$, so $\mathcal{H}_n$ can have no more than $\mu(K_n)/\varepsilon$ elements with $\mu((A \cap K_n) > \varepsilon$. Thus,

$$\mathcal{H}_n = \bigcup_{k \in \mathbb{N}} \{A \in \mathcal{H}_n : \mu(A \cap K_n) > 1/k\},$$

a countable union of finite families of sets, so $\mathcal{H}_n$ is itself countable. □

5. Let A be a Lebesgue measurable set of finite measure. For each $\varepsilon > 0$, show that there exists an open set G with $\lambda(A \triangle G) < \varepsilon$. Then, improve this to show there exists U, a finite union of open intervals, with $\lambda(A \triangle U) < \varepsilon$.

Proof. Let $\varepsilon > 0$. By the open outer-regularity of λ, we have the existence of an open set such $G \supset A$ and $\lambda(G \setminus A) < \varepsilon$. But for such a G, $G \triangle A = G \setminus A$, so this finishes the first part. (Note: actually, one doesn't even need finite measure for this part. If you go back to where we proved there is a G_δ set containing A with the same measure, you will see that it is the sigma-finiteness of λ that guarantees the existence of such a G.)

Now, if G is an open set in $\mathbb{R}^n$, then there exist a countable family $\{I_n : n \in \mathbb{N}\}$ of open intervals with $G = \bigcup_{n\in\mathbb{N}} I_n$. Then $(G \setminus \bigcup_{i\leq n} I_i) \searrow \emptyset$, so if G is of finite measure, we have $\lim_n \lambda(G \setminus \bigcup_{i\leq n} I_i)) = \lambda(\emptyset) = 0$, thus there exists n such that $\lambda(G \setminus \bigcup_{i\leq n} I_i)) = \lambda(\emptyset) < \varepsilon$. For such n, put $U = \bigcup_{i\leq n} I_i$. Then U is a finite union of open intervals with $G \supset U$ and $\lambda(G \setminus U) < \varepsilon$.

Now, let $\varepsilon > 0$ and let A have finite measure. Choose an open set $G \supset A$, with $\lambda(G \setminus A) < \varepsilon/2$. Then $\lambda(G)$ is finite. Choose a finite union U of open intervals with $G \supset U$ and $\lambda(G \setminus U) < \varepsilon/2$. Then,

$$\lambda(A \triangle U) = \lambda(A \setminus U) + \lambda(U \setminus A) \leq \lambda(G \setminus U) + \lambda(G \setminus A) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

6. Prove that $\mathcal{B}(\mathbb{R}^2) = \sigma(\{B_1 \times B_2 : B_1, B_2 \text{ Borel sets of } \mathbb{R}\})$. (Suggestion: The identity map on $\mathbb{R}^2$ can be written (π_1, π_2), where π_i is the projection onto the i^{th} coordinate.

[Note: $\sigma(\{B_1 \times B_2 : B_1, B_2 \text{ Borel sets of } \mathbb{R}\})$ is known as $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$, the product sigma-ring of $\mathcal{B}(\mathbb{R})$ with itself, also denoted $\mathcal{B}(\mathbb{R})^2$ or $\mathcal{B}^2$.]

Proof. Let $\mathcal{C} = \{B_1 \times B_2 : B_1, B_2 \in \mathcal{B}(\mathbb{R})\}$, and let $\mathcal{G}$ be the family of open sets of $\mathbb{R}^2$.

For each $B_1 \in \mathcal{B}(\mathbb{R})$, $\pi_1^{-1}(B_1) = B_1 \times \mathbb{R} \in \mathcal{C} \subset \sigma(\mathcal{C}) = \mathcal{B}^2$, so π_1 is $\mathcal{B}^2$-measurable. Similarly π_2 is $\mathcal{B}^2$-measurable. Thus the identity map (π_1, π_2) is

$$\mathcal{B}^2 - \mathcal{B}(\mathbb{R}^2)\text{- measurable.}$$

This shows that if $B \in \mathcal{B}(\mathbb{R}^2)$, then $B = (\pi_1, \pi_2)^{-1}(B) \in \mathcal{B}^2$:

$$\mathcal{B}(\mathbb{R}^2) \subset \mathcal{B}^2.$$

Now, each π_i is continuous, hence Borel measureable. Thus, if $B_1 \in \mathcal{B}(\mathbb{R})$,

$$B_1 \times \mathbb{R} = \pi_1^{-1}(B_1) \in \mathcal{B}(\mathbb{R}^2),$$

and similarly if $B_2 \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{R} \times B_2 = \pi_2^{-1}(B_2) \in \mathcal{B}(\mathbb{R}^2).$$

Thus,

$$B_1 \times B_2 = (B_1 \times \mathbb{R}) \cap (\mathbb{R} \times B_2) \in \mathcal{B}(\mathbb{R}^2).$$

This shows

$$\mathcal{C} \subset \mathcal{B}(\mathbb{R}^2),$$

and hence also

$$\mathcal{B}^2 = \sigma(\mathcal{C}) \subset \mathcal{B}(\mathbb{R}^2)$$

Since $\mathcal{B}(\mathbb{R}^2) \subset \mathcal{B}^2$ and $\mathcal{B}^2 \subset \mathcal{B}(\mathbb{R}^2)$, we have equality, as required. □

Note: For the first part of this proof, we used the fact that a function f with values in $\mathbb{R}^2$ is $\mathcal{M}$-measurable if and only if its coordinate functions f_1, f_2 are measurable. Alternatively, we can use the argument used in establishing this, as follows:

If $U = U_1 \times U_2$ is an open interval of $\mathbb{R}^2$, then U_1 and U_2 are Borel sets, so

$$U \in \mathcal{C} \subset \sigma(\mathcal{C}) = \mathcal{B}^2$$

Every open set is the union of a countable family of open intervals, so

$$\mathcal{G} \subset \sigma(\mathcal{C}),$$

and therefore,

$$\mathcal{B}(\mathbb{R}^2) = \sigma(\mathcal{G}) \subset \sigma(\mathcal{C}).$$

□

7. Using Fatou's Lemma, one can actually remove the monotonicity from the Monotone Convergence Theorem, just as long as the approximation is from below: Let (f_n) be an sequence of non-negative real-valued measurable functions converging pointwise on S to the (real-valued) function f. If $f_n(x) \le f(x)$, for all x, then $\int f_n\,d\mu \to \int f\,d\mu$.

   Proof. For each n, $0 \le f_n \le f$, so
$$\limsup_n \int f_n\,d\mu \le \int f\,d\mu.$$
   On the other hand, by Fatou's lemma
$$\int f\,d\mu = \int \liminf_n f_n\,d\mu \le \liminf_n \int f_n\,d\mu,$$
   so we have equality throughout and $\int f\,d\mu = \lim_n \int f_n\,d\mu$. □

8. If f is integrable and $a > 0$ show that $\{x : |f(x)| > a\}$ has finite measure and that $\{x : f(x) \ne 0\}$ has σ-finite measure (that is, is a union of a countable family of sets of finite measure).

   Proof. If f is integrable, so is $|f|$, so $\mu\{x : |f(x)| > a\} \le 1/a \int |f|\,d\mu < +\infty$. But $\{x : f \ne 0\} = \{x : |f| > 0\} = \bigcup_{k\in\mathbb{N}}\{|f| \ge 1/k\}$, so is the union of a countable number of sets of finite measure, as required. □

9. If f is integrable and $\varepsilon > 0$ then $\int |f - \varphi|\,d\mu < \varepsilon$, for some integrable simple function φ.

   Proof. Let f be integrable. Then f^+ and f^- are both integrable. Let $\varepsilon > 0$. By definition of $\int f^+\,d\mu$, there is a non-negative simple function $\varphi_1 \le f^+$ such that $\int \varphi_1\,d\mu > \int f^+\,d\mu - \varepsilon/2$. φ_1 is integrable, since f^+ is.

   Similarly, there exists a non-negative integrable simple function $\varphi_2 < f^-$ such that $\int \varphi_2 > \int f^- - \varepsilon/2$. Put $\varphi = \varphi_1 - \varphi_2$. Then,
$$|f - \varphi| = |f^+ - \varphi_1 - (f^- - \varphi_2)| \le |f^+ - \varphi_1| + |f^- - \varphi_2| = (f^+ - \varphi_1) + (f^- - \varphi_2)$$
   Hence,
$$\int |f - \varphi|\,d\mu \le \int f^+ - \varphi_1\,d\mu + \int f^- - \varphi_2\,d\mu < \varepsilon. \quad \square$$

10. Let f be integrable and ν be its indefinite integral: $\nu(A) = \int_A f\,d\mu$, for $A \in \mathcal{M}$. Prove $\nu(A) = 0$, for all $A \in \mathcal{M}$ iff $f = 0$ μ-a.e.

   Proof. We already know $f = 0$, μ-a.e. implies $\int f\,d\mu = 0$.

   Conversely, suppose $\nu(A) = 0$, for all $A \in \mathcal{M}$. Put $A = (f \ge 0)$. Then $f^+ = f\mathbf{1}A$, so $\int f^+\,d\mu = \int_A f\,d\mu = \nu(A) = 0$; hence, $f^+ = 0$, μ-a.e. Similarly, $\int f^-\,d\mu = -\nu(f < 0)$, so $f^- = 0$, μ-a.e. Thus, $f = f^+ - f^- = 0$, μ-a.e. □

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11. Let $\mu_1 = \mu_2$ be counting measure on all subsets of $\mathbb{N}$. Let $f(m,n) = 1$ if $n = m$, -1 if $n = m+1$ and 0 otherwise. Then $\int\int f(m,n)\mu_1(dm)\mu_2(dn) \neq \int\int f(m,n)\mu_2(dn)\,\mu_1\,(dm)$, though both are finite. Why does this not contradict the Fubini theorem?

$Soln.$. For each $m \in \mathbb{N}$,

$$\sum_{n\in\mathbb{N}} f(m,n) = \sum_{n\notin\{m,m+1\}} 0 + f(m,m) + f(m,m+1) = 0 + 1 - 1 = 0.$$

and for each $n \in \mathbb{N}\setminus\{1\}$,

$$\sum_{m\in\mathbb{N}} f(m,n) = \sum_{m\notin\{n,n-1\}} 0 + f(n,n) + f(n-1,n) = 0 - 1 + 1 = 0,$$

but for $n = 1$, $\sum_{m\in\mathbb{N}} f(m,n) = 1$, since $n-1 \notin \mathbb{N}$. Thus,

$$\begin{aligned}\int\int f(m,n)\mu_1(dm)\mu_2(dn) &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} f(m,n)\\ &= \sum_{n\in\mathbb{N}} \mathbf{1}_{\{1\}} = 1\end{aligned}$$

but

$$\begin{aligned}\int\int f(m,n)\mu_2(dn)\,\mu_1\,(dm) &= \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} f(m,n)\\ &= \sum_{m} 0 = 0\end{aligned}$$

This does not violate the Fubini Theorem, since f is not integrable with respect to product measure. Indeed, for all n, m, $\mu_1 \otimes \mu_2(\{(n,m)\} = \mu_1\{n\}\mu_2\{m\} = 1$. Since $|f|$ is 1 on an infinite set of pairs, its integral with respect to product measure is $+\infty$.

Alternatively, we could use iterated integrals on $|f|$.

$$\begin{aligned}\int\int |f(m,n)|\mu_2(dm)\mu_1(dn) &= \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} |f(m,n)|\\ &= \sum_{m\in\mathbb{N}} 2 = +\infty. \quad \square\end{aligned}$$

12. For (x, y) in the unit square $[0,1]\times[0,1]$, define

$$f(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2},$$

except at $(0,0)$, and define $f(0,0) = 0$. Prove that

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\pi/4,$$

but

$$\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \pi/4.$$

Show that this doesn't violate the Fubini Theorem, since f is not integrable on $[0,1]\times[0,1]$.

Proof. Since $\frac{\partial}{\partial x}\frac{-x}{x^2+y^2} = f(x,y)$,

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 \frac{-1}{1+y^2}\,dy = -\arctan 1 = -\pi/4.$$

Similarly, $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \arctan 1 = \pi/4$.

To compare these outcomes with the Fubini Theorem, we need a choice of measure. Let us take λ to be Lebesgue one-dimensional measure restricted to the Lebesgue measurable subsets of $[0,1]$. We know that any function defined which is defined and (improper) Riemann integrable on $[0,1]$ is also Lebesgue integrable to the same value.

Now,

$$f^+(x,y) = \begin{cases} f(x,y), & \text{if } x \ge y, \quad (x,y) \in [0,1]\times[0,1]\setminus\{(0,0)\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$f^-(x,y) = \begin{cases} -f(x,y), & \text{if } x \le y, \quad (x,y) \in [0,1]\times[0,1]\setminus\{(0,0)\} \\ 0, & \text{otherwise} \end{cases}.$$

Thus, by the Fubini Theorem,

$$\begin{aligned} \int f^+\,d\lambda\otimes\lambda &= \int\int f^+(x,y)\lambda(dy)\lambda(dx) = \int\int_0^x f(x,y)\,dy\,\lambda(dx) \\ &= \int_0^1 \left[\frac{y}{x^2+y^2}\right]_{y=0}^x dx \\ &= \int_0^1 \frac{x}{x^2+x^2}\,dx = \int_0^1 \frac{1}{2x}\,dx = \frac{1}{2}\lim_{t\to 0}(\log 1 - \log t) = +\infty \end{aligned}$$

Similarly $\int f^-\,d\,\lambda\otimes\lambda = +\infty$, so f is not integrable, and hence the Fubini Theorem is not contradicted.

By the way, $\lambda\otimes\lambda = \lambda_2$, 2 dimensional Lebesgue measure restricted to $\mathcal{M}_\lambda \otimes \mathcal{M}_\lambda$.

13. Let μ_1 be Lebesgue measure on the Borel sets of $[0,1]$ and let μ_2 be counting measure, again on the Borel sets of $[0,1]$. Let

$$f(x,y) = \begin{cases} 1, & x = y \\ 0, & x \ne y. \end{cases}$$

Prove that f is product measurable — that is, is measurable with respect to $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$ — yet

$$\int\int f(x,y)\mu_1(dx)\mu_2(dy) \ne \int\int f(x,y)\mu_2(dy)\mu_1(dx).$$

What hypothesis of the Fubini Theorem doesn't hold?

Proof. Since $B := \{(x, y) : x = y\}$ is a closed set, it is a Borel set; hence $f = \mathbf{1}_B$ is a Borel measurable. Since $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ is the Borel sigma algebra of $\mathbb{R}^2$, f is product measurable. (By the way, f is only defined on $[0, 1] \times [0, 1]$, but the definition still makes sense; we just have $f^{-1}(\mathbb{R}) = [0, 1] \times [0, 1]$. Thus, f is actually $\mathcal{B}([0, 1] \times [0, 1]) = \mathcal{B}([0, 1]) \otimes \mathcal{B}([0, 1])$ measurable.)

Now, for fixed y, $f(x, y) = \mathbf{1}_{\{y\}}(x)$, for all x, so $\int f(x, y)\mu_1(dx) = \mu_1(\{y\}) = 0$, for all y and hence,

$$\int \int f(x, y)\mu_1(dx)\mu_2(dy) = 0.$$

Similarly for fixed x, $f(x, y) = \mathbf{1}_{\{x\}}(y)$, for all y, so $\int f(x, y)\mu_2(dx) = 1$, for all x; hence,

$$\int \int f(x, y)\mu_2(dy)\mu_1(dx) = \sum_{x \in [0,1]} 1 = +\infty.$$

The reason the Fubini Theorem doesn't apply is that μ_2 is not sigma-finite. Indeed, if $[0, 1]$ is written as a union of a sequence of sets A_n of finite measure, then each A_n is finite so $[0, 1]$ would be countable, which is false. □

14. Prove the following version of **Cavellieri's Principle**. Let μ_1, μ_2 be σ-finite measures on σ-algebras $\mathcal{M}_1$, $\mathcal{M}_2$ in S_1, S_2 respectively. Prove that if E, F are subsets of $S = S_1 \times S_2$ such that $\mu_2(E(x_1)) = \mu_2(F(x_1))$, for μ_1-almost all $x_1 \in S_1$, then $(\mu_1 \otimes \mu_2)(E) = (\mu_1 \otimes \mu_2)(F)$.

Proof. With the approach we took in this course, this is a triviality.
By definition,

$$(\mu_1 \otimes \mu_2)(E) = \int \mu_2(E(x_1))\mu_1(dx_1) = \int \mu_2(F(x_1))\mu_1(dx_1) = (\mu_1 \otimes \mu_2)(F).$$

Alternatively, one could use Fubini's theorem. The result then amounts to checking that $\int \mathbf{1}_E(x_1, x_2)\mu_2(dx_1) = \mu_2(E(x_1))$ and the corresponding result for F.

15. Suppose μ is a sigma-finite measure on the σ-algebra $\mathcal{M}$ in S, and the λ is Lebesgue measure in $\mathbb{R}$, restricted to the Borel sets. For a non-negative $\mathcal{M}$-measurable function f, prove that the set $A = \{(x, y) : 0 \le y \le f(x)\}$ is measurable with respect to $\mathcal{M} \otimes \mathcal{B}(\mathbb{R})$ and $\int f\, d\mu = \mu \otimes \lambda(A)$.

Proof. Let $f_n = \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbf{1}_{[(k-1)/2^n < f \le k/2^n]}$. Now, $f_n \ge f_{n+1}$ and $f_n \to f$. Indeed, for each n, if k is the first integer with $\frac{k}{2^n} \ge f(x)$, then $f_n(x) = \frac{k}{2^n} \ge f(x)$ and there are 2 possibilities: Either,

(i) $f(x) > \frac{k}{2^n} - \frac{1}{2^{n+1}} = \frac{2k-1}{2^{n+1}}$, in which case, $f_{n+1}(x) = f_n(x) = \frac{k}{2^n}$, or

(ii) $\frac{2k-1}{2^{n+1}} \ge f(x) > \frac{2k-1-1}{2^{n+1}} = \frac{k-1}{2^n}$, in which case $f_{n+1}(x) = f_n(x) - 1/2^{n+1}$.

Thus, in all cases $f_n(x) \ge f_{n+1}(x)$.

As for convergence, the formula $(k-1)/2^n < f(x) \le k/2^n$ shows that $f_n(x) - f(x) \le \frac{1}{2^n} \to 0$, so in fact $f_n \to f$, uniformly.

Now, for each $n \in \mathbb{N}$, put $A_n = \{(x, y) : 0 \le y \le f_n(x)\}$. Then $\bigcap_n A_n = A$. Indeed, since $f_n \ge f$, $A_n \supset A$, for all n, so $\bigcap_n A_n \supset A$, and since $f_n \to f$, $\bigcap_n A_n \subset A$.

Each A_n can be written

$$\bigcup_{k\in\mathbb{N}} \{x : \tfrac{k-1}{2^n} < f(x) \le \tfrac{k}{2^n}\} \times [0, \tfrac{k}{2^n}] \cup \{x : f(x) = 0\} \times \{0\}.$$

Since f is measurable $\{x : \frac{k-1}{2^n} < f(x) \le \frac{k}{2^n}\} \in \mathcal{M}$ and $\{x : f(x) = 0\} \in \mathcal{M}$, so $A_n \in \mathcal{M} \otimes \mathcal{B}(\mathbb{R})$. Hence, A is also in the product sigma-algebra.

Finally, the x-section of A is

$$A(x) = \{y : (x, y) \in A\} = \{y : 0 \le y \le f(x)\} = [0, f(x)],$$

so by the definition of product measure,

$$\mu \otimes \lambda(A) = \int \lambda(A(x))\, \mu(dx) = \int \lambda([0, f(x)])\, \mu(dx) = \int f(x)\, \mu(dx). \quad \square$$

Note. Since not everyone uses that definition of product measure, we can invoke the Fubini Theorem instead and write $\mu \otimes \lambda(A) = \int \mathbf{1}_A\, d\mu \otimes \lambda = \int\int \mathbf{1}_A(x, y)\, \lambda(dy)\mu(dx) = \int\int \mathbf{1}_{[0,f(x)]}(y)\lambda(dy)\mu(dx) = \int \lambda([0, f(x)])\mu(dx) = \int f(x)\, \mu(dx)$.

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16. For functions $f_n, f \in \mathcal{L}^0$, $f_n \longrightarrow f$ μ-almost everywhere iff for all $\delta > 0$, $\mu(|f_n - f| > \delta,\ \text{for infinitely many } n) = 0$.

Proof. For clarity,

$$(|f_n - f| > \delta,\ \text{for infinitely many } n) = (\forall m \in \mathbb{N}, \exists n \geq m,\ |f_n - f| > \delta).$$

By definition, $f_n \to f$ μ-almost everywhere if and only if there is a μ-null set N such that $f_n \to f$ on N^c. Since f_n and f are given measurable, we can take N to be $\{x : f_n(x) \not\to f(x)\} = (f_n \not\to f)$.

Suppose $f_n \to f$, μ-almost everywhere and let $\delta > 0$ then, $(|f_n - f| > \delta, \text{for infinitely many } n\} \subset (f_n \not\to f)$, so $(|f_n - f| > \delta,\ \text{for infinitely many } n)$ has measure 0.

Conversely, if for all $\delta > 0$, $\mu(|f_n - f| > \delta,\ \text{for infinitely many } n) = 0$, then $\mu(f_n \not\to f) = \mu(\bigcup_{k \in \mathbb{N}}(|f_n - f| > 1/k,\ \text{for infinitely many } n) = 0$, so $f_n \to f$, μ-almost everwhere. □

17. For $0 < p < \infty$ deduce from the Dominated Convergence Theorem (in $\mathcal{L}^1$) the $\mathcal{L}^p$ version: If $f_n \longrightarrow f$ a.e. and there exists $g \in \mathcal{L}^p$ such that $|f_n| \leq g$ pointwise (or a.e.) then $f_n \longrightarrow f$ in $\mathcal{L}^p$.

Soln. Let $0 \leq g \in \mathcal{L}^p$. Then $g^p \in \mathcal{L}^1$. Suppose $|f_n| \leq g$ μ-a.e. and $f_n \to f$ a.e. Then $|f| \leq g$ a.e., so $|f_n - f|^p \leq (2g)^p$ a.e. and $(2g)^p \in \mathcal{L}^1$. Moreover, $|f_n - f|^p \to 0$, so $\int |f_n - f|^p \, d\mu \to 0$, by the Dominated Convergence Theorem. That is $f_n \to f$ in $\mathcal{L}^p$.

18. If φ is uniformly continuous on $\mathbb{R}$ to $\mathbb{R}$, and f_n converges in measure to f, then $\varphi \circ f_n$ converges in measure to $\varphi \circ f$.

Proof.

Let φ be uniformly continuous on $\mathbb{R}$ to $\mathbb{R}$. Then, for each $\delta > 0$, there exists $\eta > 0$ such that $|y - y'| \leq \eta$ implies $|\varphi(y) - \varphi(y')| \leq \delta$. Then,

$$\{x : |\varphi(f_n(x)) - \varphi(f(x))| > \delta\} \subset \{x : |f_n(x) - f(x)| > \eta\}$$

If $f_n \to f$ in μ-measure, then the μ-measure of the right side converges to 0, hence $\mu(|\varphi \circ f_n - \varphi \circ g| > \delta$, which is the measure of the left side also converges to 0. Thus, $\varphi \circ f_n \to \varphi \circ g$ in μ-measure. □

19. Find a sequence of functions on $[0, 1]$ that converges to 0 almost everywhere (for Lebesgue measure) but not in $\mathcal{L}^1$, and another that converges to 0 in $\mathcal{L}^1$, but does not converge anywhere.

Soln.

(a) For each n, choose $f_n = n\mathbf{1}[0, 1/n]$. Then, for $\delta > 0$, $\lambda(|f_n - 0| > \delta) \leq \lambda([0, 1/n]) = 1/n \to 0$. Thus, $f_n \to 0$ in λ-measure. On the other hand, $\|f_n\|_1 = \int f_n \, d\lambda = n\lambda([0, 1/n]) = 1$, which does not converge to 0, so f_n does not converge to 0 in $\mathcal{L}^1$.

(b) Let $g_1 = \mathbf{1}[0, 1)$, $g_2 = \mathbf{1}[0, 1/2)$, $g_2 = \mathbf{1}[1/2, 1)$, $g_3 = \mathbf{1}[0, 1/4)$, $g_4 = \mathbf{1}[1/4, 2/4)$, $g_5 = \mathbf{1}[2/4, 3/4) \ldots$. Thus, g_n, runs through the indicator functions of 1 interval of length 1, then 2 intervals of length 1/2, then 4 intervals of length 1/4, $\ldots$ k intervals of length $1/2^k$. The norm $\|g_n\|_1$, is the measure of the interval where $g_n \neq 0$, which tends to 0, but for each x, $g_n(x)$ takes on the values 0 and 1 infinitely many times, so $\{x : (g_n(x)) \text{ converges }\} = \emptyset$.

20. Let (f_n) and (h_n) be sequences in $\mathcal{L}^0(\mu)$, $f \in \mathcal{L}^0$. Suppose (h_n) is a decreasing sequence converging to 0 in μ-measure, and $|f_n - f| \leq h_n$, for all n. Prove that $f_n \longrightarrow f$ μ-almost uniformly.

Proof. The sequence (f_n) converges almost uniformly to f if and only if for each $\delta > 0$, $\mu(\exists k \geq n, |f_k - f| > \delta) \to 0$. But, by hypothesis,

$$|f_k - f| \leq h_k \leq h_n, \quad \text{for } k \geq n$$

Hence,

$$\bigcup_{k \geq n} (|f_k - f| > \delta) \subset (h_n > \delta).$$

Since (h_n) converges to 0 in measure,

$$\mu\left(\bigcup_{k \geq n} (|f_k - f| > \delta)\right) \leq \mu(h_n > \delta) \to 0,$$

as required. □

21. Concerning the $\mathcal{L}^0$ quasinorm, $\|f\|_0 = \inf\{\delta \geq 0 : \mu(|f| > \delta) \leq \delta\}$, prove

(a) $\|f\|_0 \leq \|f\|_\infty \wedge \mu(f \neq 0)$. Need there be equality?

(b) If $|\alpha| \leq 1$, $\|\alpha f\|_0 \leq \|f\|_0$

Soln.. (a) First, $\mu(|f| > \|f\|_\infty) = 0 \leq \|f\|_\infty$. Since $\|f\|_0$ is the least δ with $\mu(|f| > \delta) = 0 \leq \delta$, this shows

$$\|f\|_0 \leq \|f\|_\infty.$$

Also, if $\delta = \mu(f \neq 0)$, then $\mu(|f| > \delta) \leq \mu(f \neq 0) = \delta$, so

$$\|f\|_0 \leq \mu(f \neq 0).$$

Thus, $\|f\|_0 \leq \|f\|_\infty \wedge \mu(f \neq 0)$.

To see that there need not be equality, take μ to be Lebesgue measure on the Borel sets of $\mathbb{R}$, and put

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 2 \\ 3, & 2 < x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Then $\mu(|f| > 1) = 1 \leq 1$, and if $\delta < 1$, $\mu(|f| > \delta) = 3 > \delta$, so $\|f\|_0 = 1 < \|f\|_\infty \wedge \mu(f \neq 0) = 3$.

(b) If $|\alpha| \leq 1$, then $|\alpha f| \leq |f|$, so

$$\mu(|\alpha f| > \|f_0\|) \leq \mu(|f| > \|f\|_0) \leq \|f\|_0,$$

so $\|\alpha f\|_0 \leq \|f\|_0$. $\square$

22. Suppose μ is a finite measure. Define $\|f\|_\bullet = \| |f| \wedge 1\|_1$, for $f \in \mathcal{L}^0$ and $d(f,g) = \|f - g\|_\bullet$.

(a) Show that this d is a semimetric on $\mathcal{L}^0$.

(b) Show that a sequence (f_n) in $\mathcal{L}^0$ converges to f in measure iff $d(f_n, f) \longrightarrow 0$.

Soln.. (a) Remember the $\wedge$ means minimum. For numbers $a, b \geq 0$,

$$(a + b) \wedge 1 \leq a \wedge 1 + b \wedge 1,$$

because if one of a and b is > 1, the left side is 1 and the right side is > 1, while if both a, b are ≤ 1, then the inequality reduces to $(a + b) \wedge 1 \leq a + b$. Applying this to the functions f and g, and integrating we have

$$\||f + g| \wedge 1\|_1 \leq \||f| \wedge 1\|_1 + \||g| \wedge 1\|_1;$$

that is

$$\|f + g\|_\bullet \leq \|f\|_\bullet + \|g\|_\bullet.$$

It follows as usual that $d(f,g) = \|f - g\|_\bullet$ defines a semimetric.

(b) Now, if $f_n \to f$ in measure, then $\mu(|f_n - f| \wedge 1 > \delta) < \mu(|f_n - f| > \delta) \to 0$, and $|f_n - f| \wedge 1 \leq 1$. But the function 1 is integrable, since μ is a finite measure. Thus, by the convergence in measure version of the DCT, $\|f_n - f\|_\bullet = \int |f_n - f| \wedge 1 \, d\mu \to 0$.

Conversely, since convergence in $\mathcal{L}^1$ implies convergence in measure, if $\|f_0 - f\|_\bullet \to 0$, $|f_n - f| \wedge 1 \to 0$ in measure, and hence $|f_n - f|$ does so also.