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Denjoy vs Kurzweil-Henstock Integration in $\mathbb{R}^m$

Throughout, K will be a fixed non-degenerate compact interval of $\mathbb{R}^m$ and $\mathcal{I}$ its family of non-degenerate closed sub-intervals; $\boldsymbol{\lambda}$ will denote Lebesgue measure. A *tagged interval* is a pair (x, J) with $x \in J \in \mathcal{I}$; a finite set π of non-overlapping tagged intervals is a *partition*; it is a *partition of* I, if the union of its intervals is I. For a bounded subset A of $\mathbb{R}^m$, the *regularity* of A is the ratio of its Lebesgue measure to that of the smallest cube containing it. For (non-degenerate) intervals, which is what we will mainly consider here, one can as well use the ratio of the length of the smallest side to that of the largest.

For a family $\mathcal{H}$ of tagged interval and a subset A of K, $\mathcal{H}[A]$ is the set of those (x, J) in $\mathcal{H}$ with the tag $x \in A$, $\mathcal{H}(A)$ those with $J \subset A$. A *gauge* on E is a positive real function δ on E.

For a function φ of tagged intervals and a tagged partition π, $\varphi(\pi)$ denotes $\sum_{(x,J)\in\pi} \varphi(J)$. An interval function F yields a tagged interval function, by simply dropping the tags. Thus, $F(\pi)$ will mean $\sum_{(x,J)\in\pi} F(J)$. For a point function f, $(f\boldsymbol{\lambda})(x, J) = f(x)\boldsymbol{\lambda}(J)$ defines a tagged interval function and $(f\boldsymbol{\lambda})(\pi)$ becomes the Riemann sum $\sum_{(x,J)\in\pi} f(x)\boldsymbol{\lambda}(J)$.

By a *basis* (used for both differentiation and integration) we will mean a filterbase $\mathcal{B}$ in the space of tagged intervals such that for each $\beta \in \mathcal{B}$ and $x \in K$, there is a J with $(x, J) \in \beta$. For purposes of the talk, rather than postulate abstractly the conditions we will need, we concentrate on the following cases.

1. The *full basis*, consisting of the families

$$\beta_\delta = \{(x, J) : x \in J \in \mathcal{I},\ J \subset B(x, \delta(x))\},$$

   δ a gauge on K.

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2. The *r-regular basis* (for a fixed $r \in (0,1]$), consisting of the families

$$\beta^r_\delta = \{(x,J) : x \in J \in \mathcal{I},\ J \subset B(x,\delta(x)),\ \mathrm{reg}(J) \geq r\},$$

δ a gauge on K.

3. The *regular basis*, consisting of the families

$$\beta^\rho_\delta = \{(x,J) : x \in J \in \mathcal{I},\ J \subset B(x,\delta(x)),\ \mathrm{reg}(J) \geq \rho(x)\},$$

δ a gauge on K, ρ a function on K to $(0,1)$.

In each of these cases (provided $r < 1$ in case 2), each β contains a partition of K; and any partition contained in β of an element $I \in \mathcal{I}$ can be extended to a partition of K, still in β. These are versions of Cousin's lemma. See [Pfe86] for case 3. Unless otherwise stated, we exclude from the discussion the 1-regular (that is, cubic) base.

For a fixed basis $\mathcal{B}$, an interval function F, and an $x \in K$, we define the derivative (when it exists) by $\mathbf{D}F(x) = \mathcal{B}\text{-}\mathbf{D}F(x) = \lim_{J\to x} \frac{F(J)}{\boldsymbol{\lambda}(J)}$, in the sense that for all $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with

$$\left| \frac{F(J)}{\boldsymbol{\lambda}(J)} - \mathbf{D}F(x) \right| < \epsilon, \text{ whenever } (x,J) \in \beta.$$

(Thus, $\mathbf{D}F(x)$ is the limit of the quotient $\frac{F(J)}{\boldsymbol{\lambda}(J)}$ as J follows the filterbase $\mathcal{B}_x$ consisting of families $\{J : (x,J) \in \beta\}$, $\beta \in \mathcal{B}$.)

For a function $f : K \to \mathbb{R}$, the Kurzweil-Henstock integral of f over $I \in \mathcal{I}$ (with respect to the basis $\mathcal{B}$) is given by $\int_I f = \mathcal{B}\text{-}\int_I f = \lim_\pi (f\boldsymbol{\lambda})(\pi)$ as π runs over the tagged partitions of I following $\mathcal{B}$: for every $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with

$$\left| \sum_{(x,J)\in\pi} f(x)\boldsymbol{\lambda}(J) - \int_I f \right| < \epsilon,$$

whenever π is a partition of I contained in β.

For the bases we've mentioned, if a function f is KH integrable on K, then it is also KH integrable on each sub-interval, and thus determines an additive function F on $\mathcal{I}$, called the *indefinite integral* of f. We would like conditions under which an additive function F is the indefinite integral of a K-H integrable function f if and only if F is differentiable with derivative f (almost everywhere).

For a tagged-interval function φ, and a fixed basis, define the $\mathcal{B}$-variational measure on the subsets X of K by

$$V\varphi(X) = \mathcal{B}\text{-}V\varphi(X) = \inf_\beta \sup_{\pi\subset\beta} |\varphi|(\pi),$$

where π runs over partitions tagged in X.

Key Lemma. *For an additive function F on $\mathcal{I}$ and a function f on K,*

1. *If $\mathbf{D}F = f$ outside N where $VF(N) = V(f\boldsymbol{\lambda})(N) = 0$, then $F(I) = \int_I f$, for all $I \in \mathcal{I}$.*
2. *If $F(I) = \int_I f$, for $I \in \mathcal{I}$, then*

$$| \, |F|(\pi) - |f\boldsymbol{\lambda}|(\pi) \, | \leq |F - f\boldsymbol{\lambda}|(\pi) \to 0,$$

so that $VF = V(\boldsymbol{\lambda} f)$.

In the language of [LV00],[Bar01], $VF(N) = 0$ becomes F *is of negligible variation on* N.

Proof. (1) Suppose $\mathbf{D}F = f$ on N^c and $VF(N) = V(f\boldsymbol{\lambda})(N) = 0$. Then for each $\epsilon > 0$, and each $x \in N^c$, there exists $\beta \in \mathcal{B}$ with

$$|F(J) - f(x)\boldsymbol{\lambda}(J)| \leq \epsilon\boldsymbol{\lambda}(J),$$

for $(x, J) \in \beta$. For each partition $\pi \subset \beta$ of I, $F(I) = F(\pi)$ and

$$\begin{aligned} |F(I) - (f\boldsymbol{\lambda})(\pi)| &\leq |F - f\boldsymbol{\lambda}|(\pi[N^c]) + |F|(\pi[N]) + |f\boldsymbol{\lambda}|(\pi[N]) \\ &\leq \epsilon\boldsymbol{\lambda}(I) + |F|(\pi[N]) + |f\boldsymbol{\lambda}|(\pi[N]) \end{aligned}$$

Since the variations $VF(N)$ and $V(f\boldsymbol{\lambda})(N)$ are both 0, the latter terms can be made arbitrarily small, so that $F(I) = \int_I f$.

(2) That $|F - f\boldsymbol{\lambda}|(\pi) \to 0$, when $F(I) = \int_I f$ is Henstock's Lemma. This is merely being combined with the triangle inequality $||F(J)| - |f\boldsymbol{\lambda}|(J)| \leq |F(J) - (f\boldsymbol{\lambda})(J)|$. $\square$

For a tagged interval function φ to $\mathbb{R}$ and $X \subset \mathbb{R}^m$, we say φ is AC$_*$ on X if for every $\epsilon > 0$, there exists $\eta > 0$ such that $|\varphi|(\pi) < \epsilon$, whenever π is a partition with tags in X with $\boldsymbol{\lambda}(\pi) < \eta$; φ is ACG$_*$ on X if X can be written as the union of countably many sets on which φ is AC_*.

Lemma 1. *The tagged interval function $f\boldsymbol{\lambda}$ is ACG_* on K.*

Proof. Indeed, for π tagged in $(|f| \leq n)$, $\boldsymbol{\lambda}(\pi) < \epsilon/n$ implies $|f\boldsymbol{\lambda}|(\pi) < \epsilon$. $\square$

It may not follow that the indefinite $\mathcal{B}$-KH integral is ACG$_*$. It seems we need to strengthen the hypothesis of ACG$_*$ to get the desired descriptive characterization of KH integrals by derivatives. Say that φ is $\mathcal{B}$-AC on X, if for every $\epsilon > 0$, there exists $\eta > 0$ and a $\beta \in \mathcal{B}$, such that $|\varphi|(\pi) < \epsilon$, whenever π is a partition with tags in X, with $\boldsymbol{\lambda}(\pi) < \eta$ and with $\pi \subset \beta$; φ is $\mathcal{B}$-ACG$_*$

on X, if X can be written as the union of countably many sets on which φ is $\mathcal{B}$-ACG$_*$.

The bases $\mathcal{B}$ of our 3 cases are compatible with the topology, or "fine", in the sense that if G is open, there exists a $\beta \in \mathcal{B}$ such that for each $x \in G$, if $(x, J) \in \beta$, then $J \subset G$. They also are *σ-decomposable* [Tho02] (or *of σ-local character* [Ost86]), in the sense that if (β_n) is a sequence of members of $\mathcal{B}$ and (X_n) is a disjoint sequence of subsets of K, then there is a $\beta \in \mathcal{B}$ with $\beta[X_n] \subset \beta_n[X_n]$, for all $n \in \mathbb{N}$.

Lemma 2. *If a tagged interval function φ is $\mathcal{B}$-ACG$_*$ on K, then $\mathcal{B}$-$V\varphi(N) = 0$, for each Lebesgue nullset N.*

That is, $V\varphi$ is absolutely continuous with respect to Lebesgue measure.

Proof. One really only needs $\mathcal{B}$-ACG$_*$ on the nullset N. Write N as a disjoint union $\bigcup_n N_n$, where φ is $\mathcal{B}$-AC$_*$ on N_n. For a given $\epsilon > 0$, choose $\eta_n > 0$ and $\beta_n \in \mathcal{B}$ such that for each partition tagged in N_n of total measure $\boldsymbol{\lambda}(\pi) < \eta_n$, with $\pi \subset \beta_n$, $|\varphi|(\pi) < \epsilon/2^n$. Cover each N_n with an open G_n set of measure less that η_n, then choose a $\beta'_n \subset \beta_n$ such that $x \in G_n$ and $(x, J) \in \beta'_n$ implies $J \subset G_n$. Now find a new β' such that $\beta[X_n] \subset \beta_n[X_n]$, for all $n \in \mathbb{N}$, so that $x \in G_n$ and $(x, J) \in \beta'$ implies $J \subset G_n$. Now any partition $\pi \subset \beta'$ will have $\boldsymbol{\lambda}(\pi[N_n]) < \eta_n$, so $|\varphi|(\pi[N_n]) < \epsilon/2^n$, and hence $|\varphi|(\pi[N] < \epsilon$. This shows $V\varphi(N) = 0$, as required ☐

As a consequence, the Key Lemma yields an $\mathbb{R}^m$ version of Bartle's Theorem [LV00], 3.9.1. [Bar01], 5.12

Theorem 3. *An additive function F is the indefinite integral of a $\mathcal{B}$-KH integrable f if $\mathbf{D}F = f$ except on a nullset N with $VF(N) = 0$. In that case, VF vanishes on* every *nullset; that is, is absolutely continuous with respect to $\boldsymbol{\lambda}$.*

Proof. If N is null, then $f\boldsymbol{\lambda}(N) = 0$, since $f\boldsymbol{\lambda}$ is ACG$_*$. So if also $VF(N) = 0$, F is the indefinite integral of f, by the Key Lemma (1). But the second half then says $VF = V(f\boldsymbol{\lambda})$, which is 0 at each nullset. ☐

Call a function f *$\mathcal{B}$-Denjoy integrable* if it is almost everywhere equal to the $\mathcal{B}$-derivative of an additive function F which is $\mathcal{B}$-ACG$_*$. The function F will be called the *primitive* of f. Let $\mathcal{B}$ be the r-regular basis for $0 < r < 1$, or the regular basis, so that the Vitali Covering Theorem holds. We obtain the descriptive characterization of the $\mathcal{B}$-KH integral as a Denjoy integral.

Theorem 4. *A function f on K is $\mathcal{B}$-Denjoy integrable with primitive F if and only if it is $\mathcal{B}$-KH integrable, with indefinite integral F.*

Proof. If $\mathbf{D}F = f$ outside a nullset N and F is $\mathcal{B}$-ACG$_*$, then $VF(N) = V(f\boldsymbol{\lambda})(N) = 0$, so by the Key Lemma, F is a KH-primitive of f.

For the converse, if F is a KH-primitive of f and $N = (\mathbf{D}F \neq f)$, Henstock's Lemma and the Vitali property yield N is null as in [Ost86], page 37. $\mathbf{D}F = f$ almost everywhere and the fact that $|F|(\pi) - |\boldsymbol{\lambda} f|(\pi) \to 0$ yields F is $\mathcal{B}$-ACG$_*$. $\square$

If the r-regular base (call it $\mathcal{B}^r$) is used, we may refer to the r-derivative, the r-Denjoy integral, and the r-KH-integral. If all the r-derivatives r-$\mathbf{D}F(x)$, $0 < r < 1$, exist, they must be equal and their common value is the *ordinary derivative* [Sak64]. Actually, Kurzweil and Jarník [KJ92] have shown that r-differentiability for *some* r implies ordinary differentiability, but that r-integrability depends on the choice of r. We will call a function f ordinary Denjoy integrable if it is r-Denjoy integrable for each regularity r. It is important that this includes the condition that F is $\mathcal{B}^r$-ACG$_*$, for each r separately. If all the r-KH-integrals of f exist, they also must have a common value. This value is known as the M-integral [BDPS01],[DP01] after Mahwin, who introduced it in an equivalent form [Maw81].

Corollary 5. *A function f on K is M-integrable with indefinite integral F, if and only if it is ordinary Denjoy integrable with primitive F.*

One might note that since ACG$_*$ implies $\mathcal{B}$-ACG$_*$, for any choice of interval basis, it follows that each ACG$_*$ ordinary differentiable additive F is the indefinite M-integral of its derivative and also the KH-integral with respect to the regular basis.

Remarks

Recall that throughout, VF referred to $\mathcal{B}$-VF, the $\mathcal{B}$-variational measure. Since the talk, we learned that Di Piazza [DP01] defines $V_{\mathcal{M}}F = \sup_{r\in(0,1)} \mathcal{B}^r VF$ and proves that the ordinary derivative of F exist almost everywhere and F is its indefinite M-intgegral if and only if $V_{\mathcal{M}}F$ is absolutely continuous.

In the one-dimensional case (where all 3 bases coincide), Gordon [Gor94] refers to $\mathcal{B}$-ACG$_*$ as ACG_δ. Chew [Che90], working with the regular basis, uses the notation ACG*, with the asterisk raised instead of lowered. Chew, Lemma 1, shows that for a continuous additive function of intervals these coincide with the classical definition.

We have been unable to determine, for any of the 3 bases, whether every additive $\mathcal{B}$-ACG$_*$ function on $\mathcal{I}$ is ACG$_*$.

Open Problem. *For what bases do the notions of ACG_* and $\mathcal{B}$-ACG_* coincide for additive interval functions? (…for indefinite $\mathcal{B}$-KH integrals?)*

Further discussion of these matters appear on http://timtraynor.com (http://www.uwindsor.ca/traynor).

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