Denjoy vs Kurzweil-Henstock Integration in $\mathbb{R}^m$

Throughout, $K$ will be a fixed non-degenerate compact interval of $\mathbb{R}^m$ and $\mathcal{I}$ its family of non-degenerate closed sub-intervals; $\lambda$ will denote Lebesgue measure. A tagged interval is a pair $(x, J)$ with $x \in J \in \mathcal{I}$; a finite set $\pi$ of non-overlapping tagged intervals is a partition; it is a partition of $\mathcal{I}$, if the union of its intervals is $\mathcal{I}$. For a bounded subset $A$ of $\mathbb{R}^m$, the regularity of $A$ is the ratio of its Lebesgue measure to that of the smallest cube containing it. For (non-degenerate) intervals, which is what we will mainly consider here, one can as well use the ratio of the length of the smallest side to that of the largest.

For a family $\mathcal{H}$ of tagged interval and a subset $A$ of $K$, $\mathcal{H}[A]$ is the set of those $(x, J)$ in $\mathcal{H}$ with the tag $x \in A$, $\mathcal{H}(A)$ those with $J \subset A$. A gauge on $E$ is a positive real function $\delta$ on $E$.

For a function $\varphi$ of tagged intervals and a tagged partition $\pi$, $\varphi(\pi)$ denotes $\sum_{(x, J) \in \pi} \varphi(J)$. An interval function $F$ yields a tagged interval function, by simply dropping the tags. Thus, $F(\pi)$ will mean $\sum_{(x, J) \in \pi} F(J)$. For a point function $f$, $(f\lambda)(x, J) = f(x)\lambda(J)$ defines a tagged interval function and $(f\lambda)(\pi)$ becomes the Riemann sum $\sum_{(x, J) \in \pi} f(x)\lambda(J)$.

By a basis (used for both differentiation and integration) we will mean a filterbase $\mathcal{B}$ in the space of tagged intervals such that for each $\beta \in \mathcal{B}$ and $x \in K$, there is a $J$ with $(x, J) \in \beta$. For purposes of the talk, rather than postulate abstractly the conditions we will need, we concentrate on the following cases.

1. The full basis, consisting of the families

   $$\beta_\delta = \{(x, J) : x \in J \in \mathcal{I}, J \subset B(x, \delta(x))\},$$

   $\delta$ a gauge on $K$.

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2. The \( r \)-regular basis (for a fixed \( r \in (0, 1] \)), consisting of the families

\[
\beta^r_\delta = \{ (x, J) : x \in J \in \mathcal{I}, J \subset B(x, \delta(x)), \text{reg}(J) \geq r \},
\]

\( \delta \) a gauge on \( K \).

3. The regular basis, consisting of the families

\[
\beta^\rho_\delta = \{ (x, J) : x \in J \in \mathcal{I}, J \subset B(x, \delta(x)), \text{reg}(J) \geq \rho(x) \},
\]

\( \delta \) a gauge on \( K \), \( \rho \) a function on \( K \) to \((0, 1)\).

In each of these cases (provided \( r < 1 \) in case 2), each \( \beta \) contains a partition of \( K \); and any partition contained in \( \beta \) of an element \( I \in \mathcal{I} \) can be extended to a partition of \( K \), still in \( \beta \). These are versions of Cousin’s lemma. See [Pfe86] for case 3. Unless otherwise stated, we exclude from the discussion the 1-regular (that is, cubic) base.

For a fixed basis \( B \), an interval function \( F \), and an \( x \in K \), we define the derivative (when it exists) by

\[
D_F(x) = \lim_{J \to x} F(J) / \lambda(J),
\]

in the sense that for all \( \epsilon > 0 \), there exists \( \beta \in B \) with

\[
\left| \frac{F(J)}{\lambda(J)} - D_F(x) \right| < \epsilon,
\]

whenever \( (x, J) \in \beta \).

(Thus, \( D_F(x) \) is the limit of the quotient \( \frac{F(J)}{\lambda(J)} \) as \( J \) follows the filterbase \( B_x \) consisting of families \( \{ J : (x, J) \in \beta \}, \beta \in B \).)

For a function \( f : K \to \mathbb{R} \), the Kurzweil-Henstock integral of \( f \) over \( I \in \mathcal{I} \) (with respect to the basis \( B \)) is given by

\[
\int_I f = \lim_{\pi \to \alpha} \sum_{(x,J) \in \pi} f(x) \lambda(J),
\]

as \( \pi \) runs over the tagged partitions of \( I \) following \( B \): for every \( \epsilon > 0 \), there exists \( \beta \in B \) with

\[
\left| \sum_{(x,J) \in \pi} f(x) \lambda(J) - \int_I f \right| < \epsilon,
\]

whenever \( \pi \) is a partition of \( I \) contained in \( \beta \).

For the bases we’ve mentioned, if a function \( f \) is KH integrable on \( K \), then it is also KH integrable on each sub-interval, and thus determines an additive function \( F \) on \( I \), called the indefinite integral of \( f \). We would like conditions under which an additive function \( F \) is the indefinite integral of a KH integrable function \( f \) if and only if \( F \) is differentiable with derivative \( f \) (almost everywhere).

For a tagged-interval function \( \varphi \), and a fixed basis, define the \( B \)-variational measure on the subsets \( X \) of \( K \) by

\[
V_\varphi(X) = \inf_{\beta} \sup_{\pi \subset \beta} |\varphi(\pi)|,
\]
where \( \pi \) runs over partitions tagged in \( X \).

**Key Lemma.** For an additive function \( F \) on \( I \) and a function \( f \) on \( K \),

1. If \( DF = f \) outside \( N \) where \( VF(N) = V(f\lambda)(N) = 0 \), then \( F(I) = \int_I f \), for all \( I \in I \).

2. If \( F(I) = \int_I f \), for \( I \in I \), then
   \[
   |F(\pi) - |f\lambda|(\pi)| \leq |F - f\lambda|(\pi) \to 0,
   \]
so that \( VF = V(\lambda f) \).

In the language of [LV00],[Bar01], \( VF(N) = 0 \) becomes \( F \) is of negligible variation on \( N \).

**Proof.** (1) Suppose \( DF = f \) on \( N^c \) and \( VF(N) = V(f\lambda)(N) = 0 \). Then for each \( \epsilon > 0 \), and each \( x \in N^c \), there exists \( \beta \in B \) with
   \[
   |F(J) - f(x)\lambda(J)| \leq \epsilon \lambda(J),
   \]
for \( (x, J) \in \beta \). For each partition \( \pi \subset \beta \) of \( I \), \( F(I) = F(\pi) \) and
   \[
   |F(I) - (f\lambda)(\pi)| \leq |F - f\lambda|(\pi) + |F(\pi[N^c]) + |F\lambda|(\pi[N]) \leq \epsilon \lambda(I) + |F(\pi[N]) + |F\lambda|(\pi[N])
   \]
Since the variations \( VF(N) \) and \( V(f\lambda)(N) \) are both 0, the latter terms can be made arbitrarily small, so that \( F(I) = \int_I f \).

(2) That \( |F - f\lambda|(\pi) \to 0 \), when \( F(I) = \int_I f \) is Henstock’s Lemma. This is merely being combined with the triangle inequality \( ||F(J) - |f\lambda|(J)|| \leq |F(J) - (f\lambda)(J)|. \)

For a tagged interval function \( \varphi \) to \( R \) and \( X \subset R^m \), we say \( \varphi \) is AC on \( X \) if for every \( \epsilon > 0 \), there exists \( \eta > 0 \) such that \( |\varphi|(\pi) < \epsilon \), whenever \( \pi \) is a partition with tags in \( X \) with \( \lambda(\pi) < \eta \). \( \varphi \) is AC on \( X \) if \( X \) can be written as the union of countably many sets on which \( \varphi \) is AC.

**Lemma 1.** The tagged interval function \( f\lambda \) is AC on \( K \).

**Proof.** Indeed, for \( \pi \) tagged in \( (|f| \leq n) \), \( \lambda(\pi) < \epsilon/n \) implies \( |f\lambda|(\pi) < \epsilon \).

It may not follow that the indefinite \( B \)-KH integral is AC. It seems we need to strengthen the hypothesis of AC to get the desired descriptive characterization of KH integrals by derivatives. Say that \( \varphi \) is B-AC on \( X \), if for every \( \epsilon > 0 \), there exists \( \eta > 0 \) and a \( \beta \in B \), such that \( |\varphi|(\pi) < \epsilon \), whenever \( \pi \) is a partition with tags in \( X \), with \( \lambda(\pi) < \eta \) and with \( \pi \subset \beta \); \( \varphi \) is B-AC.
on $X$, if $X$ can be written as the union of countably many sets on which $\varphi$ is $B$-ACG*
.

The bases $B$ of our 3 cases are compatible with the topology, or “fine”, in the sense that if $G$ is open, there exists a $\beta \in B$ such that for each $x \in G$, if $(x, J) \in \beta$, then $J \subset G$. They also are $\sigma$-decomposable [Tho02] (or of $\sigma$-local character [Ost86]), in the sense that if $(\beta_n)$ is a sequence of members of $B$ and $(X_n)$ is a disjoint sequence of subsets of $K$, then there is a $\beta \in B$ with $\beta[X_n] \subset \beta_n[X_n]$, for all $n \in \mathbb{N}$.

**Lemma 2.** If a tagged interval function $\varphi$ is $B$-ACG* on $K$, then $B$-V$\varphi(N) = 0$, for each Lebesgue nullset $N$.

That is, $V\varphi$ is absolutely continuous with respect to Lebesgue measure.

**Proof.** One really only needs $B$-ACG* on the nullset $N$. Write $N$ as a disjoint union $\bigcup_n N_n$, where $\varphi$ is $B$-AC* on $N_n$.

For a given $\epsilon > 0$, choose $\eta_n > 0$ and $\beta_n \in B$ such that for each partition tagged in $N_n$ of total measure $\lambda(\pi) < \eta_n$, with $\pi \subset \beta_n$, $|\varphi(\pi)| < \epsilon/2^n$. Cover each $N_n$ with an open $G_n$ set of measure less than $\eta_n$, then choose a $\beta_n' \subset \beta_n$ such that $x \in G_n$ and $(x, J) \in \beta_n'$ implies $J \subset G_n$. Now find a new $\beta'$ such that $\beta[X_n] \subset \beta_n[X_n]$, for all $n \in \mathbb{N}$, so that $x \in G_n$ and $(x, J) \in \beta'$ implies $J \subset G_n$. Now any partition $\pi \subset \beta'$ will have $\lambda(\pi[N_n]) < \eta_n$, so $|\varphi(\pi[N_n])| < \epsilon/2^n$, and hence $|\varphi(\pi[N]) < \epsilon$. This shows $V\varphi(N) = 0$, as required.

As a consequence, the Key Lemma yields an $\mathbb{R}^n$ version of Bartle’s Theorem [LV00], 3.9.1. [Bar01], 5.12

**Theorem 3.** An additive function $F$ is the indefinite integral of a $B$-KH integrable $f$ if $DF = f$ except on a nullset $N$ with $VF(N) = 0$. In that case, $VF$ vanishes on every nullset; that is, is absolutely continuous with respect to $\lambda$.

**Proof.** If $N$ is null, then $f\lambda(N) = 0$, since $f\lambda$ is ACG*. So if also $VF(N) = 0$, $F$ is the indefinite integral of $f$, by the Key Lemma (1). But the second half then says $VF = V(f\lambda)$, which is 0 at each nullset.

Call a function $f$ $B$-Denjoy integrable if it is almost everywhere equal to the $B$-derivative of an additive function $F$ which is $B$-ACG*. The function $F$ will be called the primitive of $f$. Let $B$ be the $r$-regular basis for $0 < r < 1$, or the regular basis, so that the Vitali Covering Theorem holds. We obtain the descriptive characterization of the $B$-KH integral as a Denjoy integral.

**Theorem 4.** A function $f$ on $K$ is $B$-Denjoy integrable with primitive $F$ if and only if it is $B$-KH integrable, with indefinite integral $F$. 
Proof. If $DF = f$ outside a nullset $N$ and $F$ is $\mathcal{B}$-$\text{ACG}_*$, then $VF(N) = V(f\lambda)(N) = 0$, so by the Key Lemma, $F$ is a KH-primitive of $f$.

For the converse, if $F$ is a KH-primitive of $f$ and $N = (DF \neq f)$, Henstock’s Lemma and the Vitali property yield $N$ is null as in [Ost86], page 37. $DF = f$ almost everywhere and the fact that $|F|(\pi) - |\lambda f|(\pi) \to 0$ yields $F$ is $\mathcal{B}$-$\text{ACG}_*$.

If the $r$-regular base (call it $\mathcal{B}_r$) is used, we may refer to the $r$-derivative, the $r$-Denjoy integral, and the $r$-KH-integral. If all the $r$-derivatives $r \cdot DF(x)$, $0 < r < 1$, exist, they must be equal and their common value is the ordinary derivative [Sak64]. Actually, Kurzweil and Jarník [KJ92] have shown that $r$-differentiability for some $r$ implies ordinary differentiability, but that $r$-integrability depends on the choice of $r$. We will call a function $f$ ordinary Denjoy integrable if it is $r$-Denjoy integrable for each regularity $r$. It is important that this includes the condition that $F$ is $\mathcal{B}_r$-$\text{ACG}_*$, for each $r$ separately. If all the $r$-KH-integrals of $f$ exist, they also must have a common value. This value is known as the M-integral [BDPS01],[DP01] after Mahwin, who introduced it in an equivalent form [Maw81].

**Corollary 5.** A function $f$ on $K$ is M-integrable with indefinite integral $F$, if and only if it is ordinary Denjoy integrable with primitive $F$.

One might note that since $\text{ACG}_*$ implies $\mathcal{B}$-$\text{ACG}_*$, for any choice of interval basis, it follows that each $\text{ACG}_*$ ordinary differentiable additive $F$ is the indefinite $M$-integral of its derivative and also the KH-integral with respect to the regular basis.

**Remarks**

Recall that throughout, $VF$ referred to $\mathcal{B}$-$VF$, the $\mathcal{B}$-variational measure. Since the talk, we learned that Di Piazza [DP01] defines $V_M F = \sup_{r \in (0,1)} B^rVF$ and proves that the ordinary derivative of $F$ exist almost everywhere and $F$ is its indefinite $M$-integral if and only if $V_M F$ is absolutely continuous.

In the one-dimensional case (where all 3 bases coincide), Gordon [Gor94] refers to $\mathcal{B}$-$\text{ACG}_*$ as $\text{ACG}_\delta$. Chew [Che90], working with the regular basis, uses the notation $\text{ACG}^*$, with the asterisk raised instead of lowered. Chew, Lemma 1, shows that for a continuous additive function of intervals these coincide with the classical definition.

We have been unable to determine, for any of the 3 bases, whether every additive $\mathcal{B}$-$\text{ACG}_*$ function on $I$ is $\text{ACG}_*$.

**Open Problem.** For what bases do the notions of $\text{ACG}_*$ and $\mathcal{B}$-$\text{ACG}_*$ coincide for additive interval functions? (\ldots for indefinite $\mathcal{B}$-KH integrals?)

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References


