# On Universal Central Extensions of $\mathfrak{sl}_n(A)$

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Abstract. We describe the interplay between several (co)homology theories in the context of computing universal central extensions of Lie algebras of matrices over some associative algebras. We illustrate these methods with an explicit computation of the universal central extension of the Lie algebra of $n \times n$ matrices over a family of localizations of the Weyl algebra $A_r$.

## 1. Introduction

Central extensions arise in several contexts of mathematics and theoretical physics. They are used to enrich and elucidate representation theory, and they capture some anomalies that occur when classical theories are quantized. The universal central extensions satisfy a universal mapping property and play a prominent role in the representation theory of infinite-dimensional Lie algebras.

In most cases, however, computation of these extensions is a nontrivial problem in homological algebra. Simple and elegant presentations are known in great generality (cf. [**BM**], [**KL**], or [**vdK**], for instance), but it is seldom straightforward to explicitly compute even the dimension of a universal central extension in a concrete setting.

Much work has been done for central extensions of Lie algebras of matrices over commutative rings. (See [**BK**], [**Bl**], [**Ga**], [**He**], [**Ka**], or [**Zu**], for example.) These include the (untwisted) affine Lie algebras, the toroidal Lie algebras, and many algebras graded by abelian groups, including the algebras of currents studied by physicists.

In this short paper, we use the interplay between several (co)homology theories to calculate the universal central extensions of matrix Lie algebras over some associative algebras that are far from commutative. These algebras include the rings of differential operators on some nonsingular affine varieties. We illustrate the techniques with an explicit computation of the dimensions of the universal central extensions of the Lie algebras of matrices over a family of localizations of the Weyl algebra $A_r$.

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Throughout this paper, $\mathbb{K}$ will be a field of characteristic zero, all tensor products will be over $\mathbb{K}$, and all associative algebras will be assumed to be unital. The integers and nonnegative integers will be denoted $\mathbb{Z}$ and $\mathbb{N}$, respectively.

## 2. Central extensions and cyclic homology

**2.1. Central extensions.** In this section, we give a brief review of the basic definitions and theory of Lie algebra central extensions. More details can be found in [**Ga**], [**MP**], [**Ne**], or [**vdK**].

Let $\mathfrak{g}$ be an arbitrary Lie algebra over $\mathbb{K}$. A *central extension* of $\mathfrak{g}$ is a Lie algebra $\widetilde{\mathfrak{g}}$ and an epimorphism $\phi: \widetilde{\mathfrak{g}} \to \mathfrak{g}$ with kernel $\ker\phi$ contained in the centre of $\widetilde{\mathfrak{g}}$. Such a central extension is said to have *kernel* $\ker\phi$ and dimension $\dim(\ker\phi)$. Given two central extensions $(\widehat{\mathfrak{g}},\psi)$ and $(\widetilde{\mathfrak{g}},\phi)$ of $\mathfrak{g}$, a *morphism* $\big($from $(\widehat{\mathfrak{g}},\psi)$ to $(\widetilde{\mathfrak{g}},\phi)\big)$ in the category of central extensions is a Lie algebra homomorphism $\mu: \widehat{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ such that $\phi\circ\mu=\psi$. We say that $(\widehat{\mathfrak{g}},\psi)$ and $(\widetilde{\mathfrak{g}},\phi)$ are *isomorphic* if the morphism $\mu$ is a bijection. The central extension $(\widehat{\mathfrak{g}},\pi)$ is *universal* if there is a unique morphism from it to every other central extension of $\mathfrak{g}$. There is a well-known isomorphism between $\ker\pi$ and the (Chevalley-Eilenberg) homology group $H_2(\mathfrak{g};\mathbb{K})$ of $\mathfrak{g}$ with coefficients in the trivial module $\mathbb{K}$. (See [**We**] §7.7, for instance.)

Isomorphism classes of one-dimensional central extensions are in bijective correspondence with cohomology classes in $H^2(\mathfrak{g};\mathbb{K})$. In particular, each class $[c]$ in $H^2(\mathfrak{g};\mathbb{K})$ determines a central extension $\widetilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathbb{K}\mathbf{c}$ with the bracket $[\,\cdot\,,\cdot\,]^{\sim}$ in $\widetilde{\mathfrak{g}}$ given by $[x,y]^{\sim}=[x,y]+c(x,y)\mathbf{c}$ for $x,y\in\mathfrak{g}$, where $\mathbf{c}$ is central and $c$ is a representative of the class $[c]$. Conversely, if $c: \mathfrak{g}\times\mathfrak{g}\to\mathbb{K}$ is $\mathbb{K}$-bilinear and $\widetilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathbb{K}\mathbf{c}$ is a Lie algebra under the bracket $[\,\cdot\,,\cdot\,]^{\sim}$ defined above, then $c$ is a representative of a cohomology class $[c]$ in $H^2(\mathfrak{g};\mathbb{K})$. Direct computation with the chain complex defining $H^2(\mathfrak{g};\mathbb{K})$ shows that $[c]\in H^2(\mathfrak{g};\mathbb{K})$ if and only if $c$ satisfies the *2-cocycle conditions:*

(C1) $c(x,y)=-c(y,x)$ and
(C2) $c([x,y],z)+c([y,z],x)+c([z,x],y)=0.$

If $\mathfrak{g}$ has a universal central extension $(\widehat{\mathfrak{g}},\pi)$, then by universality, there is a bijection between isomorphism classes of one-dimensional central extensions of $\mathfrak{g}$ and $\mathbb{K}$-linear maps from $\ker\pi$ to $\mathbb{K}$. That is, $H^2(\mathfrak{g};\mathbb{K})\cong\mathrm{Hom}_{\mathbb{K}}(H_2(\mathfrak{g};\mathbb{K}),\mathbb{K})$, which is precisely the isomorphism given by the universal coefficient theorem.

A Lie algebra $\mathfrak{g}$ is *perfect* if it is equal to its commutator subalgebra $[\mathfrak{g},\mathfrak{g}]$. The following lemma appears in van der Kallen [**vdK**]:

Lemma 2.1.

1. *If* $\widehat{\mathfrak{g}}$ *and* $\widetilde{\mathfrak{g}}$ *are universal central extensions of a Lie algebra* $\mathfrak{g}$*, then* $\widehat{\mathfrak{g}}\cong\widetilde{\mathfrak{g}}$ *as central extensions.*
2. *If* $\widehat{\mathfrak{g}}$ *is the universal central extension of a Lie algebra* $\mathfrak{g}$*, then* $\widehat{\mathfrak{g}}$ *is perfect.*
3. *A Lie algebra* $\mathfrak{g}$ *has a universal central extension if and only if* $\mathfrak{g}$ *is perfect.*

□

Many of the most important infinite-dimensional Lie algebras, including the Heisenberg Lie algebras, the (untwisted) affine Lie algebras, the toroidal Lie algebras, and the Virasoro Lie algebra can be viewed as nontrivial central extensions.

**2.2. Cyclic homology.** As we have already noted, the kernel $\ker \pi$ of the universal central extension $(\widehat{L}, \pi)$ is isomorphic to the homology group $H_2(L; \mathbb{K})$. In the case where $L = \mathfrak{sl}_n(\mathbb{K}) \otimes A$ for some commutative associative algebra $A$ over $\mathbb{K}$, this homology group is isomorphic as a vector space to a module of differential forms which depends only on $A$.

For any commutative associative algebra $A$, let $\Omega^1_A$ be the left $A$-module generated by the set of symbols $dA = \{da \mid a \in A\}$ modulo the relations $d(\lambda a + b) = \lambda da + db$ and $d(ab) = a(db) + b(da)$ for all $a, b \in A$ and $\lambda \in \mathbb{K}$. Assuming only that $\mathbb{K}$ is a commutative ring of characteristic not equal to 2, S. Bloch [**Bl**] showed that $H_2(\mathfrak{sl}_n(\mathbb{K}) \otimes A; \mathbb{K}) \cong \Omega^1_A/dA$ for $n \geq 5$. In [**Ka**], this is extended to the case where $n \geq 2$, using our additional hypothesis that $\mathbb{K}$ has characteristic 0.

Proposition 2.2. [1]
*Let $A$ be a commutative associative algebra (over $\mathbb{K}$). Then as vector spaces,*

$$H_2\big(\mathfrak{sl}_n(\mathbb{K}) \otimes A; \mathbb{K}\big) \cong \Omega^1_A/dA \text{ for } n \geq 2.$$

□

If $A$ is an arbitrary associative algebra (not necessarily commutative), then let $\mathfrak{sl}_n(A)$ be the Lie algebra of $n \times n$ matrices $X$ over $A$ whose trace $\mathrm{tr}(X) \in [A, A]$, where $[A, A]$ is the $\mathbb{K}$-span of the elements $ab - ba$ for $a, b \in A$. The Lie algebra $\mathfrak{sl}_n(A)$ is perfect for $n \geq 2$, and its universal central extension is described by cyclic (co)homology, a noncommutative analogue of de Rham cohomology introduced in the early 1980s by A. Connes [**Co**].

For any associative algebra $A$ over $\mathbb{K}$, the *cyclic complex* $C_*(A)$ consists of the spaces $C_n(A) = A^{\otimes(n+1)}/\mathrm{im}(\tau - 1)$ of coinvariants of the signed permutation action

$$\tau = \tau_n : \ a_0 \otimes \cdots \otimes a_n \mapsto (-1)^n a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \tag{2.1}$$

on the Hochschild complex $A^{\otimes(n+1)}$, together with the usual Hochschild boundary maps

$$d : \ a_0 \otimes \cdots \otimes a_n \mapsto (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} + \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

Dualizing gives the *cyclic cocomplex* $C^*(A) = \mathrm{Hom}_{\mathbb{K}}(C_*(A), \mathbb{K})$ with coboundary maps $\partial : \ C^n(A) \to C^{n+1}(A)$ defined by

$$\big(\partial\gamma\big)\big((a_0 \otimes \cdots \otimes a_{n+1}) + \mathrm{im}(\tau - 1)\big) = \gamma\Big(d\big((a_0 \otimes \cdots \otimes a_{n+1}) + \mathrm{im}(\tau - 1)\big)\Big)$$

for each $\gamma \in \mathrm{Hom}_{\mathbb{K}}(C_n(A), \mathbb{K})$. Its cohomology is called *(Connes) cyclic cohomology*, and is denoted $HC^*(A)$.

Motivated by the work of S. Bloch [**Bl**] in the commutative context, C. Kassel and J.-L. Loday [**KL**] substituted the cyclic homology group $\mathrm{HC}_1(A)$ for the module $\Omega^1_A/dA$ that appeared in Bloch's commutative setting. Using a Morita equivalence argument, they proved the following theorem ([**KL**], Thm 1.7) for any associative algebra $A$ over $\mathbb{K}$. Their result also holds over fields of characteristic $p \neq 2$.

Theorem 2.3. *If $n \geq 3$, then $H_2\big(\mathfrak{sl}_n(A); \mathbb{K}\big) \cong HC_1(A)$.* □

[1] This result was also generalized by Kassel [**Ka**] to include Lie algebras of the form $\mathfrak{g} \otimes A$ for any classical Lie algebra $\mathfrak{g}$ and commutative associative algebra $A$. Berman and Krylyuk [**BK**] later extended Kassel's work to the twisted setting.

It is interesting to note that this result does not hold for $n=2$. A counterexample appears in [**Gao**].

For $n \geq 2$, the Lie algebra $\mathfrak{sl}_n(A)$ is perfect, but it need not be centreless. In general, the centre of $\mathfrak{sl}_n(A)$ is the set of central scalar matrices, $\{aI \mid a \in Z(A) \cap [A,A]\}$, where $Z(A)$ is the centre of $A$. In fact, $\mathfrak{sl}_n(A)$ is a (dimension $\dim(Z(A) \cap [A,A])$) central extension of the centreless quotient algebra $\mathfrak{sl}_n(A)/Z(\mathfrak{sl}_n(A))$. This quotient is isomorphic to $\big(\mathfrak{sl}_n(\mathbb{K}) \otimes A\big) \oplus \mathrm{ad}\,[A,A]$, which has multiplication

$$[x \otimes a, y \otimes b] = [x,y] \otimes \frac{1}{2}(a \circ b) + (x \circ y) \otimes \frac{1}{2}[a,b] + \frac{(x \,|\, y)}{n}\mathrm{ad}\,[a,b] \tag{2.2}$$

$$[\mathrm{ad}\, a, x \otimes b] = x \otimes [a,b] = -[x \otimes b, \mathrm{ad}\, a] \tag{2.3}$$

$$[\mathrm{ad}\, a, \mathrm{ad}\, b] = \mathrm{ad}\,[a,b] \tag{2.4}$$

where $x, y \in \mathfrak{sl}_n(\mathbb{K})$, $a, b \in A$, $[a,b] = ab - ba$, $a \circ b = ab + ba$, $(x \,|\, y) = \mathrm{tr}(xy)$, $[x,y] = xy - yx$ and $x \circ y = xy + yx - \dfrac{2}{n}\mathrm{tr}(xy)I_n$.

More precisely, any element $x$ in $\mathfrak{sl}_n(A)$ may be written as the sum of the matrix $x_0 = x - \dfrac{1}{n}\mathrm{tr}(x)I$ of trace 0 and the matrix $\dfrac{1}{n}\mathrm{tr}(x)I$ where $\mathrm{tr}(x) \in [A,A]$. Then the map $\psi : \ \mathfrak{sl}_n(A) \to (\mathfrak{sl}_n(\mathbb{K}) \otimes A) \oplus \mathrm{ad}\,[A,A]$ given by

$$\psi(x) = x_0 + \mathrm{ad}\ \mathrm{tr}(x) \tag{2.5}$$

is a well-defined Lie algebra homomorphism. It is clearly a surjection and $\ker \psi = \{aI \mid a \in [A,A] \cap Z(A)\} = Z(\mathfrak{sl}_n(A))$. The universal central extension of $(\mathfrak{sl}_n(\mathbb{K}) \otimes A) \oplus \mathrm{ad}\,[A,A]$ is the same as the universal central extension of $\mathfrak{sl}_n(A)$, and thus its kernel is isomorphic to the space $(Z(A) \cap [A,A]) \oplus HC_1(A)$. Computing directly at the chain level, we see that

$$HC_1(A) = \left\{ \sum_i a_i \otimes b_i + \mathcal{S}_A \in \{A,A\} \;\middle|\; \sum_i [a_i, b_i] = 0 \right\}$$

where $\mathcal{S}_A = \mathrm{Span}\{a \otimes b + b \otimes a,\ ab \otimes c + bc \otimes a + ca \otimes b \mid a,b,c \in A\}$ and $\{A,A\} = (A \otimes A)/\mathcal{S}_A$. Then $(Z(A) \cap [A,A]) \oplus HC_1(A) \cong HF(A)$ where

$$HF(A) = \left\{ \sum_i a_i \otimes b_i + \mathcal{S}_A \in \{A,A\} \;\middle|\; \sum_i \mathrm{ad}\,[a_i, b_i] = 0 \right\}.$$

THEOREM 2.4 (Allison, Benkart, and Gao [**ABG**]). *Let $A$ be an arbitrary associative algebra, and assume $n \geq 3$. Let $L$ be the centreless Lie algebra $\big(\mathfrak{sl}_n(\mathbb{K}) \otimes A\big) \oplus \mathrm{ad}\,[A,A]$ with Lie bracket given by (2.2), (2.3), and (2.4). Then the universal central extension of $L$ is $(\widehat{L}, \widehat{\pi})$ where $\widehat{L} = (\mathfrak{sl}_n(\mathbb{K}) \otimes A) \oplus \{A,A\}$ and $\widehat{L}$ has Lie bracket*

$$[x \otimes a, y \otimes b]\widehat{\ } = [x,y] \otimes \frac{1}{2}(a \circ b) + (x \circ y) \otimes \frac{1}{2}[a,b] + \mathrm{tr}(xy)\{a,b\}$$

$$[\{a,b\}, x \otimes c]\widehat{\ } = x \otimes \widehat{\pi}(\{a,b\})c$$

$$[\{a,b\}, \{c,d\}]\widehat{\ } = \{\widehat{\pi}(\{a,b\})c, d\} + \{c, \widehat{\pi}(\{a,b\})d\},$$

*where $x, y \in \mathfrak{sl}_n(\mathbb{K})$, $a,b,c,d \in A$, $\{a,b\} = a \otimes b + \mathcal{S}_A \in (A \otimes A)/\mathcal{S}_A = \{A,A\}$. The covering map $\widehat{\pi} : \ \widehat{L} \to L$ is given by $\widehat{\pi}(x \otimes a) = x \otimes a$ and $\widehat{\pi}(\{a,b\}) = \dfrac{1}{n}\mathrm{ad}\,[a,b]$, and the kernel of $\widehat{\pi}$ is $HF(A)$.*

## 3. Connes sequence and Künneth theorem

For any (unital) associative algebra $A$, the cyclic cochains are the invariants of the natural action (2.1) of the signed cyclic permutations $\tau = \tau_n$ on the spaces $\mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n+1)}, \mathbb{K})$. There is a vector space isomorphism

$$\psi : \ \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n+1)}, \mathbb{K}) \to \mathrm{Hom}_{\mathbb{K}}(A^{\otimes n}, A^*)$$

given by

$$(\psi f)(a_1 \otimes \cdots a_n)(a) = f(a \otimes a_1 \otimes \cdots \otimes a_n)$$

for all $f \in \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n+1)}, \mathbb{K})$ and $a, a_1, \dots a_n \in A$.

The space $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ has an $A$-bimodule structure given by

$$(a\gamma b)(c) = \gamma(bca) \text{ for } \gamma \in A^* \text{ and } a, b, c \in A. \tag{3.1}$$

This lets us interpret $\mathrm{Hom}_{\mathbb{K}}(A^{\otimes n}, A^*)$ as a Hochschild cocomplex $C^*_H(A; A^*)$ with coboundary map $b : \ C^n_H(A; A^*) \to C^{n+1}_H(A; A^*)$ given by

$$\begin{aligned}
&(b\gamma)(a_1 \otimes \cdots \otimes a_{n+1}) \\
&\qquad = \quad a_1\gamma(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i \gamma(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\
&\qquad\qquad + (-1)^{n+1}\gamma(a_1 \otimes \cdots \otimes a_n)a_{n+1}
\end{aligned}$$

for $\gamma \in C^n_H(A; M)$.

Note that some care is needed to properly interpret the terms in the low dimensions. The space $C^0_H(A; A^*) = \mathrm{Hom}_{\mathbb{K}}(A, A^*)$ can be identified with $A^*$ by identifying each $\gamma \in C^0_H(A; A^*)$ with $\gamma(1) \in A^*$. Then the proper interpretation of the coboundary $b : \ C^0_H(A; A^*) \to C^1_H(A; A^*)$ is $(b\gamma)(a) = a\gamma - \gamma a$ for any $a \in A$ and $\gamma \in A^*$.

Since the cyclic coboundary $\psi \circ \partial \circ \psi^{-1}$ is the restriction of the Hochschild coboundary $b$ to the image $\psi(C^*(A))$ of the cyclic cocomplex in $C^*_H(A; A^*)$, $\psi$ gives an embedding of $C^*(A)$ into $C^*_H(A; A^*)$ *as chain cocomplexes*. The embedding gives a short exact sequence

$$0 \longrightarrow C^*(A) \xrightarrow{\ \iota\ } C^*_H(A; A^*) \longrightarrow \frac{C^*_H(A; A^*)}{\iota(C^*(A))} \longrightarrow 0.$$

Connes ([**Co**], Cor 38) observed that the $n$th cohomology group of the quotient complex $\dfrac{C^*_H(A; A^*)}{\iota(C^*(A))}$ is isomorphic to $HC^{n-1}(A)$, so the short exact sequence induces the following long exact sequence, called the *Connes sequence*, relating the cyclic cohomology $HC^*(A)$ to the Hochschild cohomology $H^*(A; A^*)$:

THEOREM 3.1. *Let $A$ be an associative algebra over the field $\mathbb{K}$. Then the following is a long exact sequence.*

$$\cdots \longrightarrow H^n(A; A^*) \longrightarrow HC^{n-1}(A) \longrightarrow HC^{n+1}(A) \longrightarrow H^{n+1}(A; A^*) \longrightarrow \cdots$$

□

A trivial consequence of the Connes sequence is that $\mathrm{HC}^0(A) \cong H^0(A; A^*)$. By definition, the Hochschild cohomology group

$$\begin{aligned} H^0(A;A^*) &= \frac{\ker\left(b:\ C^0_H(A,A^*) \to C^1_H(A,A^*)\right)}{\mathrm{im}\left(b:\ 0 \to C^0_H(A,A^*)\right)} \\ &= \ker\left(b:\ C^0_H(A,A^*) \to C^1_H(A,A^*)\right) \\ &= \{\gamma \in A^* \mid a\gamma - \gamma a = 0 \text{ for all } a \in A\} \\ &= \{\gamma \in A^* \mid \gamma(ba) - \gamma(ab) = 0 \text{ for all } a,b \in A\}, \end{aligned}$$

and this expression is clearly 0 if $A = [A,A]$ as a Lie algebra under the canonical bracket $[a,b] = ab - ba$. When $A = [A,A]$, we say that $A$ is *Lie-perfect*. In this case, $HC^0(A) \cong H^0(A;A^*) = 0$, so the Connes sequence gives $HC^1(A) \cong H^1(A;A^*)$. Another straightforward computation at the chain level shows that $H^1(A;A^*) = \dfrac{\mathrm{Der}(A,A^*)}{\mathrm{Inn}(A,A^*)}$, where $\mathrm{Der}(A,A^*)$ (resp. $\mathrm{Inn}(A,A^*)$) is the space of derivations[2] $d$ (resp. inner derivations) from $A$ to the $A$-bimodule $A^*$. Since we are working over a field, $HC^k(A) \cong \mathrm{Hom}_{\mathbb{K}}(HC_k(A), \mathbb{K})$ (for every $k$) by the universal coefficient theorem. This proves the following useful proposition:

Proposition 3.2. *Let $A$ be an associative algebra that is Lie-perfect. Then as vector spaces,*

$$HC_0(A) = HC^0(A) = 0 \text{ and } HC^1(A) \cong \frac{\mathrm{Der}(A,A^*)}{\mathrm{Inn}(A,A^*)}.$$

□

We will also use the Künneth theorem for cyclic homology (cf. [**Lo**], Cor 4.3.12):

Theorem 3.3. *Let $A$ and $B$ be associative algebras (over $\mathbb{K}$). Then there is a natural long exact sequence:*

$$\cdots \longrightarrow HC_n(A\otimes B) \longrightarrow \bigoplus_{r+s=n} HC_r(A) \otimes HC_s(B) \longrightarrow \bigoplus_{i+j=n-2} HC_i(A)\otimes HC_j(B)$$
$$\longrightarrow HC_{n-1}(A\otimes B) \longrightarrow \bigoplus_{r+s=n-1} HC_r(A) \otimes HC_s(B) \longrightarrow \cdots$$

□

## 4. Localized Weyl algebras

For $r \geq 1$ and $0 \leq s \leq r$, let $A_{r,s}$ be the (unital) associative algebra generated by $\{p_i, q_i, q_j^{-1} \mid 1 \leq i \leq r \text{ and } 1 \leq j \leq s\}$, modulo the relations

$$[p_k, q_\ell] = \delta_{k,\ell}, \quad [p_k, p_\ell] = [q_k, q_\ell] = 0, \quad \text{and} \quad q_m q_m^{-1} = q_m^{-1} q_m = 1$$

for all $1 \leq k, \ell \leq r$ and $1 \leq m \leq s$. Then $A_{r,s}$ is the Weyl algebra $A_r = A_{r,0}$, localized at the multiplicative set $\{q_1^{a_1} q_2^{a_2} \cdots q_s^{a_s} \mid a_1, a_2, \ldots, a_s \in \mathbb{N}\}$. An easy computation shows that $A_{r,s}$ is *central simple*—that is, its centre $Z(A_{r,s}) = \mathbb{K}1$, and it has no nontrivial ideals. It has a natural irreducible, faithful representation on $\mathbb{K}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}, t_{s+1}, \ldots, t_r]$ by letting the $p_i$ act as differentiation operators $\dfrac{\partial}{\partial t_i}$ and the $q_i^{\pm 1}$ act as multiplication by $t_i^{\pm 1}$. If $\mathbb{K}$ is algebraically closed, then $A_{r,s}$

[2] $d: A \to A^*$ is a *derivation* if $d(ab) = (da).b + a.(db)$ for each $a, b \in A$. The map $d$ is *inner* if there is a fixed $\gamma \in A^*$ such that $d(a) = \gamma.a - a.\gamma$ for all $a \in A$.

is also the ring of differential operators on the variety $\underbrace{S^1 \times \cdots \times S^1}_{s\ terms} \times \underbrace{\mathbb{K} \times \cdots \times \mathbb{K}}_{r-s\ terms}$, where $S^1$ is the circle in $\mathbb{K}^2$ given by the equation $x^2 + y^2 = 1$ (cf. [**MR**], §15.5).

For $x = q$ or $p$, we use the multi-index notation $x^\alpha$ to denote $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{Z}^s \times \mathbb{N}^{r-s}$. For each $\gamma \in \mathbb{N}^r$, we write $\gamma!$ for the product $\gamma_1! \gamma_2! \cdots \gamma_r!$, and $\binom{\alpha}{\gamma}$ for the product $\prod_{i=1}^{r} \binom{\alpha_i}{\gamma_i}$. It is straightforward to verify the following lemma using induction:

Lemma 4.1. *Let $\alpha \in \mathbb{N}^r$ and $\beta \in \mathbb{Z}^s \times \mathbb{N}^{r-s}$. Then in $A_{r,s}$,*

$$p^\alpha q^\beta = \sum_{\gamma \in \mathbb{N}^r} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} q^{\beta-\gamma} p^{\alpha-\gamma}.$$

□

Lemma 4.1 lets us write any element of $A_{r,s}$ as a $\mathbb{K}$-linear combination of terms of the form $q^\alpha p^\beta$. Each element has a unique such representation, called its *standard form*. For each standard form monomial $q^\alpha p^\beta$ of $A_{r,s}$, we have

$$[p_i, q^\alpha p^\beta] = \alpha_i q^{\alpha - e_i} p^\beta$$

and

$$[q_i, q^\alpha p^\beta] = -\beta_i q^\alpha p^{\beta - e_i},$$

where $e_i = (0, \ldots, 1, \ldots, 0)$ with the nonzero entry 1 in the $i$th position. Since each element in $A_{r,s}$ has a unique standard form expression, we say that

$$\operatorname{ad} p_i = \frac{\partial}{\partial q_i} \tag{4.1}$$

$$\operatorname{ad} q_i = -\frac{\partial}{\partial p_i} \tag{4.2}$$

as operators on $A_{r,s}$ (for $1 \leq i \leq r$).

Since the standard form monomials

$$q^\alpha p^\beta = \left[ -\frac{1}{\beta_1 + 1} q_1, q^\alpha p^{\beta + e_1} \right]$$

span $A_{r,s}$, we see that

Lemma 4.2. *$A_{r,s}$ is Lie-perfect.* □

Let $\mathcal{W}$ be the associative algebra $A_{1,1}$. Viewed as a Lie algebra in the usual way, $\mathcal{W}$ arises as a limit of objects called $W_n$-algebras (see [**Za**], for instance), and is sometimes denoted $\mathcal{W}_{1+\infty}$. Observe that

$$A_{r,s} = \underbrace{\mathcal{W} \otimes \cdots \otimes \mathcal{W}}_{s\ terms} \otimes \underbrace{A_1 \otimes \cdots \otimes A_1}_{r-s\ terms},$$

where $A_1$ is the Weyl algebra $A_{1,0}$.

By Proposition 3.2 and Lemma 4.2, $HC_0(A_{r,s}) = 0$ for every $r$ and $s$. We use this fact, the Künneth theorem, and the decomposition $A_{r,s} = A_{r-1,s-1} \otimes \mathcal{W}$ to compute $HC_1(A_{r,s})$ for $r \geq 2$, $s \geq 1$:

Proposition 4.3. *If $r \geq 2$ and $s \geq 1$, then $HC_1(A_{r,s}) = 0$.*

Proof. Let $A = A_{r-1,s-1}$. Then by the Künneth theorem for cyclic homology (Theorem 3.3), the following is part of a long exact sequence:

$$\bigoplus_{k+\ell=0} HC_k(A)\otimes HC_\ell(\mathcal{W}) \longrightarrow HC_1(A\otimes\mathcal{W}) \longrightarrow \bigoplus_{k+\ell=1} HC_k(A)\otimes HC_\ell(\mathcal{W}).$$

But since $A$ and $\mathcal{W}$ are Lie-perfect, $HC_0(A) = HC_0(\mathcal{W}) = 0$, so both sums in the sequence above are 0. Hence $HC_1(A_{r,s}) = HC_1(A\otimes\mathcal{W}) = 0$. □

It remains only to compute $HC_1(A)$ for $A = A_r = A_{r,0}$ and $A = A_{1,1} = \mathcal{W}$, which we will do using a derivations computation and another (fairly trivial) application of the Künneth theorem.

For our derivations computations, we will need to define the formal integral, a linear operator on $A_1$ (resp. $\mathcal{W}$). For any $k,\ell\in\mathbb{N}$ (resp. $k\in\mathbb{Z}$ and $\ell\in\mathbb{N}$), let

$$\begin{aligned}\int q^k p^\ell\ dp &= \frac{1}{\ell+1}q^k p^{\ell+1},\\ \int q^k p^\ell\ dq &= \begin{cases}\frac{1}{k+1}q^{k+1}p^\ell & \text{if } k\neq -1\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Proposition 4.4. $\dim HC_1(A_1) = 0$ *and* $\dim HC_1(\mathcal{W}) = 1$.

Proof. By Lemma 4.2, Proposition 3.2, and the universal coefficient theorem, it is enough to show that every derivation from $A_1$ to $A_1^*$ is inner, and that $\mathrm{Der}(\mathcal{W},\mathcal{W}^*) = \mathrm{Inn}(\mathcal{W},\mathcal{W}^*)\oplus\mathbb{K}D$ for some non-inner derivation $D$. We begin by identifying such a non-inner derivation $D\in\mathrm{Der}(\mathcal{W},\mathcal{W}^*)\setminus\mathrm{Inn}(\mathcal{W},\mathcal{W}^*)$.

We will write $p,q,q^{-1}$ for the generators $p_1,q_1,q_1^{-1}$, respectively, of the algebras $A_1$ or $\mathcal{W}$. Observe that any derivation $\eta\in\mathrm{Der}(\mathcal{W},\mathcal{W}^*)$ must satisfy $0=\eta(1)=\eta(qq^{-1}) = (\eta q)q^{-1}+q(\eta q^{-1})$, so $\eta q^{-1} = -q^{-1}(\eta q)q^{-1}$. The relation $q^{-1}q = 1$ forces the same formula for $\eta q^{-1}$ in terms of $\eta q$. Thus any $\mathcal{W}^*$-valued map defined on the generators $p$ and $q$ can be extended to a derivation in $\mathrm{Der}(\mathcal{W},\mathcal{W}^*)$ if its (unique) extension respects the relation $pq-qp=1$.

Let $D(p)=0$ and $D(q)(q^rp^s) = \delta_{r,-1}\delta_{s,0}$ for all $r\in\mathbb{Z}$ and $s\in\mathbb{N}$. Then for any such $r$ and $s$,

$$\begin{aligned}D(pq-qp)(q^rp^s) &= \big((Dp)q + p(Dq) - (Dq)p - q(Dp)\big)(q^rp^s)\\ &= (Dq)([q^rp^s,p]) \;=\; -(Dq)(rq^{r-1}p^s) \;=\; 0.\end{aligned}$$

(Here we have used the definition of $\mathcal{W}^*$ as a $\mathcal{W}$-bimodule (Equation (3.1)).) Thus $D$ defines a derivation in $\mathrm{Der}(\mathcal{W},\mathcal{W}^*)$.

Suppose $D$ is inner. Then there exists some $\lambda\in\mathcal{W}^*$ such that $Dq = \lambda q - q\lambda$. But $1 = (Dq)(q^{-1}) = (\lambda q - q\lambda)(q^{-1}) = \lambda(qq^{-1}-q^{-1}q) = 0$, a contradiction. Hence $D\in\mathrm{Der}(\mathcal{W},\mathcal{W}^*)\setminus\mathrm{Inn}(\mathcal{W},\mathcal{W}^*)$.

For the remainder of the proof, let $A = A_1$ or $\mathcal{W}$ be fixed. For the sake of notational convenience, we will view $D$ as a derivation in $\mathrm{Der}(A,A^*)$ by defining $D=0$ if $A=A_1$. Let $\partial\in\mathrm{Der}(A,A^*)$, and define $\gamma = \partial - (\partial q)(q^{-1})D$. It suffices to show that $\gamma\in\mathrm{Inn}(A,A^*)$.

Let $m(x) = -\gamma(q)(\int x\ dp)$ for each $x\in A$. Note that $m\in A^*$ and $\mu(x) = m\circ\mathrm{ad}\,x = mx - xm$ defines an inner derivation $\mu\in\mathrm{Inn}(A,A^*)$. We will show that $\gamma=\mu$.

For any $y \in A_r$, we see that $y = \frac{\partial c}{\partial p}$ where $c = \int y \; dp$. Then using (4.1) and (4.2), we have

$$\begin{aligned} \mu(p)(y) &= m([p,y]) \\ &= m\left(\frac{\partial^2 c}{\partial q \partial p}\right) \\ &= m\left(\frac{\partial^2 c}{\partial p \partial q}\right) \\ &= -\gamma(q)\left(\int \frac{\partial^2 c}{\partial p \partial q}\; dp\right) \\ &= -\gamma(q)\left(\frac{\partial c}{\partial q} + f(q)\right). \end{aligned}$$

for some Laurent polynomial $f \in \mathbb{K}[q, q^{-1}]$.

Since $\gamma$ is a derivation, $0 = \gamma(1) = \gamma(0)$. Thus for every $a \in A$,

$$\begin{aligned} 0 \;=\; \gamma([p,q])(a) &= \big(\gamma(p)q + p\gamma(q) - \gamma(q)p - q\gamma(p)\big)(a) \\ &= \gamma(p)[q,a] - \gamma(q)[p,a]. \end{aligned}$$

Then by (4.1) and (4.2),

$$\gamma(p)\left(\frac{\partial a}{\partial p}\right) = -\gamma(q)\left(\frac{\partial a}{\partial q}\right). \tag{4.3}$$

Applying (4.3) gives

$$\begin{aligned} \mu(p)(y) &= -\gamma(q)\left(\frac{\partial c}{\partial q} + f(q)\right) \\ &= -\gamma(q)\left(\frac{\partial c}{\partial q}\right) - \gamma(q)\left(\frac{\partial}{\partial q}\int f(q)\; dq\right) \\ &= \gamma(p)\left(\frac{\partial c}{\partial p}\right) + \gamma(p)\left(\frac{\partial}{\partial p}\int f(q)\; dq\right) \\ &= \gamma(p)\left(\frac{\partial c}{\partial p}\right) \\ &= \gamma(p)(y). \end{aligned}$$

By similar arguments,

$$\mu(q)(y) = \gamma(q)(y).$$

Since $\mu$ and $\gamma$ agree on the generators of $A$, the derivations $\gamma$ and $\mu$ are equal. In particular, $\gamma \in \mathrm{Inn}(A, A^*)$. □

Since $A_r \cong \underbrace{A_1 \otimes \cdots \otimes A_1}_{r \text{ terms}}$, the following corollary now follows from the Künneth theorem (Theorem 3.3):

Corollary 4.5. $\dim HC_1(A_r) = 0$. □

Combining Propositions 4.3 and 4.4, and Corollary 4.5, we obtain the following theorem.

THEOREM 4.6. *For any $r \geq 1$ and $0 \leq s \leq r$,*

$$HC_1(A_{r,s}) \cong \begin{cases} \mathbb{K} & \text{if } (r,s) = (1,1) \\ 0 & \text{otherwise.} \end{cases}$$

□

Since $A_{r,s}$ is central simple and Lie-perfect, Theorem 4.6 also gives the skew-dihedral homology $HF(A_{r,s}) \cong Z(A_{r,s}) \cap [A_{r,s}, A_{r,s}]$.

COROLLARY 4.7.

$$\dim HF(A_{r,s}) = \begin{cases} 2 & \text{if } (r,s) = (1,1) \\ 1 & \text{otherwise.} \end{cases}$$

□

Explicit formulas for the 2-cocycles corresponding to the universal central extension of the centreless Lie algebra $\big(\mathfrak{sl}_n(\mathbb{K}) \otimes A_{r,s}\big) \oplus \operatorname{ad} A_{r,s}$ may be found in an elementary (though tedious) way (cf. [**La2**]) by choosing a basis for the skew-dihedral homology and then computing the projection formulas associated with the splitting $\{A, A\} = \operatorname{ad}[A, A] \oplus HF(A)$, in the notation of Theorem 2.4.

As we noted in the discussion before Theorem 2.4, the Lie algebra $\mathfrak{sl}_n(A_{r,s})$ is a central extension of $\big(\mathfrak{sl}_n(\mathbb{K}) \otimes A_{r,s}\big) \oplus \operatorname{ad} A_{r,s}$, and this central extension is universal if $(r,s) \neq (1,1)$ by Corollary 4.7. In Theorems 4.8 and 4.9 below, we give an explicit formula for the 2-cocycle corresponding to this central extension.

We also give a formula for the more interesting cocycle $\lambda$ corresponding to the universal central extension of $\mathfrak{sl}_n(A_{1,1})$. This cocycle was first described by Kac-Peterson [**KP**], though they do not indicate their method of computation. A representation-theoretic approach appears in [**La1**] and a conformal algebra approach in [**Ro**]. A description of the universal central extension using generating functions is given in [**VV**]. The universality of the resulting extensions follows from Corollary 4.7 and Theorem 2.4. Li [**Li**] was the first to give a proof of the universality of the extension in Theorem 4.9.

THEOREM 4.8. *Let $L = \big(\mathfrak{sl}_n(\mathbb{K}) \otimes A_{r,s}\big) \oplus \operatorname{ad} A_{r,s}$ where $(r,s) \neq (1,1)$ and $n \geq 3$. Then the map $\mu : L \times L \to \mathbb{K}$ given by*

(i) $\mu(x \otimes a, y \otimes b) = (x|y)\langle a, b\rangle$
(ii) $\mu(\operatorname{ad} a, \operatorname{ad} b) = n\langle a, b\rangle$
(iii) $\mu(x \otimes a, \operatorname{ad} b) = \mu(\operatorname{ad} b, x \otimes a) = 0$

*is a 2-cocycle defining the universal central extension of $L$, where $(x|y)$ is the trace $Tr(xy)$ and*

$$\langle q^\alpha p^\beta, q^\gamma p^\eta \rangle = \begin{cases} (\gamma)_\beta \ \delta_{\alpha+\gamma,\beta} & \text{if } \beta \neq 0 \text{ and } \eta = 0 \\ -(\alpha)_\eta \ \delta_{\alpha+\gamma,\eta} & \text{if } \beta = 0 \text{ and } \eta \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

*where $\alpha, \gamma \in \mathbb{Z}^s \times \mathbb{N}^{r-s}$, $\beta, \eta \in \mathbb{N}^r$, and*

$$(\alpha)_\eta = \prod_{i=1}^{r} (\alpha_i)(\alpha_i - 1) \cdots (\alpha_i - (\eta_i - 1)).$$

□

THEOREM 4.9. *Let $n \geq 3$ and $\mathcal{W} = A_{1,1}$. Then the universal central extension of $L = \big(\mathfrak{sl}_n(\mathbb{K}) \otimes \mathcal{W}\big) \oplus \operatorname{ad}\mathcal{W}$ is $\widetilde{L} := L \oplus \mathbb{K}\mathbf{c} \oplus \mathbb{K}\mathbf{d}$ with bracket $[\,\cdot\,,\cdot\,]^{\sim}$ given by*

$$[\alpha,\beta]^{\sim} = [\alpha,\beta] + \lambda(\alpha,\beta)\mathbf{c} + \mu(\alpha,\beta)\mathbf{d}$$
$$\mathbf{c},\mathbf{d} \in Z(\widetilde{L})$$

*where $[\,\cdot\,,\cdot\,]$ is the bracket of the Lie algebra $L$ defined in the discussion before Theorem 2.4, and $\lambda,\mu:\ L\times L \to \mathbb{K}$ are the 2-cocycles defined by the following formulas (in the notation of Theorem 4.8):*

(1a) $\lambda(x\otimes a, y\otimes b) = (x\,|\,y)\langle a,b\rangle_\lambda$

(1b) $\lambda(\operatorname{ad} a, \operatorname{ad} b) = n\langle a,b\rangle_\lambda$

(1c) $\lambda(x\otimes a, \operatorname{ad} b) = \lambda(\operatorname{ad} b, x\otimes a) = 0$

*and*

(2a) $\mu(x\otimes a, y\otimes b) = (x|y)\langle a,b\rangle_\mu$

(2b) $\mu(\operatorname{ad} a, \operatorname{ad} b) = n\langle a,b\rangle_\mu$

(2c) $\mu(x\otimes a, \operatorname{ad} b) = \lambda(\operatorname{ad} b, x\otimes a) = 0$

*where*

$$\langle q^k p^\ell, q^r p^s\rangle_\lambda = (-1)^{s+1}\ell!s!\binom{r}{\ell+s+1}\delta_{k+r,\ell+s}$$

$$\langle q^k p^\ell, q^r p^s\rangle_\mu = \begin{cases} (r)_\ell\,\delta_{k+r,\ell} & \text{if } \ell\neq 0 \text{ and } s=0\\ -(k)_s\,\delta_{k+r,s} & \text{if } \ell=0 \text{ and } s\neq 0\\ 0 & \text{otherwise.}\end{cases}$$

□

## 5. Conclusion

Since the ring of differential operators on any nonsingular affine variety is a homomorphic image of a subalgebra of a Weyl algebra $A_r$, the techniques described in this paper should be applicable to a large family of matrix Lie algebras. Our restriction to characteristic 0 is necessary, however, since rings of differential operators need not be Lie-perfect in positive characteristic. For example, if the base field $\mathbb{K}$ has characteristic $p > 0$, then none of the algebras $A_{r,s}$ satisfies the Lie-perfect condition. But in characteristic 0, we conjecture that the ring of differential operators on any nonsingular affine variety *is* Lie-perfect, and thus amenable to the approach described in this paper.

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