Differential Conformal Superalgebras and their Forms

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Abstract. We introduce the formalism of differential conformal superalgebras, which we show leads to the "correct" automorphism group functor and accompanying descent theory in the conformal setting. As an application, we classify forms of $N = 2$ and $N = 4$ conformal superalgebras by means of Galois cohomology.

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0 Introduction

The families of superconformal algebras described in the work of A. Schwimmer and N. Seiberg [17] bear a striking resemblance to the loop realization of the affine Kac-Moody algebras [8]. All of these algebras belong to a more general class known as Γ-*twisted formal distribution superalgebras*, where Γ is a subgroup of $\mathbb{C}$ containing $\mathbb{Z}$ [10, 11].

In a little more detail, a superconformal algebra[1] (or more generally, any twisted formal distribution algebra), is encoded by a conformal superalgebra $\mathcal{A}$ and an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$. Recall that $\mathcal{A}$ has a $\mathbb{C}[\partial]$-module structure and n-products $a_{(n)}b$, satisfying certain axioms [9]. Let σ be a diagonalizable automorphism of $\mathcal{A}$ with eigenspace decomposition

$$\mathcal{A} = \bigoplus_{\overline{m} \in \Gamma/\mathbb{Z}} \mathcal{A}_{\overline{m}},$$

where $\mathcal{A}_{\overline{m}} = \{a \in \mathcal{A} \mid \sigma(a) = e^{2\pi i m} a\}$, Γ is an additive subgroup of $\mathbb{C}$ containing $\mathbb{Z}$, and $\overline{m} \in \mathbb{C}/\mathbb{Z}$ is the coset $m + \mathbb{Z} \subset \mathbb{C}$.[2] Then the associated Γ-twisted formal distribution superalgebra $\mathrm{Alg}\,(\mathcal{A}, \sigma)$ is constructed as follows.

[1]e.g., the Virasoro algebra or its superanalogues

[2]If $\sigma^M = 1$, then this construction can be performed over an arbitrary algebraically closed field k of characteristic zero in the obvious way, by letting Γ be the group $\frac{1}{M}\mathbb{Z}$ and replacing $e^{2\pi i/M}$ with a primitive Mth root of 1 in k. This is the situation that will be considered in the present work.

Let $\mathcal{L}(\mathcal{A},\sigma) = \bigoplus_{m\in\Gamma} \left(\mathcal{A}_{\overline{m}} \otimes_{\mathbb{C}} t^m\right)$, and let

$$\mathrm{Alg}\,(\mathcal{A},\sigma) = \mathcal{L}(\mathcal{A},\sigma) \,/\, (\partial + \delta_t)\mathcal{L}(\mathcal{A},\sigma),$$

where ∂ denotes the map $\partial \otimes 1$ and δ_t is $1 \otimes \frac{d}{dt}$. For each $a \in \mathcal{A}_{\overline{m}}$ and $m \in \Gamma$, let a_m be the image of the element $a \otimes t^m \in \mathcal{L}(\mathcal{A},\sigma)$ in $\mathrm{Alg}\,(\mathcal{A},\sigma)$. These elements span $\mathrm{Alg}\,(\mathcal{A},\sigma)$, and there is a well-defined product on this space, given by

$$a_m b_n = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} \left(a_{(j)}b\right)_{m+n-j}, \tag{0.1}$$

for all $a \in \mathcal{A}_{\overline{m}}$ and $b \in \mathcal{A}_{\overline{n}}$.

The name *twisted formal distribution algebra* comes from the fact that the superalgebra $\mathrm{Alg}(\mathcal{A},\sigma)$ is spanned by the coefficients of the family of twisted pairwise local formal distributions

$$\mathcal{F} = \bigcup_{\overline{m}\in\Gamma/\mathbb{Z}} \left\{ a(z) = \sum_{k\in\overline{m}} a_k z^{-k-1} \mid a \in \mathcal{A}_{\overline{m}} \right\}.$$

For $\sigma = 1$ and $\Gamma = \mathbb{Z}$, we recover the maximal non-twisted formal distribution superalgebra associated with the conformal superalgebra $\mathcal{A}$. See [9, 10] for details.

For example, let A be an ordinary superalgebra over $\mathbb{C}$. The current conformal superalgebra $\mathcal{A} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} A$ is defined by letting $a_{(n)}b = \delta_{n,0}ab$ for $a, b \in A$ and extending these n-products to $\mathcal{A}$ using the conformal superalgebra axioms. The associated *loop algebra* $A \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ is then encoded by the current superconformal algebra $\mathcal{A}$. Taking σ to be an automorphism of $\mathcal{A}$ extended from a finite order (or, more generally, semisimple) automorphism of A, we recover the construction of a σ-twisted loop algebra associated to the pair $(\mathcal{A},\sigma)$. When A is a Lie algebra, this is precisely the construction of σ-twisted loop algebras described in [8].

Under the correspondence described above, the superconformal algebras on Schwimmer and Seiberg's lists are the Γ-twisted formal distribution algebras associated with the $N = 2$ and $N = 4$ Lie conformal superalgebras [9, 11]. Prior to Schwimmer and Seiberg's work, it was generally assumed that the $N = 2$ family of superconformal algebras consisted of infinitely many distinct isomorphism classes. However, it was later recognized that this family contains (at most) two distinct isomorphism classes. A similar construction with $N = 4$ superconformal algebras was believed to yield an infinite family of distinct isomorphism classes [17].

The connection of the construction of the superalgebra $\mathrm{Alg}\,(\mathcal{A},\sigma)$ to the theory of differential conformal superalgebras is as follows. The $\mathbb{C}[\partial]$-module $\mathcal{L}(\mathcal{A},\sigma)$ carries the structure of a differential conformal superalgebra with derivation $\delta=\delta_t$ and n-products given by

$$(a\otimes t^k)_{(n)}(b\otimes t^\ell)=\sum_{j\in\mathbb{Z}_+}\binom{k}{j}\big(a_{(n+j)}b\big)\otimes t^{k+\ell-j}. \tag{0.2}$$

Then $(\partial+\delta)\mathcal{L}(\mathcal{A},\sigma)$ is an ideal of $\mathcal{L}(\mathcal{A},\sigma)$ with respect to the 0-product, which induces the product given by (0.1) on $\mathrm{Alg}\,(\mathcal{A},\sigma)$. Moreover, the differential conformal superalgebra $\mathcal{L}(\mathcal{A},\sigma)$ is a twisted form of the affinization $L(\mathcal{A})=\mathcal{L}(\mathcal{A},id)$ of $\mathcal{A}$.

Thus, there are two steps to the classification of twisted superconformal algebras $\mathrm{Alg}\,(\mathcal{A},\sigma)$ up to isomorphism. First, we classify the twisted forms of the differential conformal superalgebra $L(\mathcal{A})$. In light of the above discussion, this gives a complete (but possibly redundant) list of superconformal algebras, obtained by factoring by the image of $\partial+\delta$ and retaining only the 0-product. Second, we should figure out which of these resulting superconformal algebras are non-isomorphic.

The second step of the classification is rather straightforward. For example, the twisted $N=4$ superconformal algebras are distinguished by the eigenvalues of the Virasoro operator L_0 on the odd part. The remainder of the paper will consider the first step, namely the classification of the $\mathcal{L}(\mathcal{A},\sigma)$ up to isomorphism.

Recently, the classification (up to isomorphism) of affine Kac-Moody algebras has been given in terms of torsors and non-abelian étale cohomology [16]. The present paper develops conformal analogues of these techniques, and lays the foundation for a classification of forms of conformal superalgebras by cohomological methods. These general results are then applied to classify the twisted $N=2$ and $N=4$ conformal superalgebras up to isomorphism.

To illustrate our methods, let us look at the case of the twisted loop algebras as they appear in the theory of affine Kac-Moody Lie algebras. Any such $\mathcal{L}$ is naturally a Lie algebra over $R:=\mathbb{C}[t^{\pm1}]$ and

$$\mathcal{L}\otimes_R S\simeq\mathfrak{g}\otimes_{\mathbb{C}} S\simeq(\mathfrak{g}\otimes_{\mathbb{C}} R)\otimes_R S \tag{0.3}$$

for some (unique) finite-dimensional simple Lie algebra $\mathfrak{g}$, and some (finite, in this case) étale extension S/R. In particular, $\mathcal{L}$ is an S/R-form of the R-algebra $\mathfrak{g}\otimes_{\mathbb{C}} R$, with respect to the étale topology of $\mathrm{Spec}(R)$. Thus $\mathcal{L}$

corresponds to a torsor over $\mathrm{Spec}(R)$ under $\mathbf{Aut}(\mathfrak{g})$ whose isomorphism class is an element of the pointed set $H^1_{\text{ét}}\big(R, \mathbf{Aut}(\mathfrak{g})\big)$.

Similar considerations apply to forms of the R-algebra $A \otimes_k R$ for any finite-dimensional algebra A over an algebraically closed field k of characteristic 0. The crucial point in the classification of forms of $A \otimes_k R$ by cohomological methods is that in the exact sequence of pointed sets

$$H^1_{\text{ét}}\big(R, \mathbf{Aut}^0(A)\big) \to H^1_{\text{ét}}\big(R, \mathbf{Aut}(A)\big) \xrightarrow{\psi} H^1_{\text{ét}}\big(R, \mathbf{Out}(A)\big), \tag{0.4}$$

where $\mathbf{Out}(A)$ is the (finite constant) group of connected components of A, the map ψ is injective [16].

Grothendieck's theory of the algebraic fundamental group allows us to identify $H^1_{\text{ét}}\big(R, \mathbf{Out}(A)\big)$ with the set of conjugacy classes of the corresponding finite (abstract) group $\mathrm{Out}(A)$. The injectivity of the map

$$H^1_{\text{ét}}\big(R, \mathbf{Aut}(A)\big) \xrightarrow{\psi} H^1_{\text{ét}}\big(R, \mathbf{Out}(A)\big)$$

means that to any form $\mathcal{L}$ of $A \otimes_k R$, we can attach a conjugacy class of the finite group $\mathrm{Out}(A)$ that characterizes $\mathcal{L}$ up to R-isomorphism. In particular, if $\mathbf{Aut}(A)$ is connected, then all forms (and consequently, all twisted loop algebras) of A are *trivial*–that is, isomorphic to $A \otimes_k R$ as R-algebras.

With the previous discussion as motivation, we now consider the $N = 2, 4$ Lie conformal superalgebras $\mathcal{A}$ described in [9]. The automorphism groups of these objects are as follows:

Table 1

N	$\mathrm{Aut}(\mathcal{A})$
2	$\mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$
4	$\big(\mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C})\big)/\pm I$

It was originally believed that the standard $N = 2$ algebra lead to an infinite family of non-isomorphic superconformal algebras (arising as Γ-twisted formal distribution algebras of the different $\mathcal{L}(\mathcal{A}, \sigma)$, as we explained above). This is somewhat surprising, for since $\mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$ has two connected components, one would expect (by analogy with the finite-dimensional case) that there would be only two non-isomorphic twisted loop algebras attached to $\mathcal{A}$. Indeed, Schwimmer and Seiberg later observed that all of the superconformal algebras in one of these supposedly infinite families are isomorphic [17], and that (at most) two such isomorphism classes existed.

On the other hand, since the automorphism group of the $N = 4$ conformal superalgebra is connected, one would expect all twisted loop algebras in this case to be trivial and, a fortiori, that all resulting superconformal algebras would be isomorphic. Yet Schwimmer and Seiberg aver in this case the existence of an infinite family of non-isomorphic superconformal algebras!

The explanation of how, in the case of conformal superalgebras, a connected automorphism group allows for an infinite number of non-isomorphic loop algebras is perhaps the most striking consequence of our work. Briefly speaking, the crucial point is as follows. A twisted loop algebra $\mathcal{L}$ of a k-algebra A is always split by an extension $S_m := k[t^{\pm 1/m}]$ of $R := k[t^{\pm 1}]$, for some positive integer m. The extension S_m/R is Galois, and its Galois group can be identified with $\mathbb{Z}/m\mathbb{Z}$ by fixing a primitive mth root of 1 in k. The cohomology class corresponding to $\mathcal{L}$ can be computed using the usual Galois cohomology $H^1\big(\mathcal{G}al(S_m/R), \mathbf{Aut}(A)(S_m)\big)$, where $\mathbf{Aut}(A)(S_m)$ is the automorphism group of the S_m-algebra $A \otimes_k S_m$. One can deal with all loop algebras at once by considering the direct limit $\widehat{S}$ of the S_m, which plays the role of the "separable closure" of R. In fact, $\widehat{S}$ is the simply connected cover of R (in the algebraic sense), and the algebraic fundamental group $\pi_1(R)$ of R at its generic point can thus be identified with $\widehat{\mathbb{Z}}$, namely the inverse limit of the groups $\mathcal{G}al(S_m/R) = \mathbb{Z}/m\mathbb{Z}$.

Finding the "correct" definitions of conformal superalgebras over rings and of their automorphisms leads to an explanation of how Schwimmer and Seiberg's infinite series in the $N = 4$ case is possible. In our framework, rings are replaced by rings equipped with a k-linear derivation (differential k-rings). The resulting concept of differential conformal superalgebra is central to our work, and one is forced to rewrite all the faithfully flat descent formalism in this setting. Under some natural finiteness conditions, we recover the situation that one encounters in the classical theory, namely that the isomorphism classes of twisted loop algebras of $\mathcal{A}$ are parametrized by $H^1\big(\widehat{\mathbb{Z}}, \mathbf{Aut}(\mathcal{A})(\widehat{S})\big)$, with $\widehat{\mathbb{Z}} = \mathcal{G}al(\widehat{S}/R)$ acting continuously via automorphisms of $\mathcal{A} \otimes_k \widehat{S}$.

In the $N = 2$ case, the automorphism group $\mathbf{Aut}(\mathcal{A})(\widehat{S}) = \widehat{S}^{\times} \rtimes \mathbb{Z}/2\mathbb{Z}$, and the cohomology set $H^1\big(\widehat{\mathbb{Z}}, \mathbf{Aut}(\mathcal{A})(\widehat{S})\big) \simeq \mathbb{Z}/2\mathbb{Z}$, as expected. By contrast, in the $N = 4$ case, $\mathbf{Aut}(\mathcal{A})(\widehat{S})$ is *not* $\big(\mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(\widehat{S})\big)/\pm I$ as we would expect from Table 1 above. In fact,

$$\mathbf{Aut}(\mathcal{A})(\widehat{S}) = \big(\mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(\mathbb{C})\big)/\pm I.$$

The relevant H^1 vanishes for $\mathbf{SL}_2(\widehat{S})$, but it is the somehow surprising appearance of the "constant" infinite group $\mathbf{SL}_2(\mathbb{C})$ in which the action of $\pi_1(R) = \widehat{\mathbb{Z}}$ is trivial that is ultimately responsible for an infinite family of non-isomorphic twisted conformal superalgebras that are parametrized by the conjugacy classes of elements of finite order of $\mathbf{PGL}_2(\mathbb{C})$.

In this paper, we introduce differential conformal (super)algebras, and show how these can be used for the study of forms of conformal (super)algebras. However, the theory of differential conformal (super)algebras reaches far beyond. For example, it is an adequate tool in the study of differential (super)algebras; see Remark 2.7d in [9]. The ordinary conformal (super)algebras do not quite serve this purpose since they allow only translationally invariant differential (super)algebras. Another area of applicability of differential conformal (super)algebras is the theory of not necessarily translation invariant evolution PDEs.

Notation: Throughout this paper, k will be a field of characteristic zero. Unless mentioned to the contrary, we denote $\otimes_k$ simply by $\otimes$. We will denote by $k - alg$ the category of unital commutative associative k-algebras. If k is algebraically closed, we also fix a primitive mth root of unity $\xi_m \in k$ for each $m > 0$. We assume that these roots of unity are chosen in a *compatible* fashion, that is, $\xi_{\ell m}^{\ell} = \xi_m$ for all positive integers ℓ and m.

The integers, nonnegative integers, and rationals will be denoted $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{Q}$, respectively. For pairs a, b of elements in a superalgebra, we let $p(a,b) = (-1)^{p(a)p(b)}$ where $p(a)$ (resp., $p(b)$) is the parity of a (resp., b).

Finally, for any linear transformation T of a given k-space V, and for any nonnegative integer n, we follow the usual convention for divided powers and define $T^{(n)} := \frac{1}{n!}T^n$.

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1 Conformal superalgebras

This section contains basic definitions and results about conformal superalgebras over differential rings. We recall that k denotes a field of characteristic 0, and $k - alg$ the category of unital commutative associative k-algebras.

1.1 Differential rings

For capturing the right concept of conformal superalgebras over rings, each object in the category of base rings should come equipped with a derivation.

This leads us to consider the category $k - \delta alg$ whose objects are pairs $\mathcal{R} = (R, \delta_R)$ consisting of an object R of $k - alg$ together with a k-linear derivation δ_R of R (a differential k–ring). A morphism from $\mathcal{R} = (R, \delta_R)$ to $\mathcal{S} = (S, \delta_S)$ is a k-algebra homomorphism $\tau : R \to S$ that commutes with the respective derivations. That is, the diagram

$$\begin{array}{ccc} R & \xrightarrow{\tau} & S \\ {\scriptstyle \delta_R}\downarrow & & \downarrow{\scriptstyle \delta_S} \\ R & \xrightarrow{\tau} & S. \end{array} \tag{1.1}$$

commutes.

For a fixed $\mathcal{R} = (R, \delta_R)$ as above, the objects in the category $\mathcal{R} - ext$ of *extensions of* $\mathcal{R}$ are the pairs $(\mathcal{S}, \tau)$, where $\mathcal{S} = (S, \delta_S) \in k - \delta alg$ and $\tau : R \to S$ satisfies (1.1). Each extension (S, δ_S) admits an obvious R-algebra structure: $s \cdot r = r \cdot s := \tau(r)s$ for all $r \in R$ and $s \in S$. The morphisms in $\mathcal{R} - ext$ are the R-algebra homorphisms commuting with derivations. That is, for any $S_1, S_2 \in \mathcal{R} - ext$, $\mathrm{Hom}_{\mathcal{R}-ext}(S_1, S_2)$ is the set of R-algebra homomorphisms in $\mathrm{Hom}_{k-\delta alg}(S_1, S_2)$.

Let $\mathcal{S}_i = \{(S_i, \delta_i) \mid 1 \leq i \leq n\}$ be a family of extensions of $\mathcal{R} = (R, \delta_R)$. Then

$$\delta := \sum_{i=1}^{n} \mathrm{id} \otimes \cdots \otimes \delta_i \otimes \cdots \otimes \mathrm{id}$$

is a k-linear derivation of $S_1 \otimes_R S_2 \otimes_R \cdots \otimes_R S_n$. The resulting extension $(S_1 \otimes_R S_2 \otimes_R \cdots \otimes_R S_n, \delta)$ of (R, δ_R) is called the *tensor product* of the $\mathcal{S}_i$ and is denoted by $\mathcal{S}_1 \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathcal{S}_n$.

Similarly, we define the *direct product* $\mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ by considering the k-derivation $\delta_1 \times \cdots \times \delta_n$ of $S_1 \times \cdots \times S_n$.

Example 1.2 Consider the Laurent polynomial ring $R = k[t, t^{-1}]$. For each positive integer m, we set $S_m = k[t^{1/m}, t^{-1/m}]$ and $\widehat{S} = \varinjlim S_m$.[3] We can think of $\widehat{S}$ as the ring $k[t^q \,|\, q \in \mathbb{Q}]$ spanned by all rational powers of the variable t. The k-linear derivation $\delta_t = \frac{d}{dt}$ of R is also a derivation of S_m and $\widehat{S}$. Thus $\mathcal{R} = (R, \delta_t)$, $\mathcal{S}_m = (S_m, \delta_t)$, and $\widehat{\mathcal{S}} = (\widehat{S}, \delta_t)$ are objects in $k - \delta alg$. Clearly $\mathcal{S}_m$ is an extension of $\mathcal{R}$, and $\widehat{\mathcal{S}}$ is an extension of $\mathcal{S}_m$ (hence also of $\mathcal{R}$). These differential k-rings will play a crucial role in our work.

[3] In [3], [4], and [5], where the multivariable case is considered, the rings R, S_m and $\widehat{S}$ were denoted by R_1, $R_{1,m}$ and $R_{1,\infty}$ respectively.

1.2 Differential conformal superalgebras

Throughout this section, $\mathcal{R} = (R, \delta_R)$ will denote an object of $k - \delta alg$.

Definition 1.3 *An $\mathcal{R}$-conformal superalgebra* is a triple $\big(\mathcal{A}, \partial_{\mathcal{A}}, (-{}_{(n)}-)_{n\in\mathbb{Z}_+}\big)$ consisting of

(i) a $\mathbb{Z}/2\mathbb{Z}$-graded R-module $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$,
(ii) an element $\partial_{\mathcal{A}} \in \operatorname{End}_k(\mathcal{A})$ stabilizing the even and odd parts of $\mathcal{A}$,
(iii) a k-bilinear product $(a, b) \mapsto a_{(n)}b$ for each $n \in \mathbb{Z}_+$,

satisfying the following axioms for all $r \in R$, $a, b, c \in \mathcal{A}$ and $m, n \in \mathbb{Z}_+$:

(CS0) $a_{(n)}b = 0$ for $n \gg 0$

(CS1) $\partial_{\mathcal{A}}(a)_{(n)}b = -na_{(n-1)}b$ and $a_{(n)}\big(\partial_{\mathcal{A}}(b)\big) = \partial_{\mathcal{A}}\big(a_{(n)}b\big) + na_{(n-1)}b$

(CS2) $\partial_{\mathcal{A}}(ra) = r\partial_{\mathcal{A}}(a) + \delta_R(r)a$

(CS3) $a_{(n)}(rb) = r(a_{(n)}b)$ and $(ra)_{(n)}b = \sum_{j\in\mathbb{Z}_+} \delta_R^{(j)}(r)\big(a_{(n+j)}b\big)$.

If $n = 0$, (CS1) should be interpreted as $\partial_{\mathcal{A}}(a)_{(0)}b = 0$. Note that $\partial_{\mathcal{A}}$ is a derivation of all n–products, called the *derivation of* $\mathcal{A}$. The binary operation $(a, b) \mapsto a_{(n)}b$ is called the *n-product of* $\mathcal{A}$.

If the $\mathcal{R}$-conformal superalgebra $\mathcal{A}$ also satisfies the following two axioms, $\mathcal{A}$ is said to be an *$\mathcal{R}$-Lie conformal superalgebra*:

(CS4) $a_{(n)}b = -p(a, b) \sum_{j=0}^{\infty}(-1)^{j+n}\partial_{\mathcal{A}}^{(j)}(b_{(n+j)}a)$

(CS5) $a_{(m)}(b_{(n)}c) = \sum_{j=0}^{m} \binom{m}{j}(a_{(j)}b)_{(m+n-j)}c + p(a, b)b_{(n)}(a_{(m)}c)$.

Remark 1.4 For a given $r \in R$ we will denote the corresponding homothety $a \mapsto ra$ by $r_{\mathcal{A}}$. Axiom (CS2) can then be rewritten as follows:

(CS2) $\partial_{\mathcal{A}} \circ r_{\mathcal{A}} = r_{\mathcal{A}} \circ \partial_{\mathcal{A}} + \delta_R(r)_{\mathcal{A}}$

Remark 1.5 If $R = k$, then δ_R is necessarily the zero derivation. Axioms (CS2) and (CS3) are then superfluous, as they simply say that $\partial_{\mathcal{A}}$ and the (n)-products are k-linear. The above definition thus specializes to the usual definition of conformal superalgebra over fields (cf. [9] in the case of complex numbers). Henceforth when referring to a conformal superalgebra over k, it will always be understood that k comes equipped with the trivial derivation.

Example 1.6 *(Affinization of a conformal superalgebra)* Let $\mathcal{A}$ be a conformal superalgebra over k. As in Kac [9],[4] we define the affinization $L(\mathcal{A})$ of $\mathcal{A}$ as follows. The underlying space of $L(\mathcal{A})$ is $\mathcal{A} \otimes_k k[t, t^{-1}]$, with the $\mathbb{Z}/2\mathbb{Z}$-grading given by assigning even parity to the indeterminate t . The derivation of $L(\mathcal{A})$ is

$$\partial_{L(\mathcal{A})} = \partial_{\mathcal{A}} \otimes 1 + 1 \otimes \delta_t$$

where $\delta_t = \frac{d}{dt}$, and the n-product is given by

$$(a \otimes f)_{(n)}(b \otimes g) = \sum_{j \in \mathbb{Z}_+} (a_{(n+j)}b) \otimes \delta_t^{(j)}(f)g \tag{1.7}$$

for all $n \in \mathbb{Z}_+$, $a, b \in \mathcal{A}$, and $f, g \in k[t, t^{-1}]$. It is immediate to verify that $\mathcal{A}$ is a k-conformal superalgebra. In fact, $\mathcal{A}$ is in the obvious way a $(k[t, t^{-1}], \delta_t)$–conformal superalgebra.

If $R = k[t, t^{-1}]$ and $\mathcal{R} = (R, \delta_t)$ are as in Example 1.2, then the affinization $L(\mathcal{A}) = \mathcal{A} \otimes_k R$ also admits an $\mathcal{R}$-conformal structure via the natural action of R given by $r'(a \otimes r) := a \otimes r'r$ for all $a \in \mathcal{A}$ and $r, r' \in R$. The only point that needs verification is Axiom (CS2), and this is straightforward to check.

Thus $\mathcal{A} \otimes_k R$ is both a k- and an $\mathcal{R}$-conformal superalgebra. We will need both of these structures in what follows. From a physics point of view, it is the k-conformal structure that matters; from a cohomological point of view, the $\mathcal{R}$-conformal structure is crucial.

We will also refer to the affinization $L(\mathcal{A}) = \mathcal{A} \otimes_k R$ as the *(untwisted) loop algebra of* $\mathcal{A}$. It will always be made explicit whether $L(\mathcal{A})$ is being viewed as a k- or as an $\mathcal{R}$-conformal superalgebra.

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{R} = (R, \delta_R)$-conformal superalgebras. A map $\phi : \mathcal{A} \to \mathcal{B}$ is called a *homomorphism* of $\mathcal{R}$-conformal superalgebras if it is an R-module homomorphism that is homogeneous of degree $\overline{0}$, respects the n-products, and commutes with the action of the respective derivations. That is, it satisfies the following three properties:

(H0) ϕ is R-linear and $\phi(\mathcal{A}_{\overline{\imath}}) \subseteq \mathcal{B}_{\overline{\imath}}$ for $\overline{\imath} = \overline{0}, \overline{1}$

(H1) $\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b)$ for all $a, b \in \mathcal{A}$ and $n \in \mathbb{Z}_+$

(H2) $\partial_{\mathcal{B}} \circ \phi = \phi \circ \partial_{\mathcal{A}}$.

[4] The definition given in [9] is an adaptation of affinization of vertex algebras, as defined by Borcherds [2].

By means of these morphisms we define the category of $\mathcal{R}$-conformal superalgebras, which we denote by $\mathcal{R} - conf$.

An $\mathcal{R}$-conformal superalgebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is an *isomorphism* if it is bijective; it is an *automorphism* if also $\mathcal{A} = \mathcal{B}$. The set of all automorphisms of an $\mathcal{R}$-conformal superalgebra $\mathcal{A}$ will be denoted $\mathrm{Aut}_{\mathcal{R}-\mathrm{conf}}(\mathcal{A})$, or simply $\mathrm{Aut}_{\mathcal{R}}(\mathcal{A})$.

Remark 1.8 To simplify some of the longer computations, it will be convenient to use the λ-product. This is the generating function $a_\lambda b$ defined as

$$a_\lambda b := \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} a_{(n)} b$$

for any pair of conformal superalgebra elements a, b and indeterminate λ, with $\lambda^{(n)} := \frac{1}{n!}\lambda^n$. In the case of Lie conformal superalgebras, we denote $a_\lambda b$ by $[a_\lambda b]$.

The condition (H1) that homomorphisms ϕ respect all n-products is equivalent to

(H1') $\phi(a_\lambda b) = \phi(a)_\lambda \phi(b)$

for all a, b.

Axiom (CS0) is equivalent to the property that $a_\lambda b$ is polynomial in λ. Axiom (CS1) is equivalent to:

$$\big(\partial_{\mathcal{A}}(a)\big)_\lambda b = -\lambda a_\lambda b \quad \text{and} \quad a_\lambda \partial_{\mathcal{A}} b = (\partial_{\mathcal{A}} + \lambda)(a_\lambda b). \tag{1.9}$$

Axiom (CS2) is equivalent to

$$a_\lambda rb = r(a_\lambda b) \quad \text{and} \quad (ra)_\lambda b = (a_{\lambda+\delta_R} b)_{\to} r, \tag{1.10}$$

where $\to$ means that δ_R is moved to the right and applied to r.

Remark 1.11 Note that the homotheties $r_{\mathcal{A}} :\ a \mapsto ra$ (for $r \in R$) are typically *not* $\mathcal{R}$-conformal superalgebra homomorphisms, and that the map $\partial_{\mathcal{A}} :\ a \mapsto \partial_{\mathcal{A}}(a)$ is a $\mathcal{R}$-conformal superalgebra derivation of the λ-product: $\partial_{\mathcal{A}}(a_\lambda b) = (\partial_{\mathcal{A}} a)_\lambda b + a_\lambda \partial_{\mathcal{A}} b$.

1.3 Base change

Let $\mathcal{S} = (S, \delta_S)$ be an extension of a base ring $\mathcal{R} = (R, \delta_R) \in k - \delta alg$. Given an $\mathcal{R}$-conformal superalgebra $\mathcal{A}$, the S-module $\mathcal{A} \otimes_R S$ admits an $\mathcal{S}$-conformal structure, which we denote by $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}$, that we now describe.

The derivation $\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}}$ is given by

$$\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}}(a\otimes s) := \partial_{\mathcal{A}}(a)\otimes s + a\otimes \delta_S(s) \tag{1.12}$$

for all $a\in\mathcal{A}$, $s\in S$. The $\mathbb{Z}/2Z$-grading is inherited from that of $\mathcal{A}$ by setting

$$(\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S})_{\bar{\iota}} := \mathcal{A}_{\bar{\iota}}\otimes_{\mathcal{R}}\mathcal{S}.$$

The n-products are defined via

$$(a\otimes r)_{(n)}(b\otimes s) = \sum_{j\in\mathbb{Z}_+}(a_{(n+j)}b)\otimes\delta_S^{(j)}(r)s \tag{1.13}$$

for all $a,b\in\mathcal{A}$, $r,s\in S$, and $n\in\mathbb{Z}_+$. Axioms (CS0)–(CS3) hold, as can be verified directly. (If $\mathcal{A}$ is also a *Lie* conformal superalgebra, then (CS4)–(CS5) hold, and $\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}$ is also Lie.) The $\mathcal{S}$-conformal superalgebra on $\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}$ described above is said to be obtained from $\mathcal{A}$ by *base change* from $\mathcal{R}$ to $\mathcal{S}$.

Example 1.14 The affinization $\widehat{\mathcal{A}}$ of a k-conformal superalgebra $\mathcal{A}$ (Example 1.6), viewed as a conformal superalgebra over $\mathcal{R} := (k[t,t^{-1}],\delta_t)$, is obtained from $\mathcal{A}$ by base change from k to $\mathcal{R}$.

Remark 1.15 It is straightforward to verify that the tensor products used in defining change of base are associative. More precisely, assume that $\mathcal{S} = (S,\delta_S)$ is an extension of both $\mathcal{R} = (R,\delta_R)$ and $\mathcal{T} = (T,\delta_T)$, and $\mathcal{U} = (U,\delta_U)$ is also an extension of $\mathcal{T} = (T,\delta_T)$. Then for any $\mathcal{R}$-conformal superalgebra $\mathcal{A}$, the map $(a\otimes s)\otimes u\mapsto a\otimes(s\otimes u)$ defines a $\mathcal{U}$-conformal isomorphism

$$(\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S})\otimes_{\mathcal{T}}\mathcal{U}\cong\mathcal{A}\otimes_{\mathcal{R}}(\mathcal{S}\otimes_{\mathcal{T}}\mathcal{U}).$$

Here $u\in U$ acts on $\mathcal{A}\otimes_R(S\otimes_T U)$ by multiplication, namely

$$u(a\otimes(s\otimes u')) := a\otimes(s\otimes uu')$$

for $a\in\mathcal{A}$, $s\in S$ and $u'\in U$; the derivation $\partial_{\mathcal{A}\otimes(\mathcal{S}\otimes\mathcal{U})}$ acts on $\mathcal{A}\otimes_R(S\otimes_T U)$ as

$$\partial_{\mathcal{A}\otimes_{\mathcal{R}}(\mathcal{S}\otimes\mathcal{U})} = \partial_{\mathcal{A}}\otimes\mathrm{id}_{S\otimes U} + \mathrm{id}_{\mathcal{A}}\otimes\delta_{S\otimes U},$$

where $\delta_{S\otimes U} := \delta_S\otimes\mathrm{id}_U + \mathrm{id}_S\otimes\delta_U$. The associativity of tensor products will be useful when working with $\mathcal{S}/\mathcal{R}$-forms (§2 and §3 below).

Extension functor: Each extension $\mathcal{S} = (S, \delta_S)$ of $\mathcal{R} = (R, \delta_R)$ defines an *extension functor*

$$\mathfrak{E} = \mathfrak{E}_{\mathcal{S}/\mathcal{R}} : \mathcal{R} - \mathrm{conf} \longrightarrow \mathcal{S} - \mathrm{conf}$$

as follows:

Given an $\mathcal{R}$-conformal superalgebra $\mathcal{A}$, let $\mathfrak{E}(\mathcal{A})$ be the $\mathcal{S}$-conformal superalgebra $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}$. For each $\mathcal{R}$-conformal superalgebra homomorphism $\psi : \mathcal{A} \to \mathcal{B}$, the unique S-linear map satisfying

$$\begin{aligned} \mathfrak{E}(\psi) : \ \mathfrak{E}(\mathcal{A}) \quad &\to \quad \mathfrak{E}(\mathcal{B}) && (1.16) \\ a \otimes s \quad &\mapsto \quad \psi(a) \otimes s && (1.17) \end{aligned}$$

is clearly a homomorphism of $\mathcal{S}$-conformal superalgebras, and it is straightforward to verify that $\mathfrak{E}$ is a functor.

Restriction functor: Likewise, any $\mathcal{S}$-conformal superalgebra $\mathcal{B}$ can be viewed as an $\mathcal{R}$-conformal superalgebra by *restriction of scalars from* $\mathcal{S}$ to $\mathcal{R}$:

If the extension $\mathcal{S}/\mathcal{R}$ corresponds to a $k - \delta alg$ morphism $\phi : \ R \to S$, we view $\mathcal{B}$ as an R-module via ϕ. Then $\mathcal{B}$ is naturally an $\mathcal{R}$-conformal superalgebra. The only nontrivial axiom to verify is (CS2). Using the notation of Remark 1.11, we have

$$\begin{aligned} \partial_{\mathcal{B}} \circ r_{\mathcal{B}} &= \partial_{\mathcal{B}} \circ \phi(r)_{\mathcal{B}} \\ &= \phi(r)_{\mathcal{B}} \circ \partial_{\mathcal{B}} + \delta_S(\phi(r))_{\mathcal{B}} \\ &= r_{\mathcal{B}} \circ \partial_{\mathcal{B}} + \phi(\delta_R(r))_{\mathcal{B}} \\ &= r_{\mathcal{B}} \circ \partial_{\mathcal{B}} + \delta_R(r)_{\mathcal{B}}. \end{aligned}$$

This leads to the *restriction functor*

$$\mathfrak{R} = \mathfrak{R}_{\mathcal{S}/\mathcal{R}} : \mathcal{S} - \mathrm{conf} \longrightarrow \mathcal{R} - \mathrm{conf},$$

which attaches to an $\mathcal{S}$-conformal superalgebra $\mathcal{B}$ the same $\mathcal{B}$ viewed as an $\mathcal{R}$-conformal superalgebra; likewise, to any $\mathcal{S}$-superconformal homomorphism $\psi : \ \mathcal{B} \to \mathcal{C}$, $\mathfrak{R}$ attaches the $\mathcal{R}$-conformal superalgebra morphism ψ.

1.4 The automorphism functor of a conformal superalgebra

Let $\mathcal{A}$ be an $\mathcal{R} = (R, \delta_R)$-conformal superalgebra. We now define the automorphism group functor $\mathbf{Aut}(\mathcal{A})$. For each extension $\mathcal{S} = (S, \delta_S)$ of $\mathcal{R}$, consider the group

$$\mathbf{Aut}(\mathcal{A})(\mathcal{S}) := \mathrm{Aut}_{\mathcal{S}}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}) \tag{1.18}$$

of automorphisms of the $\mathcal{S}$-conformal superalgebra $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}$. For each morphism $\psi : \mathcal{S}_1 \to \mathcal{S}_2$ between two extensions $\mathcal{S}_1 = (S_1, \delta_1)$ and $\mathcal{S}_2 = (S_2, \delta_2)$ of $\mathcal{R}$, and each automorphism $\theta \in \mathbf{Aut}(\mathcal{A})(\mathcal{S}_1)$, let $\mathbf{Aut}(\mathcal{A})(\psi)(\theta)$ be the unique S_2-linear map determined by

$$\begin{aligned} \mathbf{Aut}(\mathcal{A})(\psi)(\theta) :\ \mathcal{A} \otimes_R S_2 &\ \to\ \mathcal{A} \otimes_R S_2 && (1.19) \\ a \otimes 1 &\ \mapsto\ \sum_i a_i \otimes \psi(s_i) && (1.20) \end{aligned}$$

for $a \in \mathcal{A}$, where $\theta(a \otimes 1) = \sum_i a_i \otimes s_i$.

Proposition 1.21 $\mathbf{Aut}(\mathcal{A})$ *is a functor from the category of extensions of* (R, δ_R) *to the category of groups.*

Proof Let $\theta_1, \theta_2 \in \mathrm{Aut}_{\mathcal{S}_1}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}_1)$, and write $\theta_2(a \otimes 1) = \sum_i a_i \otimes s_i$ for some $a_i \in \mathcal{A}$ and $s_i \in S_1$. Then for any morphism $\psi : \mathcal{S}_1 \to \mathcal{S}_2$, we have (in the notation above):

$\mathbf{Aut}(\mathcal{A})(\psi)(\theta_1) \circ \mathbf{Aut}(\mathcal{A})(\psi)(\theta_2)(a \otimes 1)$

$$\begin{aligned} &= \mathbf{Aut}(\mathcal{A})(\psi)(\theta_1)(1 \otimes \psi)\theta_2(a \otimes 1) \\ &= \mathbf{Aut}(\mathcal{A})(\psi)(\theta_1) \sum_i a_i \otimes \psi(s_i) \\ &= \sum_i \psi(s_i) \mathbf{Aut}(\mathcal{A})(\psi)(\theta_1)(a_i \otimes 1) \\ &= (1 \otimes \psi) \sum_i s_i \theta_1(a_i \otimes 1) \\ &= (1 \otimes \psi)\theta_1 \sum_i a_i \otimes s_i \\ &= (1 \otimes \psi)\theta_1\theta_2(a \otimes 1) \\ &= \mathbf{Aut}(\mathcal{A})(\psi)(\theta_1\theta_2)(a \otimes 1). \end{aligned}$$

Using the S_2-linearity of the $\mathcal{S}_2$-conformal automorphisms $\mathbf{Aut}(\mathcal{A})(\psi)(\theta_1)$, $\mathbf{Aut}(\mathcal{A})(\psi)(\theta_2)$, and $\mathbf{Aut}(\mathcal{A})(\psi)(\theta_1\theta_2)$, we have

$$\mathbf{Aut}(\mathcal{A})(\psi)(\theta_1) \circ \mathbf{Aut}(\mathcal{A})(\psi)(\theta_2) = \mathbf{Aut}(\mathcal{A})(\psi)(\theta_1\theta_2).$$

In particular, note that for any $\theta \in \mathbf{Aut}(\mathcal{A})(\mathcal{S}_1)$, we have

$$\mathbf{Aut}(\mathcal{A})(\psi)(\theta^{-1}) \circ \mathbf{Aut}(\mathcal{A})(\psi)(\theta) = \mathbf{Aut}(\mathcal{A})(\psi)(\mathrm{id}_{\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}_1}). \qquad (1.22)$$

It is clear from the definition of $\mathbf{Aut}(\mathcal{A})(\psi)$ that $\mathbf{Aut}\,\mathcal{A}(\psi)(\mathrm{id}_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}_1})$ is the identity map on $\mathcal{A}\otimes 1$, hence also on $\mathcal{A}\otimes_R S_2$ by S_2-linearity. Thus $\mathbf{Aut}(\mathcal{A})(\psi)(\theta)$ is invertible, so it is bijective.

That $\mathbf{Aut}(\mathcal{A})(\psi)(\theta)$ is an $\mathcal{S}_2$-conformal superalgebra homomorphism follows easily from the assumption that $\theta \in \mathrm{Aut}_{\mathcal{S}_1}(\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}_1)$ and $\psi : \mathcal{S}_1 \to \mathcal{S}_2$ is a morphism of $\mathcal{R}$-extensions. Therefore, $\mathbf{Aut}(\mathcal{A})(\psi)(\theta) \in \mathrm{Aut}_{\mathcal{S}_2}(\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}_2)$, and we have now shown that $\mathbf{Aut}(\mathcal{A})(\psi)$ is a group homomorphism

$$\mathbf{Aut}(\mathcal{A})(\psi):\ \mathbf{Aut}(\mathcal{A})(\mathcal{S}_1) \to \mathbf{Aut}(\mathcal{A})(\mathcal{S}_2). \tag{1.23}$$

Clearly $\mathbf{Aut}(\mathcal{A})$ sends the identity morphism $\mathrm{id}_{\mathcal{S}}$ to the identity map on $\mathbf{Aut}(\mathcal{A})(\mathcal{S})$ for any extension $\mathcal{S}$ of $\mathcal{R}$. To finish proving that $\mathbf{Aut}(\mathcal{A})$ is a functor, it remains only to note that if $\psi_1:\ \mathcal{S}_1 \to \mathcal{S}_2$ and $\psi_2:\ \mathcal{S}_2 \to \mathcal{S}_3$ are morphisms between extensions $\mathcal{S}_i = (S_i, \delta_i)_{1\le i\le 3}$ of $\mathcal{R}$, then for $a \in \mathcal{A}$ and $\theta(a\otimes 1) = \sum_i a_i \otimes s_i$, we have

$$\mathbf{Aut}(\mathcal{A})(\psi_2\psi_1)(\theta):\ a\otimes 1 \mapsto \sum_i a_i \otimes \psi_2\psi_1(s_i),$$

which defines precisely the same map (via S_3-linearity) as $\mathbf{Aut}(\mathcal{A})(\psi_2) \circ \mathbf{Aut}(\mathcal{A})(\psi_1)(\theta)$. Hence $\mathbf{Aut}(\mathcal{A})(\psi_2\psi_1) = \mathbf{Aut}(\mathcal{A})(\psi_2)\circ\mathbf{Aut}(\mathcal{A})(\psi_1)$, which completes the proof of the proposition. □

2 Forms of conformal superalgebras and Čech cohomology

Given R in $k-alg$ and a (not necessarily commutative, associative, or unital) R–algebra A, recall that a *form of* A (for the fppf–topology on $\mathrm{Spec}(R)$) is an R-algebra F such that $F\otimes_R S \cong A\otimes_R S$ (as S-algebras) for some fppf (faithfully flat and finitely presented) extension S/R in $k-alg$. There is a correspondence between R-isomorphism classes of forms of A and the pointed set of non-abelian cohomology $H^1_{\mathrm{fppf}}(R, \mathbf{Aut}(A))$ defined à la Čech. Here $\mathbf{Aut}(A) := \mathbf{Aut}(A)_R$ denotes the sheaf of groups over $\mathrm{Spec}(R)$ that attaches to an extension R'/R in $k-alg$ the group $\mathrm{Aut}_S(A\otimes_R R')$ of automorphisms of the R'–algebra $A\otimes_R R'$. For any extension S/R in $k-alg$, there is a canonical map

$$H^1_{\mathrm{fppf}}(R, \mathbf{Aut}(A)) \to H^1_{\mathrm{fppf}}(S, \mathbf{Aut}(A)_S)$$

The kernel of this map is denoted by $H^1_{\mathrm{fppf}}(S/R, \mathbf{Aut}(A))$; these are the forms of A that are trivialized by the base change S/R. One has

$$H^1_{\text{fppf}}(R, \mathbf{Aut}(A)) = \varinjlim H^1_{\text{fppf}}(S/R, \mathbf{Aut}(A)),$$

where the limit is taken over all "isomorphisms classes" of fppf extensions S/R in $k-alg$.

In trying to recreate this construction for an $\mathcal{R}$-conformal superalgebra $\mathcal{A}$, we encounter a fundamental obstacle: Unlike in the case of algebras, the n-products (1.13) in $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}$ are *not* obtained by S-linear extension of the n-products in $\mathcal{A}$ (unless the derivation of $\mathcal{S}$ is trivial). This prevents the automorphism functor $\mathbf{Aut}(\mathcal{A})$ from being representable in the naïve way, and the classical theory of forms cannot be applied blindly. Nonetheless, we will show in the next section that the expected correspondence between forms and cohomology continues to hold, even in the case of conformal superalgebras.

In the case of algebras, when the extension S/R is Galois, isomorphism classes of S/R–forms have an interpretation in terms of non-abelian Galois cohomology. (See [20], for instance.) We will show in §2.2 that, just as in the case of algebras, the Galois cohomology $H^1\left(\mathcal{G}al(\mathcal{S}/\mathcal{R}), \mathrm{Aut}_{\mathcal{S}}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S})\right)$ still parametrizes the $\mathcal{S}/\mathcal{R}$-forms of $\mathcal{A}$ (with the appropriate definition of Galois extension and $\mathrm{Aut}_{\mathcal{S}}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S})$).

Throughout this section, $\mathcal{A}$ will denote a conformal superalgebra over $\mathcal{R} = (R, \delta_R)$.

2.1 Forms split by an extension

Definition 2.1 Let $\mathcal{S}$ be an extension of $\mathcal{R}$. An $\mathcal{R}$-conformal superalgebra $\mathcal{F}$ is an *$\mathcal{S}/\mathcal{R}$-form* of $\mathcal{A}$ (or *form of $\mathcal{A}$ split by $\mathcal{S}$*) if

$$\mathcal{F} \otimes_{\mathcal{R}} \mathcal{S} \cong \mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}$$

as $\mathcal{S}$-conformal superalgebras.

For us, the most interesting examples of forms split by a given extension are the conformal superalgebras that are obtained via the type of twisted loop construction that one encounters in the theory of affine Kac-Moody Lie algebras.

Example 2.2 Assume k is algebraically closed. Suppose that $\mathcal{A}$ is a k-conformal superalgebra, equipped with an automorphism σ of period m. For each $i \in \mathbb{Z}$ consider the eigenspace

$$\mathcal{A}_i = \{x \in \mathcal{A} \mid \sigma(x) = \xi_m^i x\}.$$

with respect to our fixed choice (ξ_m) of compatible primitive roots of unity in k. (The space $\mathcal{A}_i$ depends only on the class of i modulo m, of course.) Let $R = k[t, t^{-1}]$, $S_m = k[t^{1/m}, t^{-1/m}]$ and $\widehat{S} := \varinjlim S_m$. We consider the extensions $\mathcal{S}_m = (S_m, \delta_t)$ and $\widehat{\mathcal{S}} = (\widehat{S}, \delta_t)$ of $\mathcal{R} = (R, \delta_t)$ where $\delta_t = \frac{d}{dt}$.

Consider the subspace $\mathcal{L}(\mathcal{A}, \sigma) \subseteq \mathcal{A} \otimes_k S_m \subset \mathcal{A} \otimes_k \widehat{S}$ given by

$$\mathcal{L}(\mathcal{A}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i \otimes t^{i/m}. \tag{2.3}$$

Each eigenspace $\mathcal{A}_i$ is stable under $\partial_{\mathcal{A}}$ because σ is a conformal automorphism. From this, it easily follows that $\mathcal{L}(\mathcal{A}, \sigma)$ is stable under the action of $\partial_{\mathcal{A} \otimes \mathcal{S}_m} = \partial_{\mathcal{A}} \otimes 1 + 1 \otimes \delta_t$. Since $\mathcal{L}(\mathcal{A}, \sigma)$ is also closed under the n-products of $\mathcal{A} \otimes_k \mathcal{S}_m$, it is a k-conformal subalgebra of $\mathcal{A} \otimes_k \mathcal{S}_m$ called the *(twisted) loop algebra* of $\mathcal{A}$ with respect to σ. Note that the definition of $\mathcal{L}(\mathcal{A}, \sigma)$ as a subalgebra of $\mathcal{A} \otimes_k \widehat{S}$ does not depend on the choice of the period m of the given automorphism σ. Because of this, it will be at times convenient when comparing loop algebras corresponding to automorphisms of different periods to think of loop algebras as k–conformal subalgebras of $\mathcal{A} \otimes_k \widehat{\mathcal{S}}$. It is clear that $\mathcal{L}(\mathcal{A}, \sigma)$ is stable under the natural action of R on $\mathcal{A} \otimes_k \mathcal{S}_m$. As in Example 1.6, one checks that $\partial_{\mathcal{A} \otimes \mathcal{S}_m} \circ r_{\mathcal{A} \otimes_k \mathcal{S}_m} - r_{\mathcal{A} \otimes_k \mathcal{S}_m} \circ \partial_{\mathcal{A} \otimes \mathcal{S}_m} = \delta_t(r)_{\mathcal{A} \otimes_k \mathcal{S}_m}$ for all $r \in R$. This shows that just as in the untwisted case, the twisted loop algebra $\mathcal{L}(\mathcal{A}, \sigma)$ is both a k- and an $\mathcal{R}$-conformal superalgebra.

Proposition 2.4 *Let σ be an automorphism of period m of a k-conformal superalgebra $\mathcal{A}$. Then the twisted loop algebra $\mathcal{L}(\mathcal{A}, \sigma)$ is an $\mathcal{S}_m/\mathcal{R}$-form of $\mathcal{A} \otimes_k \mathcal{R}$.*

Proof For ease of notation, we will write $\mathcal{S} = (S, \delta_S)$ for $\mathcal{S}_m = (S_m, \delta_{S_m})$ in this proof. By the associativity of the tensor products used in scalar extension (Remark 1.15), the multiplication map

$$\psi : \ (\mathcal{A} \otimes_k \mathcal{R}) \otimes_{\mathcal{R}} \mathcal{S} \ \rightarrow \ \mathcal{A} \otimes_k \mathcal{S} \tag{2.5}$$

$$(a \otimes r) \otimes s \ \mapsto \ a \otimes rs \tag{2.6}$$

(for $a \in \mathcal{A}$, $r \in R$, and $s \in S$) is an isomorphism of $\mathcal{S}$-conformal superalgebras.

Likewise, it is straightforward to verify that the multiplication map

$$\mu : \ \big(\mathcal{A} \otimes_k \mathcal{S}\big) \otimes_{\mathcal{R}} \mathcal{S} \ \rightarrow \ \mathcal{A} \otimes_k \mathcal{S} \tag{2.7}$$

$$(a \otimes s_1) \otimes s_2 \ \mapsto \ a \otimes s_1 s_2 \tag{2.8}$$

(for $a \in \mathcal{A}$ and $s_1, s_2 \in S$) is a homomorphism of $\mathcal{S}$-conformal superalgebras. Indeed, μ is the composition of the "associativity isomorphism"

$$\big(\mathcal{A} \otimes_k \mathcal{S}\big) \otimes_{\mathcal{R}} \mathcal{S} \to \mathcal{A} \otimes_k \big(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}\big)$$

with the superconformal homomorphism defined by multiplication:

$$\begin{array}{rcl} \mathcal{A} \otimes_k (\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}) & \to & \mathcal{A} \otimes_k \mathcal{S} \\ a \otimes (s_1 \otimes s_2) & \mapsto & a \otimes s_1 s_2. \end{array}$$

To complete the proof of Proposition 2.4, it suffices to prove that the restriction

$$\mu:\ \mathcal{L}(\mathcal{A}, \sigma) \otimes_{\mathcal{R}} \mathcal{S} \to \mathcal{A} \otimes_k \mathcal{S} \tag{2.9}$$

is bijective. For $a_i \in \mathcal{A}_i$, we have

$$a_i \otimes t^{j/m} = \mu(a_i \otimes t^{i/m} \otimes t^{(j-i)/m}),$$

so μ is clearly surjective. To see that μ is also injective, assume (without loss of generality) that a k-basis $\{a_\lambda\}$ of $\mathcal{A}$ is chosen so that each $a_\lambda \in \mathcal{A}_{i(\lambda)}$ for some unique $0 \leq i(\lambda) < m$. Let $x \in \mathcal{L}(\mathcal{A}, \sigma) \otimes_R S$. Since S is a free R-module with basis $\{t^{i/m} \mid 0 \leq i \leq m-1\}$, we can uniquely write

$$x = \sum_{i=0}^{m-1} x_i \otimes t^{i/m}$$

where $x_i = \sum_\lambda a_\lambda \otimes f_{\lambda i}$ and $f_{\lambda i} \in t^{i(\lambda)/m} k[t, t^{-1}]$. Then if $\mu(x) = 0$, we have

$$\sum_{\lambda i} a_\lambda \otimes f_{\lambda i} t^{i/m} = 0,$$

and thus

$$\sum_{i=0}^{m-1} f_{\lambda, i} t^{i/m} = 0$$

for all λ. Then $f_{\lambda,i} = 0$ for all λ and i. Hence $x = 0$, so μ is injective, and

$$\psi^{-1} \circ \mu:\ \mathcal{L}(\mathcal{A}, \sigma) \otimes_{\mathcal{R}} \mathcal{S} \to \big(\mathcal{A} \otimes_k \mathcal{R}\big) \otimes_{\mathcal{R}} \mathcal{S} \tag{2.10}$$

is an $\mathcal{S}$-conformal superalgebra isomorphism as desired. $\square$

2.2 Cohomology and forms

Throughout this section $\mathcal{S} = (S, \delta_S)$ will denote an extension of $\mathcal{R} = (R, \delta_R)$, and $\mathcal{A}$ will be an $\mathcal{R}$-conformal superalgebra.

Lemma 2.11 *Let $\psi : \mathcal{A} \to \mathcal{B}$ be an $\mathcal{R}$-conformal superalgebra homomorphism and $\gamma : \mathcal{S} \to \mathcal{S}$ a morphism of extensions. The canonical map*

$$\psi \otimes \gamma : \mathcal{A} \otimes_R S \to \mathcal{B} \otimes_R S \tag{2.12}$$

is R-linear, commutes with the action of $\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}}$, and preserves n-products. In particular, $\psi \otimes \gamma$ is an $\mathcal{R}$-conformal superalgebra homomorphism via restriction of scalars from $\mathcal{S}$ to $\mathcal{R}$:

$$\psi \otimes \gamma : \mathfrak{R}_{\mathcal{S}/\mathcal{R}}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}) \to \mathfrak{R}_{\mathcal{S}/\mathcal{R}}(\mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}).$$

Proof Let $x \in \mathcal{A}$ and $s \in S$. Then

$$\begin{aligned}
\psi \otimes \gamma\big(\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}}(x \otimes s)\big) &= \psi \otimes \gamma\big(\partial_{\mathcal{A}}(x) \otimes s + x \otimes \delta_S(s)\big) \\
&= \partial_{\mathcal{B}}(\psi(x)) \otimes \gamma(s) + \psi(x) \otimes \delta_S\big(\gamma(s)\big) \\
&= \partial_{\mathcal{B}\otimes_{\mathcal{R}}\mathcal{S}}\big(\psi(x) \otimes \gamma(s)\big).
\end{aligned}$$

Also, for $x, y \in \mathcal{A}$ and $s, t \in S$, we have

$$\begin{aligned}
\psi \otimes \gamma(x \otimes s_{(n)} y \otimes t) &= \psi \otimes \gamma \left(\sum_{j \in \mathbb{Z}_+} x_{(n+j)} y \otimes \delta_S^{(j)}(s) t \right) \\
&= \sum_{j \in \mathbb{Z}_+} \psi(x)_{(n+j)} \psi(y) \otimes \delta_S^{(j)}\big(\gamma(s)\big)\gamma(t) \\
&= \psi(x) \otimes \gamma(s)_{(n)} \psi(y) \otimes \gamma(t).
\end{aligned}$$

□

Corollary 2.13 *The map $\psi \otimes 1 : \mathcal{A} \otimes_{\mathcal{R}} \mathcal{S} \to \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}$ is an $\mathcal{S}$-conformal superalgebra homomorphism.*

Proof It is enough to note that the map $\psi \otimes 1$ commutes with the action of S. □

For $1 \le i \le 2$ and $1 \le j < k \le 3$, consider the following R-linear maps:

$$\begin{aligned}
d_i &: S \to S \otimes_R S \\
d_{jk} &: S \otimes_R S \to S \otimes_R S \otimes_R S,
\end{aligned}$$

defined by $d_1(s) = s\otimes 1$, $d_2 = 1\otimes s$, $d_{12}(s\otimes t) = s\otimes t\otimes 1$, $d_{13}(s\otimes t) = s\otimes 1\otimes t$, and $d_{23}(s\otimes t) = 1\otimes s\otimes t$ for all $s, t \in S$. It is straightforward to verify that these induce $\mathcal{R} - ext$ morphisms $d_i : \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}$ and $d_{jk} : \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}$. (See §1.1). Let $\mathcal{A}$ be an $\mathcal{R}$-conformal superalgebra. By functoriality (Proposition 1.21), we obtain group homomorphisms (also denoted by d_i and d_{jk})

$$\begin{aligned} d_i :\ & \mathbf{Aut}(\mathcal{A})(\mathcal{S}) \to \mathbf{Aut}(\mathcal{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}) \\ d_{jk} :\ & \mathbf{Aut}(\mathcal{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}) \to \mathbf{Aut}(\mathcal{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}) \end{aligned}$$

for $1 \le i \le 2$ and $1 \le j < k \le 3$.

Recall that $u \in \mathbf{Aut}(\mathcal{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S})$ is called a 1-*cocycle*[5] if

$$d_{13}(u) = d_{23}(u)d_{12}(u). \tag{2.14}$$

On the set $Z^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}))$ of 1-cocycles, one defines an equivalence relation by declaring two cocycles u and v to be *equivalent* (or *cohomologous*) if there exists an automorphism $\lambda \in \mathbf{Aut}(\mathcal{A})(\mathcal{S})$ such that

$$v = \big(d_2(\lambda)\big)\, u \,\big(d_1(\lambda)\big)^{-1}. \tag{2.15}$$

Formula (2.15) defines an action of the group $\mathbf{Aut}(\mathcal{A})(\mathcal{S})$ on the set of cocycles $Z^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}))$. The corresponding quotient set is denoted by $H^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}))$, and this is the (nonabelian) Čech cohomology of $\mathbf{Aut}(\mathcal{A})$ relative to the covering $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. There is no natural group structure on the set $H^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}))$, but it has a distinguished element, namely the equivalence class of the identity element of the group $\mathbf{Aut}(\mathcal{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S})$. We will denote this class by 1, and write $[u]$ for the equivalence class of an arbitrary cocycle $u \in Z^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}))$.

Theorem 2.16 *Assume that the extension $\mathcal{S} = (S, \delta_S)$ of $\mathcal{R} = (R, \delta_R)$ is faithfully flat (i.e., S is a faithfully flat R-module). Then for any $\mathcal{R}$-conformal superalgebra $\mathcal{A}$, the pointed set $H^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}))$ parametrizes the set of $\mathcal{R}$-isomorphism classes of $\mathcal{S}/\mathcal{R}$-forms of $\mathcal{A}$. Under this correspondence, the distinguished element 1 corresponds to the isomorphism class of $\mathcal{A}$ itself.*

Proof It suffices to check that the standard descent formalism for modules is compatible with the conformal superalgebra structures.

[5]For faithfully flat ring extensions S/R, the 1-cocycle condition is motivated by patching data on open coverings of $\mathrm{Spec}(R)$. See the discussion in [20, §17.4], for instance.

Throughout this proof, fix an $\mathcal{S}/\mathcal{R}$-form $\mathcal{B}$ of the $\mathcal{R}$-conformal superalgebra $\mathcal{A}$. Let $\eta:\ S\otimes_R S\to S\otimes_R S$ be the "switch" map given by $\eta(s\otimes t)=t\otimes s$ for all $s,t\in S$. Let

$$\begin{aligned}\eta_{\mathcal{A}}:=\mathrm{id}_{\mathcal{A}}\otimes\eta:\ \mathcal{A}\otimes_R S\otimes_R S &\to \mathcal{A}\otimes_R S\otimes_R S\\ \eta_{\mathcal{B}}:=\mathrm{id}_{\mathcal{B}}\otimes\eta:\ \mathcal{B}\otimes_R S\otimes_R S &\to \mathcal{B}\otimes_R S\otimes_R S.\end{aligned}$$

We now check that the key points from the classical formalism of faithfully flat descent hold in the conformal setting:

(1) *Let $\psi:\ \mathcal{B}\otimes_{\mathcal{R}}\mathcal{S}\to\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}$ be an isomorphism of $\mathcal{S}$-conformal superalgebras. Let*

$$u_{\psi,\mathcal{B}}:=(\eta_{\mathcal{A}})(\psi\otimes 1)(\eta_{\mathcal{B}})(\psi^{-1}\otimes 1). \tag{2.17}$$

Then $u_{\psi,\mathcal{B}}\in Z^1(\mathcal{S}/\mathcal{R},\mathbf{Aut}(\mathcal{A}))$.

Proof (1): That $u_{\psi,\mathcal{B}}:\ \mathcal{A}\otimes_R S\otimes_R S\to\mathcal{A}\otimes_R S\otimes_R S$ is $S\otimes_R S$-linear and bijective is clear, and it is straightforward to verify that $u_{\psi,\mathcal{B}}$ satisfies the cocycle condition (2.14). Therefore it is enough to check that $u_{\psi,\mathcal{B}}$ commutes with the derivation $\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}\otimes_R\mathcal{S}}$, and that it preserves the n-products.

By Corollary 2.13 and the associativity of the tensor product, $\psi\otimes 1$ and $\psi^{-1}\otimes 1$ are $\mathcal{S}\otimes_R\mathcal{S}$-superconformal homomorphisms, so it is only necessary to check that $\eta_{\mathcal{A}}$ and $\eta_{\mathcal{B}}$ commute with $\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}\otimes_R\mathcal{S}}$ and preserve n-products. By Lemma 2.11, applied to $\eta_{\mathcal{A}}=\mathrm{id}_{\mathcal{A}}\otimes\eta$, it is enough to check that η commutes with $\delta_{S\otimes S}$, which is clear since

$$\begin{aligned}\eta(\delta_{S\otimes S}(s\otimes t)) &= \eta\big(\delta_S(s)\otimes t+s\otimes\delta_S(t)\big) &(2.18)\\ &= t\otimes\delta_S(s)+\delta_S(t)\otimes s &(2.19)\\ &= \delta_{S\otimes S}\big(\eta(s\otimes t)\big) &(2.20)\end{aligned}$$

for all $s,t\in S$.

(2) *The class of $u_{\psi,\mathcal{B}}$ in $H^1(\mathcal{S}/\mathcal{R},\mathbf{Aut}(\mathcal{A}))$ is independent of the choice of automorphism ψ in Part (1). If we denote this class by $[u_{\mathcal{B}}]$, then $\alpha:\ \mathcal{B}\mapsto[u_{\mathcal{B}}]$ is a map from the set of $\mathcal{R}$-isomorphism classes of $\mathcal{S}/\mathcal{R}$-forms of $\mathcal{A}$ to the pointed set $H^1(\mathcal{S}/\mathcal{R},\mathbf{Aut}(\mathcal{A}))$.*

Proof (2): Suppose that ϕ is another $\mathcal{S}$-superconformal isomorphism

$$\phi : \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S} \to \mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}.$$

Let $\lambda = \phi\psi^{-1} \in \mathbf{Aut}(\mathcal{A})(S)$. Note that $d_2(\lambda)\eta_{\mathcal{A}} = \eta_{\mathcal{A}} d_1(\lambda)$. Thus

$$\begin{aligned} d_2(\lambda) u_{\psi,\mathcal{B}} d_1(\lambda)^{-1} &= d_2(\lambda)\eta_{\mathcal{A}}(\psi \otimes 1)\eta_{\mathcal{B}}(\psi^{-1} \otimes 1)(\psi\phi^{-1} \otimes 1) \\ &= \eta_{\mathcal{A}}(\phi\psi^{-1} \otimes 1)(\psi \otimes 1)\eta_{\mathcal{B}}(\phi^{-1} \otimes 1) \\ &= u_{\phi,\mathcal{B}}. \end{aligned}$$

(3) *Let $u \in \mathbf{Aut}(\mathcal{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S})$. Then*

- **(a)** *The subset $\mathcal{A}_u := \{x \in \mathcal{A} \otimes_R S \mid \; u(x \otimes 1) = \eta_{\mathcal{A}}(x \otimes 1)\}$ is an $\mathcal{R}$-conformal subalgebra of $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}$.*
- **(b)** *The canonical map*

$$\begin{aligned} \mu_u : \; \mathcal{A}_u \otimes_{\mathcal{R}} \mathcal{S} &\to \mathcal{A} \otimes_{\mathcal{R}} \mathcal{S} \\ x \otimes s &\mapsto s.x \end{aligned}$$

  is an $\mathcal{S}$-conformal superalgebra isomorphism.
- **(c)** *If u and v are cohomologous cocycles in $Z^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}))$, then $\mathcal{A}_u$ and $\mathcal{A}_v$ are isomorphic as $\mathcal{R}$-conformal superalgebras.*

Proof (3a): Clearly $\mathcal{A}_u$ is an R-submodule of $\mathcal{A} \otimes_R S$. Next we verify that $\mathcal{A}_u$ is stable under the action of $\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}}$.

Recall (2.18) that $\eta_{\mathcal{A}}$ commutes with the derivation $\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}\otimes_{\mathcal{R}}\mathcal{S}}$. Thus for all $x \in \mathcal{A}_u$,

$$\begin{aligned} u(\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}}(x) \otimes 1) &= u\, \partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}\otimes_{\mathcal{R}}\mathcal{S}}(x \otimes 1) \\ &= \partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}\otimes_{\mathcal{R}}\mathcal{S}}\, u(x \otimes 1) \\ &= \partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}\otimes_{\mathcal{R}}\mathcal{S}}\, \eta_{\mathcal{A}}(x \otimes 1) \\ &= \eta_{\mathcal{A}}\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}\otimes_{\mathcal{R}}\mathcal{S}}(x \otimes 1) \\ &= \eta_{\mathcal{A}}\partial_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}}(x) \otimes 1. \end{aligned}$$

To complete the proof of (3a), it remains only to show that $\mathcal{A}_u$ is closed under n-products. For x and y in $\mathcal{A}_u$ we have

$$\begin{aligned} u\big((x_{(n)}y) \otimes 1\big) &= u(x \otimes 1_{(n)} y \otimes 1) \\ &= u(x \otimes 1)_{(n)} u(y \otimes 1) \\ &= \eta_{\mathcal{A}}(x \otimes 1)_{(n)} \eta_{\mathcal{A}}(y \otimes 1) \\ &= \eta_{\mathcal{A}}\big(x \otimes 1_{(n)} y \otimes 1\big) \\ &= \eta_{\mathcal{A}}\big((x_{(n)}y) \otimes 1\big). \end{aligned}$$

Proof (3b): The map $\mu_u : \mathcal{A}_u \otimes_R S \to \mathcal{A} \otimes_R S$ is an S-module isomorphism by the classical descent theory for modules. (See [20, Chap 17], for instance.) We need only show that it is a homomorphism of $\mathcal{S}$-conformal superalgebras.

Let μ be the multiplication map

$$\begin{array}{rcl} \mu : \mathcal{A} \otimes_R S \otimes_R S & \to & \mathcal{A} \otimes_R S \\ a \otimes s \otimes t & \mapsto & a \otimes st. \end{array}$$

It is straightforward to verify that

$$\mu \circ \partial_{\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S} \otimes \mathcal{S}} = \partial_{\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}} \circ \mu, \text{ and } \mu \text{ preserves } n\text{−products.} \tag{2.21}$$

Since μ_u is the restriction of μ to $\mathcal{A}_u \otimes_R S$ it follows from (2.21) that μ_u preserves n-products. It remains only to show that μ_u commutes with $\partial_{\mathcal{A}_u \otimes_{\mathcal{R}} \mathcal{S}}$. But since $\partial_{\mathcal{A}_u \otimes_{\mathcal{R}} \mathcal{S}}$ acts on $\mathcal{A}_u \otimes_R S$ as the restriction of the derivation $\partial_{\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}}$ of $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}$ to the subalgebra $\mathcal{A}_u \otimes_{\mathcal{R}} \mathcal{S}$ and $\mu_u = \mu\, \iota$, where ι is the inclusion map $\iota : \mathcal{A}_u \otimes_R S \hookrightarrow \mathcal{A} \otimes_R S \otimes_R S$, we can again appeal to (2.21). This shows that μ_u is an $\mathcal{S}$-conformal superalgebra isomorphism.

Proof (3c) Write $v = d_2(\lambda)\, u\, d_1(\lambda)^{-1}$ for some $\lambda \in \mathbf{Aut}(\mathcal{A})(\mathcal{S})$. Then since $d_2(\lambda)\, \eta_{\mathcal{A}} = \eta_{\mathcal{A}}\, d_1(\lambda)$, we see that

$$\begin{array}{rcl} v\,(\lambda \otimes 1)(a_u \otimes 1) & = & v\, d_1(\lambda)(a_u \otimes 1) \\ & = & d_2(\lambda)\, u(a_u \otimes 1) \\ & = & d_2(\lambda)\, \eta_{\mathcal{A}}(a_u \otimes 1) \\ & = & \eta_{\mathcal{A}}\, d_1(\lambda)(a_u \otimes 1) \\ & = & \eta_{\mathcal{A}}\,(\lambda \otimes 1)(a_u \otimes 1) \end{array}$$

for all $a_u \in \mathcal{A}_u$. Thus,

$$\lambda(\mathcal{A}_u) \subseteq \mathcal{A}_v. \tag{2.22}$$

Likewise, $\lambda^{-1}(\mathcal{A}_v) \subseteq \mathcal{A}_u$, so applying λ^{-1} to both sides of (2.22) gives:

$$\mathcal{A}_u \subseteq \lambda^{-1}(\mathcal{A}_v) \subseteq \mathcal{A}_u,$$

and $\lambda(\mathcal{A}_u) = \mathcal{A}_v$.

Since λ is an $\mathcal{S}$-automorphism of $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}$, it commutes with the actions of S and $\partial_{\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}}$, and it preserves n-products. Thus its restriction to $\mathcal{A}_u$ commutes with $\mathcal{R}$ and preserves n-products. Furthermore $\lambda \partial_{\mathcal{A}_u} = \partial_{\mathcal{A}_v} \lambda$ since $\partial_{\mathcal{A}_u}$ and $\partial_{\mathcal{A}_u}$ are the restrictions of $\partial_{\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}}$ to $\mathcal{A}_u$ and $\mathcal{A}_v$ respectively. Thus λ is an $\mathcal{R}$-conformal superalgebra isomorphism from $\mathcal{A}_u$ to $\mathcal{A}_v$.

By (3c), there is a well-defined map $\beta:\ [u] \mapsto [\mathcal{A}_u]$ from the cohomology set $H^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}))$ to the set of $\mathcal{R}$-isomorphism classes of $\mathcal{S}/\mathcal{R}$-forms of $\mathcal{A}$. We have also seen that β and the map α defined in (2) are inverses of each other. That the distinguished element $1 \in H^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}))$ corresponds to the algebra $\mathcal{A}$ is clear from the definition of $\mathcal{A}_u$ given in (3a). □

Remark 2.23 For any isomorphism ψ as in Part (1) of the proof of Theorem 2.16, the cocycle $u_{\psi,\mathcal{B}}$ can be rewritten in terms of the maps $d_i:\ S \to S \otimes_R S$:

$$u_{\psi,\mathcal{B}} = d_2(\psi) d_1(\psi)^{-1}. \tag{2.24}$$

Proof If $\psi(b \otimes 1) = \sum_i a_i \otimes w_i$, then

$$\begin{aligned}
\eta_{\mathcal{A}}(\psi \otimes 1)\eta_{\mathcal{B}}(b \otimes s \otimes t) &= \eta_{\mathcal{A}}(\psi \otimes 1)(b \otimes t \otimes s) \\
&= \eta_{\mathcal{A}}\big(\psi(b \otimes t) \otimes s\big) \\
&= \eta_{\mathcal{A}}\big((t.\psi(b \otimes 1)) \otimes s\big) \\
&= \eta_{\mathcal{A}}\left(\sum_i a_i \otimes tw_i \otimes s\right) \\
&= \sum_i a_i \otimes s \otimes tw_i \\
&= d_2(\psi)(b \otimes s \otimes t)
\end{aligned}$$

for all $s, t \in S$. Thus

$$\begin{aligned}
u_{\psi,\mathcal{B}} &= d_2(\psi)(\psi^{-1} \otimes 1) \\
&= d_2(\psi)(\psi \otimes 1)^{-1} \\
&= d_2(\psi) d_1(\psi)^{-1}.
\end{aligned}$$

□

Remark 2.25 In the notation of Part (1) of the proof of Theorem 2.16, the algebra $\mathcal{B}$ is isomorphic to the $\mathcal{R}$-conformal superalgebra $\mathcal{B} \otimes 1 \subseteq \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}$ by the faithful flatness of S/R. This means that there is an isomorphic copy of each $\mathcal{S}/\mathcal{R}$-form of $\mathcal{A}$ *inside* $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}$, and the algebra $\mathcal{B}$ can be recovered (up to $\mathcal{R}$-conformal isomorphism) from the cocycle $u_{\psi,\mathcal{B}}$, since

$$\psi(\mathcal{B} \otimes 1) = \{x \in \mathcal{A} \otimes_R S \mid u_{\psi,\mathcal{B}}(x \otimes 1) = \eta_{\mathcal{A}}(x \otimes 1)\}. \tag{2.26}$$

Remark 2.27 Let $\mathcal{S} = (S, \delta_S)$ be an extension of $\mathcal{R} = (R, \delta_R)$. Let Γ be a finite group of automorphisms of $\mathcal{S}$ (as an extension of $\mathcal{R}$). We say that $\mathcal{S}$

is a *Galois extension of* $\mathcal{R}$ *with group* Γ if S is a Galois extension of R with group Γ. (See [12] for definition). The unique R-module map

$$\psi : S \otimes_R S \to S \times \cdots \times S \qquad (|\Gamma| \text{ copies of } S)$$

satisfying

$$a \otimes b \mapsto \big(\gamma(a)b\big)_{\gamma \in \Gamma}$$

is easily seen to be an isomorphism of $\mathcal{R}$-ext (see §1.1). This induces a group isomorphism

$$\begin{gathered}\mathbf{Aut}(\mathcal{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}) \simeq \mathbf{Aut}(\mathcal{A})(\mathcal{S}) \times \cdots \times \mathbf{Aut}(\mathcal{A})(\mathcal{S}) \\ u \mapsto (u_\gamma)_{\gamma \in \Gamma}\end{gathered}$$

The cocycle condition $u \in Z^1\big(S/R,\, \mathbf{Aut}(\mathcal{A})\big)$ translates, just as in the classical situation, into the usual cocycle condition $u_{\gamma\rho} = u_\gamma\, {}^\gamma u_\rho$ where Γ acts on $\mathbf{Aut}(\mathcal{A})(\mathcal{S}) = \mathrm{Aut}_{\mathcal{S}}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S})$ by conjugation, i.e., ${}^\gamma\theta = (1 \otimes \gamma)\theta(1 \otimes \gamma^{-1})$. This leads to a natural isomorphism

$$H^1\big(\mathcal{S}/\mathcal{R},\, \mathbf{Aut}(\mathcal{A})\big) \simeq H^1\big(\Gamma, \mathrm{Aut}_{\mathcal{S}}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S})\big)$$

where the right-hand side is the usual Galois cohomology.

2.3 Limits

Throughout this section, $R := k[t^{\pm 1}]$*,* $S_m := k[t^{\pm 1/m}]$*,* $\widehat{S} := \varinjlim S_m$*, and* $\delta_t := \frac{d}{dt}$.

We are interested in classifying all twisted loop algebras of a given conformal superalgebra $\mathcal{A}$ over k. If $\sigma \in \mathrm{Aut}_{k-\mathrm{conf}}(\mathcal{A})$ is of period m, then $L(\mathcal{A}, \sigma)$ is trivialized by the extension $\mathcal{S}_m/\mathcal{R}$, where $\mathcal{R} = (R, \delta_t)$ and $\mathcal{S}_m = (S_m, \delta_t)$. (See Example 2.2.) To compare $L(\mathcal{A}, \sigma)$ with another $L(\mathcal{A}, \sigma')$ where σ' is of period m', we may consider a common refinement $\mathcal{S}_{mm'}$. An elegant way of taking care of all such refinements at once is by considering the limit $\widehat{\mathcal{S}} = (\widehat{S}, \delta_t)$. For algebras over k, and under some finiteness assumptions, $\widehat{S}$ plays the role of the separable closure of R (see [3] and [5] for details). We follow this philosophy in the present situation.

Let $m \in \mathbb{Z}_+$, and let $\bar{\ } : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the canonical map. Each extension $\mathcal{S}_m/\mathcal{R}$ is Galois with Galois group $\mathbb{Z}/m\mathbb{Z}$, where $\bar{1}(t^{1/m}) = \xi_m t^{1/m}$. (Our choice of roots of unity ensures that the action of $\mathbb{Z}/\ell m\mathbb{Z}$ on S_m is compatible with that of $\mathbb{Z}/m\mathbb{Z}$ under the canonical map $\mathbb{Z}/\ell m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.) Fix an algebraic closure $\overline{k(t)}$ of $k(t)$ containing all of the rings S_m, and let $\pi_1(R)$ be the algebraic fundamental group of $\mathrm{Spec}(R)$ at the geometric point $a = \mathrm{Spec}\left(\overline{k(t)}\right)$. (See [18] for details.) Then $\widehat{S}$ is the algebraic simply-connected cover of R (in the algebraic sense), and $\pi_1(R) = \widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/m\mathbb{Z}$, where $\widehat{\mathbb{Z}}$ acts continuously on $\widehat{S}$ via ${}^1t^{p/q} = \xi_q^p t^{p/q}$. Let $\pi : \pi_1(R) \to \mathrm{Aut}_R(\widehat{S})$ be the corresponding group homomorphism.

Let $\mathcal{A}$ be an $\mathcal{R} = (R, \delta_t)$-conformal superalgebra. As in Remark 2.27, $\pi_1(R)$ acts on $\mathbf{Aut}(\mathcal{A})(\widehat{\mathcal{S}}) = \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_{\mathcal{R}} \widehat{\mathcal{S}})$ by means of π. That is, if $\gamma \in \pi_1(R)$ and $\theta \in \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_{\mathcal{R}} \widehat{\mathcal{S}})$, then

$$^{\gamma}\theta = \big(1 \otimes \pi(\gamma)\big)\theta\big(1 \otimes \pi(\gamma)^{-1}\big).$$

Let $\mathcal{A}$ be an $\mathcal{R}$-conformal superalgebra. Given another $\mathcal{R}$-conformal superalgebra $\mathcal{B}$, we have

$$\begin{aligned} \mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}_m \simeq_{\mathcal{S}_m} \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m &\Longrightarrow A \otimes_{\mathcal{R}} \mathcal{S}_m \otimes_{\mathcal{S}_\mathbf{m}} \mathcal{S} \simeq_{\mathcal{S}} \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m \otimes_{\mathcal{S}_m} \mathcal{S} \\ &\Longrightarrow A \otimes_{\mathcal{R}} \mathcal{S} \simeq_{\mathcal{S}} B \otimes_{\mathcal{R}} \mathcal{S} \end{aligned}$$

for all extensions $\mathcal{S}$ of $\mathcal{S}_m$. This yields inclusions

$$H^1\big(\mathcal{S}_m/\mathcal{R}, \mathbf{Aut}(\mathcal{A})\big) \subseteq H_1(\mathcal{S}_n/\mathcal{R}, \mathbf{Aut}(\mathcal{A})\big) \subseteq H^1(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{A})\big)$$

for all $m|n$, hence a natural injective map

$$\eta : \varinjlim H^1(\mathcal{S}_m/\mathcal{R}, \mathbf{Aut}(\mathcal{A})\big) \to H^1(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{A})\big). \tag{2.28}$$

In the classical situation, namely when $\mathcal{A}$ is an algebra, the surjectivity of the map η is a delicate problem (see [14] for details and references). The following result addresses this issue for an important class of conformal superalgebras.

Proposition 2.29 *Assume that $\mathcal{A}$ satisfies the following finiteness condition:*

(**Fin**) *There exist $a_1, \ldots, a_n \in \mathcal{A}$ such that the set $\{\partial_{\mathcal{A}}^{\ell}(ra_i) \mid r \in R,\ \ell \geq 0\}$ spans $\mathcal{A}$.*

Then.

(1) *For all $\mathcal{R}$-superconformal algebra $\mathcal{B}$ the natural map* $\varinjlim \mathrm{Hom}_{\mathcal{S}_m}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}_m, \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m\big) \to \mathrm{Hom}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_{\mathcal{R}} \widehat{\mathcal{S}}, \mathcal{B} \otimes_{\mathcal{R}} \widehat{\mathcal{S}}\big)$ *is bijective.*

(2) *The natural map* $\eta : \varinjlim H^1(\mathcal{S}_m/\mathcal{R}, \mathbf{Aut}(\mathcal{A})\big) \to H^1(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{A})\big)$ *is bijective.*

(3) *The action of the profinite group* $\pi_1(R)$ *acts continuously on* $\mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_{\mathcal{R}} \widehat{\mathcal{S}}) = \mathbf{Aut}(\mathcal{A})(\widehat{\mathcal{S}})$ *and*

$$H^1\big(\widehat{\mathcal{S}}/\mathcal{R},\, \mathbf{Aut}(\mathcal{A})\big) \simeq H^1_{\mathrm{ct}}\big(\pi_1(R),\, \mathbf{Aut}\,(\mathcal{A})(\widehat{\mathcal{S}})\big),$$

where the right H^1_{ct} *denotes the continuous non-abelian cohomology of the profinite group* $\pi_1(R)$ *acting (continuously) on the group* $\mathrm{Aut}\,(\mathcal{A})(\widehat{\mathcal{S}})$

Proof (1) To establish the surjectivity of our map we must show that every $\widehat{\mathcal{S}}$-conformal superalgebra homomorphism

$$\psi : \mathcal{A} \otimes_{\mathcal{R}} \widehat{\mathcal{S}} \to \mathcal{B} \otimes_{\mathcal{R}} \widehat{\mathcal{S}}$$

is obtained by base change from an $\mathcal{S}_m$-homomorphism

$$\psi_m : \mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}_m \to \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m.$$

Let $m > 0$ be sufficiently large so that $\psi(a_i \otimes 1) \in \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m$ for all i. Since $\partial_{\mathcal{B}\otimes_{\mathcal{R}}\widehat{\mathcal{S}}} = \partial_{\mathcal{B}} \otimes 1 + 1 \otimes \delta_t$ stabilizes $\mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m$, we have

$$\begin{aligned} \psi\big(\partial^{\ell}_{\mathcal{A}}(ra_i) \otimes 1\big) &= \psi\big(\partial^{\ell}_{\mathcal{A}\otimes_{\mathcal{R}}\widehat{\mathcal{S}}}(ra_i \otimes 1)\big) \\ &= \partial^{\ell}_{\mathcal{B}\otimes_{\mathcal{R}}\widehat{\mathcal{S}}}\big(\psi(ra_i \otimes 1)\big) = \partial^{\ell}_{\mathcal{B}\otimes_{\mathcal{R}}\widehat{\mathcal{S}}}\big(\psi(a_i \otimes r)\big) \\ &\in \partial^{\ell}_{\mathcal{B}\otimes_{\mathcal{R}}\widehat{\mathcal{S}}}(\mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m) \subseteq \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m\,. \end{aligned}$$

Thus $\psi(\mathcal{A} \otimes 1) \subseteq \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m$. Since ψ is $\mathcal{S}_m$-linear, we get

$$\psi(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}_m) \subseteq \mathcal{B} \otimes_R \mathcal{S}_m\,.$$

By restriction, we then have an $\mathcal{S}_m$-conformal superalgebra homomorphism

$$\psi_m : \mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}_m \to \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m,$$

which by base change induces ψ.

As for the injectivity, let $i = 1, 2$ and consider two elements $\psi_i \in \mathrm{Hom}_{\mathcal{S}_{m_i}}\big(\mathcal{A}\otimes_{\mathcal{R}}\mathcal{S}_{m_i}, \mathcal{B}\otimes_{\mathcal{R}}\mathcal{S}_{m_i}\big)$ that map to the same element ψ of $\mathrm{Hom}_{\widehat{\mathcal{S}}}\big(\mathcal{A}\otimes_{\mathcal{R}} \widehat{\mathcal{S}}, \mathcal{B} \otimes_{\mathcal{R}} \widehat{\mathcal{S}}\big)$. Let $m = m_1 m_2$ and consider the images ψ_i' of the ψ_i in $\mathrm{Hom}_{\mathcal{S}_m}\big(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}_m, \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m\big)$. Then both ψ_1' and ψ_2' map to ψ via the base change $\mathcal{S}_m \to \widehat{\mathcal{S}}$. Since the extension $\widehat{S}/S_m$ in k-alg is faithfully flat we conclude that $\psi_1' = \psi_2'$. This shows that ψ_1 and ψ_2 yield the same element of $\varinjlim \mathrm{Hom}_{\mathcal{S}_m}\big(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}_m, \mathcal{B} \otimes_{\mathcal{R}} \mathcal{S}_m\big)$.

(2) If we think of the relevant H^1 as measuring isomorphism classes of forms, then that the map η is bijective is an easy application of (1).

(3) The reasoning of (1) above shows that an automorphism ψ of the $\widehat{\mathcal{S}}$-conformal superalgebra $\mathcal{A} \otimes_{\mathcal{R}} \widehat{\mathcal{S}}$ is determined by its values on the $a_i \otimes 1$. Since $\widehat{\mathcal{S}}$ is a free R-module admitting the set $\{t^q : 0 \leq q < 1\}_{q \in \mathbb{Q}}$ as a basis we can write $\psi(a_i \otimes 1) = \Sigma a_q(i) \otimes t^q$ for some unique $a_q(i) \in \mathcal{A}$. It follows that for $\gamma \in \widehat{\mathbb{Z}} = \pi_1(R)$ we have ${}^{\gamma}\psi = \psi$ if and only if ${}^{\gamma}(t^q) = t^q$ whenever $a_q(i) \neq 0$. Since the action of $\pi_1(R)$ on $\widehat{\mathcal{S}}$ is continuous the stabilizer in $\pi_1(R)$ of each t^q is open. Thus the stabilizer of ψ in $\pi_1(R)$ is a finite intersection of open subgroups, hence open. This shows that the action of $\pi_1(R)$ on $\mathbf{Aut}(\mathcal{A})(\widehat{\mathcal{S}})$ is continuous. Now (3) follows from (2) and Remark 2.27 $\square$

Remark 2.30 We shall later see that all of the conformal superalgebras that interest us do satisfy the above finiteness condition. Computing $H^1_{\mathrm{ct}}\big(\pi_1(R),\, \mathbf{Aut}\,(\mathcal{A})(\widehat{\mathcal{S}})\big)$ is thus central to the classification of forms. The following two results are therefore quite useful.

(1) *If* $\mathbf{G}$ *is a linear algebraic group over* k *whose identity connected component is reductive, then the canonical map*

$$H^1_{\mathrm{ct}}\big(\pi_1(R),\, \mathbf{G}(\widehat{\mathcal{S}})\big) \to H^1_{\text{ét}}\big(R,\, \mathbf{G}\big)$$

is bijective.

(2) *If* $\mathfrak{G}$ *is a reductive group scheme over* $\mathrm{Spec}(R)$ *then* $H^1_{\text{ét}}\big(R,\, \mathfrak{G}\big) = 1$.

The first result follows from Corollary 2.16.3 of [5], while (2) is the main result of [16].

2.4 The centroid trick

Analogous to work done for Lie (super)algebras [1, 6], it is possible to study the more delicate question of k-conformal isomorphism, as opposed to the stronger condition of R-conformal isomorphism, using a technique that we will call the *centroid trick*.[6] In this section, we collect some general facts about centroids and the relationship between these two types of conformal isomorphism. In the next section, we will apply the results of this section to some interesting examples.

Except where otherwise explicitly noted, we assume throughout §2.4 that $\mathcal{R} = (R, \delta_R)$ is an arbitrary object of $k - \delta alg$. Recall that if $R = k$, then $\delta_k = 0$.

[6] This name was suggested by B.N. Allison to emphasize the idea's widespread applicability.

For any $\mathcal{R}$-conformal superalgebra $\mathcal{A}$, let $\mathrm{Ctd}_{\mathcal{R}}(\mathcal{A})$ be the set

$$\{\chi \in \mathrm{End}_{R-smod}(\mathcal{A}) \mid \chi(a_{(n)}b) = a_{(n)}\chi(b) \text{ for all } a, b \in \mathcal{A},\ n \in \mathbb{Z}_+\},$$

where $\mathrm{End}_{R-smod}(\mathcal{A})$ is the set of homogeneous R-supermodule endomorphisms $\mathcal{A} \to \mathcal{A}$ of degree $\overline{0}$.

Recall that for $r \in R$ we use $r_{\mathcal{A}}$ to denote the homothety $a \mapsto ra$. By Axiom (CS3), we have $r_{\mathcal{A}} \in \mathrm{Ctd}_{\mathcal{R}}(\mathcal{A})$. Let $R_{\mathcal{A}} = \{r_{\mathcal{A}} : r \in R\}$. We have a canonical morphism of associative k- (and R-) algebras

$$R \to R_{\mathcal{A}} \subseteq \mathrm{Ctd}_{R}(\mathcal{A}). \tag{2.31}$$

Via restriction of scalars, our $\mathcal{R}$-conformal superalgebra $\mathcal{A}$ admits a k-conformal structure (where, again, k is viewed as an object of $k - \delta alg$ by attaching the zero derivation). This yields the inclusion

$$\mathrm{Ctd}_{\mathcal{R}}(\mathcal{A}) \subseteq \mathrm{Ctd}_{k}(\mathcal{A}). \tag{2.32}$$

Lemma 2.33 *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{R}$-conformal superalgebras such that their restrictions are isomorphic k-conformal superalgebras. That is, suppose there is a k-conformal superalgebra isomorphism $\phi : \mathcal{A} \to \mathcal{B}$ (where $\mathcal{A}$ and $\mathcal{B}$ are viewed as k-conformal superalgebras by restriction). Then the following properties hold.*

(i) *The map $\chi \to \phi\chi\phi^{-1}$ defines an associative k-algebra isomorphism $\mathrm{Ctd}(\phi) : \mathrm{Ctd}_k(\mathcal{A}) \to \mathrm{Ctd}_k(\mathcal{B})$. Moreover, ϕ is an $\mathcal{R}$-conformal superalgebra isomorphism if and only if $\mathrm{Ctd}(\phi)(r_{\mathcal{A}}) = r_{\mathcal{B}}$ for all $r \in R$.*

(ii) *The map $\delta_{\mathrm{Ctd}_k(\mathcal{A})} : \chi \mapsto [\partial_{\mathcal{A}}, \chi] = \partial_{\mathcal{A}}\chi - \chi\partial_{\mathcal{A}}$ is a derivation of the associative k-algebra $\mathrm{Ctd}_k(\mathcal{A})$. Furthermore, the diagram*

$$\begin{array}{ccc} \mathrm{Ctd}_k(\mathcal{A}) & \xrightarrow{\mathrm{Ctd}(\phi)} & \mathrm{Ctd}_k(\mathcal{B}) \\ {\scriptstyle \delta_{\mathrm{Ctd}_k(\mathcal{A})}}\downarrow & & \downarrow{\scriptstyle \delta_{\mathrm{Ctd}_k(\mathcal{B})}} \\ \mathrm{Ctd}_k(\mathcal{A}) & \xrightarrow{\mathrm{Ctd}(\phi)} & \mathrm{Ctd}_k(\mathcal{B}). \end{array} \tag{2.34}$$

commutes.

Proof (i) This is a straightforward consequence of the various definitions.

(ii) For any $\chi \in \mathrm{Ctd}_k(\mathcal{A})$, $a, b \in \mathcal{A}$ and $n \geq 0$,

$$\begin{aligned}
[\partial_{\mathcal{A}}, \chi]\left(a_{(n)}b\right) &= \left(\partial_{\mathcal{A}}\chi - \chi\partial_{\mathcal{A}}\right)\left(a_{(n)}b\right) \\
&= \partial_{\mathcal{A}}\left(a_{(n)}\chi(b)\right) - \chi\left(\partial_{\mathcal{A}}(a)_{(n)}b + a_{(n)}\partial_{\mathcal{A}}(b)\right) \\
&= \partial_{\mathcal{A}}(a)_{(n)}\chi(b) + a_{(n)}\partial_{\mathcal{A}}(\chi(b)) - \partial_{\mathcal{A}}(a)_{(n)}(\chi b) - a_{(n)}\chi(\partial_{\mathcal{A}}(b)) \\
&= a_{(n)}[\partial_{\mathcal{A}}, \chi](b).
\end{aligned}$$

The commutativity of Diagram 2.34 is easy to verify. □

For some of the algebras which interest us the most (e.g. the conformal superalgebras in §3), the natural ring homomorphisms $R \to \mathrm{Ctd}_k(\mathcal{A})$ are isomorphisms. This makes the following result relevant.

Proposition 2.35 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be conformal superalgebras over $\mathcal{R} = (R, \delta_R)$. Assume that $\mathrm{Aut}_k(\mathcal{R}) = 1$, i.e., the only k-algebra automorphism of R that commutes with the derivation δ_R is the identity. Also assume that the canonical maps $R \to \mathrm{Ctd}_k(\mathcal{A}_i)$ are k-algebra isomorphisms for $i = 1, 2$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic as k-conformal superalgebras if and only if they are isomorphic as $\mathcal{R}$-conformal superalgebras.*

Proof Clearly, if $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism of $\mathcal{R}$-conformal superalgebras, then it is also an isomorphism of k-conformal superalgebras when $\mathcal{A}_1$ and $\mathcal{A}_2$ are viewed as k-conformal superalgebras by restriction of scalars.

Now suppose that $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a k-conformal isomorphism, and consider the resulting k-algebra isomorphism $\mathrm{Ctd}(\phi) : \mathrm{Ctd}_k(\mathcal{A}_1) \to \mathrm{Ctd}_k(\mathcal{A}_2)$ of Lemma 2.33. Under the identification $R_{\mathcal{A}_1} = R = R_{\mathcal{A}_2}$, the commutativity of Diagram (2.34), together with $[\partial_{\mathcal{A}_1}, r_{\mathcal{A}_1}] = \delta_R(r)_{\mathcal{A}_1}$ and $[\partial_{\mathcal{A}_2}, r_{\mathcal{A}_2}] = \delta_R(r)_{\mathcal{A}_2}$, yields that $\mathrm{Ctd}(\phi)$, when viewed as an element of $\mathrm{Aut}_{k-alg}(R)$, commutes with the action of δ_R. By hypothesis, $\mathrm{Ctd}(\phi) = \mathrm{id}_R$, and therefore ϕ is R-linear by Lemma 2.33(i). Since ϕ also commutes with $\partial_{\mathcal{A}_1}$ and preserves n-products, it is an $\mathcal{R}$-conformal isomorphism. □

Corollary 2.36 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be conformal superalgebras over $\mathcal{R} = (R, \partial_t)$ where $R = k[t, t^{-1}]$ and $\delta_t = \frac{d}{dt}$. Assume that the canonical maps $R \to \mathrm{Ctd}_k(\mathcal{A}_i)$ are k-algebra isomorphisms, for $i = 1, 2$. Then*

$$\mathcal{A}_1 \cong_k \mathcal{A}_2 \quad \textit{if and only if} \quad \mathcal{A}_1 \cong_{\mathcal{R}} \mathcal{A}_2.$$

Proof The only associative k-algebra automorphisms of R are given by maps $t \mapsto \alpha t^{\epsilon}$, where α is a nonzero element of k and $\epsilon = \pm 1$. Thus the only k-algebra automorphism commuting with the derivation δ_t is the identity map, so the conditions of Proposition 2.35 are satisfied. □

Remark 2.37 The previous corollary is in sharp contrast to the situation that arises in the case of twisted loop algebras of finite-dimensional simple Lie algebras, where k-isomorphic forms need not be R-isomorphic. (See [1] and [16].) The rigidity encountered in the conformal case is due to the presence of the derivation δ_t.

2.5 Central extensions

In this section we assume that our conformal superalgebras are Lie-conformal, i.e. they satisfy axioms (CS4) and (CS5). Following standard practice we denote the λ-product $a_\lambda b$ by $[a_\lambda b]$ (which is then called the λ-bracket.)

A *central extension* of a k-conformal superalgebra $\mathcal{A}$ is a k-conformal superalgebra $\widetilde{\mathcal{A}}$ and a conformal epimorphism $\pi : \widetilde{\mathcal{A}} \to \mathcal{A}$ with kernel $\ker \pi$ contained in the *centre* $Z(\widetilde{\mathcal{A}}) = \{a \in \widetilde{\mathcal{A}} \mid [a_\lambda b] = 0 \text{ for all } b \in \widetilde{\mathcal{A}}\}$ of $\widetilde{\mathcal{A}}$. Given two central extensions $(\widetilde{\mathcal{A}}, \pi)$ and $(\widetilde{\mathcal{B}}, \mu)$ of $\mathcal{A}$, a *morphism* (from $(\widetilde{\mathcal{A}}, \pi)$ to $(\widetilde{\mathcal{B}}, \mu)$) in the category of central extensions is a conformal homomorphism $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ such that $\mu \circ \phi = \pi$. A central extension of $\mathcal{A}$ is *universal* if there is a unique morphism from it to every other central extension of $\mathcal{A}$.

Proposition 2.38 *Let $(\widetilde{\mathcal{A}_i}, \pi_i)$ be central extensions of k-conformal superalgebras $\mathcal{A}_i$ with $\ker \pi_i = Z(\widetilde{\mathcal{A}_i})$ for $i = 1, 2$. Suppose $\widetilde{\psi} : \widetilde{\mathcal{A}_1} \to \widetilde{\mathcal{A}_2}$ is a k-conformal isomorphism. Then $\mathcal{A}_1 \cong_{k-conf} \mathcal{A}_2$.*

Proof In the category of k-vector spaces, fix sections $\sigma_i : \mathcal{A}_i \to \widetilde{\mathcal{A}_i}$ of $\pi_i : \widetilde{\mathcal{A}_i} \to \mathcal{A}_i$. We verify that

$$\psi = \pi_2 \circ \widetilde{\psi} \circ \sigma_1 : \mathcal{A}_1 \to \mathcal{A}_2$$

is a k-conformal isomorphism.

Note that

$$\begin{aligned} \pi_1 \left[\sigma_1(x)_\lambda \sigma(y)\right] &= \left[\pi_1\sigma_1(x)_\lambda \pi_1\sigma_1(y)\right] \\ &= [x_\lambda y] \\ &= \pi_1(\sigma_1 \left[x_\lambda y\right]), \end{aligned}$$

so $\sigma_1 [x_\lambda y] = [\sigma_1(x)_\lambda \sigma_1(y)] + w(\lambda)$ for some polynomial $w(\lambda)$ in the formal variable λ with coefficients in $\ker \pi_1 = Z(\widetilde{\mathcal{A}_1})$.

Moreover, $\left[\widetilde{\psi}(u)_\lambda \widetilde{\psi}(a)\right] = \widetilde{\psi}\,[u_\lambda a] = 0$, for all $u \in Z(\widetilde{\mathcal{A}_1})$ and $a \in \widetilde{\mathcal{A}_1}$, so it is easy to see that $\widetilde{\psi}$ restricts to a bijection between $Z(\widetilde{\mathcal{A}_1})$ and $Z(\widetilde{\mathcal{A}_2})$.

Therefore,

$$\begin{aligned}
\psi\,[x_\lambda y] &= \pi_2 \circ \widetilde{\psi} \circ \sigma_1\,[x_\lambda y] \\
&= \pi_2 \circ \widetilde{\psi}\big(\,[\sigma_1(x)_\lambda \sigma_1(y)]\,\big) \\
&= \pi_2 \circ \widetilde{\psi}\,[\sigma_1(x)_\lambda \sigma_1(y)] \\
&= \Big[\pi_2 \circ \widetilde{\psi} \circ \sigma_1(x)_\lambda \pi_2 \circ \widetilde{\psi} \circ \sigma_1(y)\Big] \\
&= [\psi(x)_\lambda \psi(y)]\,.
\end{aligned}$$

To see that ψ commutes with the derivations, note that $\pi_1\big(\partial_{\widetilde{\mathcal{A}_1}}(x)\big) = \partial_{\mathcal{A}_1}\big(\pi_1(x)\big)$ for all $x \in \mathcal{A}_1$. Thus $\pi_1\Big(\partial_{\widetilde{\mathcal{A}_1}}\big(\sigma_1(x)\big)\Big) = \partial_{\mathcal{A}_1}\big(\pi_1\sigma_1(x)\big) = \partial_{\mathcal{A}_1}(x)$, so $\partial_{\widetilde{\mathcal{A}_1}}\big(\sigma_1(x)\big) = \sigma_1\big(\partial_{\mathcal{A}_1}(x)\big) + u$ for some $u \in \ker \pi_1$. Hence

$$\begin{aligned}
\partial_{\mathcal{A}_2}\psi(x) &= \partial_{\mathcal{A}_2}\pi_2 \circ \widetilde{\psi} \circ \sigma_1(x) \\
&= \pi_2 \circ \widetilde{\psi}\big(\partial_{\widetilde{\mathcal{A}_1}}\sigma_1(x)\big) \\
&= \pi_2 \circ \widetilde{\psi} \circ \sigma_1\big(\partial_{\mathcal{A}_1}(x)\big) + \pi_2 \circ \widetilde{\psi}(u) \\
&= \psi\big(\partial_{\mathcal{A}_1}(x)\big)
\end{aligned}$$

since $\widetilde{\psi}:\ Z(\widetilde{\mathcal{A}_1}) \to Z(\widetilde{\mathcal{A}_2})$. Hence $\psi:\ \mathcal{A}_1 \to \mathcal{A}_2$ is a homomorphism of k-conformal superalgebras.

The map ψ is clearly injective: if $x \in \ker\psi$, then $\widetilde{\psi} \circ \psi_1(x) \in \ker \pi_2 = Z(\mathcal{A}_2)$, so $\sigma_1(x) \in Z(\mathcal{A}_1)$ and $x = \pi_1\psi_1(x) = 0$.

To see that ψ is surjective, let $y \in \mathcal{A}_2$. Then let $x = \pi_1 \circ \widetilde{\psi}^{-1} \circ \sigma_2(y)$. For any $\widetilde{x} \in \widetilde{\mathcal{A}_1}$, we have $\sigma_1\pi_1(\widetilde{x}) = \widetilde{x} + v$ for some $v \in \ker \pi_1$. Thus

$$\begin{aligned}
\psi(x) &= \pi_2 \circ \widetilde{\psi}\big(\sigma_1 \circ \pi_1(\widetilde{\psi}^{-1} \circ \sigma_2(y))\big) \\
&= \pi_2 \circ \widetilde{\psi}(\widetilde{\psi}^{-1} \circ \sigma_2(y) + v)
\end{aligned}$$

for some $v \in \ker \pi_1$. Then

$$\psi(x) = \pi_2 \circ \sigma_2(y) + \pi_2 \circ \widetilde{\psi}(v) = y,$$

and $\psi:\ \mathcal{A}_1 \to \mathcal{A}_2$ is surjective. Hence ψ is an isomorphism of k-conformal superalgebras. □

Corollary 2.39 *Suppose that $(\widetilde{\mathcal{A}_i}, \pi_i)$ are universal central extensions of k-conformal superalgebras $\mathcal{A}_i$ with $Z(\mathcal{A}_i) = 0$ for $i = 1, 2$. Then*

$$\widetilde{\mathcal{A}_1} \cong_{k-conf} \widetilde{\mathcal{A}_2} \quad \textit{if and only if} \quad \mathcal{A}_1 \cong_{k-conf} \mathcal{A}_2.$$

□

3 Examples and applications

In this section, we compute the automorphism groups of some important conformal superalgebras. It is then easy to explicitly classify forms of these algebras by computing the relevant cohomology sets introduced in §2.2.

For all of this section, we fix the notation

$$\begin{aligned}\mathcal{R} = (R, \delta_R) &:= \left(k[t, t^{-1}], \frac{d}{dt}\right)\\ \widehat{\mathcal{S}} = (\widehat{S}, \delta_{\widehat{S}}) &:= \left(k[t^q \mid q \in \mathbb{Q}], \frac{d}{dt}\right),\end{aligned}$$

where k is an algebraically closed field of characteristic zero.

By definition, every k-conformal superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded module $\mathcal{A}$ over the polynomial ring $k[\partial]$ where ∂ acts on $\mathcal{A}$ via $\partial_{\mathcal{A}}$. For the applications below, we work with k-conformal superalgebras $\mathcal{A}$ which are free $k[\partial]$-supermodules. That is, there exists a $\mathbb{Z}/2\mathbb{Z}$-graded subspace $V = V_{\overline{0}} \oplus V_{\overline{1}} \subseteq \mathcal{A}$ so that

$$\mathcal{A}_{\overline{\iota}} = k[\partial] \otimes_k V_{\overline{\iota}}$$

for $\overline{\iota} = \overline{0}, \overline{1}$.

The following result is extremely useful in computing automorphism groups of conformal superalgebras.

Lemma 3.1 *Let $\mathcal{A} = k[\partial] \otimes_k V$ be a k-conformal superalgebra which is a free $k[\partial]$-supermodule. Let $\mathcal{S} = (S, \delta_S)$ be an arbitrary object of $k - \delta alg$. Then*

(i) Every automorphism of the $\mathcal{S}$-conformal superalgebra $\mathcal{A} \otimes_k \mathcal{S}$ is completely determined by its restriction to $V \simeq (1 \otimes V) \otimes 1 \subseteq \mathcal{A} \otimes_k \mathcal{S}$.

(ii) Assume $\phi : V \otimes_k S \to V \otimes_k S$ is a bijective parity-preserving S-linear map such that $\phi([v \otimes 1_\lambda w \otimes 1]) = [\phi(v \otimes 1)_\lambda \phi(w \otimes 1)]$ for all $v, s \in V$. Then there is a unique automorphism $\widehat{\phi} \in \mathbf{Aut}(\mathcal{A})(\mathcal{S})$ extending ϕ.

Proof Let $\{v_i \mid i \in I\}$ be a k-basis of V consisting of homogeneous elements relative to its $\mathbb{Z}/2\mathbb{Z}$-grading, and $\{s_j \mid j \in J\}$ a k-basis of S. Since $\partial_{\mathcal{A}\otimes\mathcal{S}} = \partial_A \otimes 1 + 1 \otimes \delta_S$ and since the v_i form a $k[\partial]$-basis of $\mathcal{A}$, the set

$$\{\partial^\ell_{\mathcal{A}\otimes\mathcal{S}}(v_i \otimes s_j) \mid i \in I, j \in J, \ell \geq 0\}$$

is a k-basis of $\mathcal{A} \otimes_k S$. Because any $\mathcal{S}$-automorphism must commute with $\partial_{\mathcal{A}\otimes\mathcal{S}}$, we have no choice but to define $\widehat{\phi}$ to be the unique k-linear map on the vector space $\mathcal{A} \otimes_k S$ satisfying $\widehat{\phi}(\partial^{\ell}_{\mathcal{A}\otimes\mathcal{S}}(v_i \otimes s_j)) = \partial^{\ell}_{\mathcal{A}\otimes\mathcal{S}}(\phi(v_i \otimes s_j))$. In particular, we have

$$\widehat{\phi}(\partial^{\ell}_{\mathcal{A}\otimes\mathcal{S}}(x)) = \partial^{\ell}_{\mathcal{A}\otimes\mathcal{S}}(\phi(x)). \tag{3.2}$$

for all $x \in V \otimes_k S$. We claim that $\widehat{\phi} \in \mathbf{Aut}(\mathcal{A})(\mathcal{S})$. It is immediate from the definition that $\widehat{\phi}$ is invertible, and that it commutes with the action of $\partial_{\mathcal{A}\otimes\mathcal{S}}$.

For any $v, w \in V$, $r, s \in R$, and $n \in \mathbb{Z}_+$,

$$\begin{aligned}
\phi(v \otimes r_{(n)} w \otimes s) &= s\phi(v \otimes r_{(n)} w \otimes 1) \\
&= -s\, p(v,w) \sum_{j=0}^{\infty} (-1)^{n+j} \partial^{(j)}_{\mathcal{A}\otimes\mathcal{S}} \phi(w \otimes 1_{(n+j)} v \otimes r) \\
&= -s\, p(v,w) \sum_{j=0}^{\infty} (-1)^{n+j} \partial^{(j)}_{\mathcal{A}\otimes\mathcal{S}} r\phi(w \otimes 1_{(n+j)} v \otimes 1) \\
&= -s\, p(v,w) \sum_{j=0}^{\infty} (-1)^{n+j} \partial^{(j)}_{\mathcal{A}\otimes\mathcal{S}} r\phi(w \otimes 1)_{(n+j)} \phi(v \otimes 1) \\
&= -s\, p(v,w) \sum_{j=0}^{\infty} (-1)^{n+j} \partial^{(j)}_{\mathcal{A}\otimes\mathcal{S}} \phi(w \otimes 1)_{(n+j)} r\phi(v \otimes 1) \\
&= s\,(r\phi(v \otimes 1))_{(n)} \phi(w \otimes 1) \\
&= \phi(v \otimes r)_{(n)} \phi(w \otimes s),
\end{aligned}$$

by (CS4) and (CS3). Similar arguments using (CS1) and (CS4) show that for any homogeneous $x, y \in V \otimes S$ and $\ell, m, n \in \mathbb{Z}_+$,

$$\begin{aligned}
\widehat{\phi}\big(\partial^{(m)}_{\mathcal{A}\otimes\mathcal{S}}(x)_{(n)} \partial^{(\ell)}_{\mathcal{A}\otimes\mathcal{S}}(y)\big) &= \partial^{(m)}_{\mathcal{A}\otimes\mathcal{S}} \phi(x)_{(n)} \partial^{(\ell)}_{\mathcal{A}\otimes\mathcal{S}} \phi(y) \\
&= \widehat{\phi}\big(\partial^{(m)}_{\mathcal{A}\otimes\mathcal{S}}(x)\big)_{(n)} \widehat{\phi}\big(\partial^{(\ell)}_{\mathcal{A}\otimes\mathcal{S}}(y)\big),
\end{aligned}$$

so $\widehat{\phi}$ preserves n-products. Finally, to see that $\widehat{\phi}$ is S-linear, we first observe that

$$s_{\mathcal{A}\otimes\mathcal{S}} \circ \partial^{(n)}_{\mathcal{A}\otimes\mathcal{S}} = \sum_{i=0}^{n} (-1)^i \partial^{(i)}_{\mathcal{A}\otimes\mathcal{S}} \circ \delta_S(s)^{(n-i)}_{\mathcal{A}\otimes\mathcal{S}}. \tag{3.3}$$

This follows from repeated use of Axiom (CS2) applied to the $\mathcal{S}$-conformal superalgebra $\mathcal{A} \otimes_k \mathcal{S}$ when taking into account that $\partial_{\mathcal{A}\otimes\mathcal{S}} = \partial_{\mathcal{A}} \otimes 1 + 1 \otimes \delta_S$.

For all $s \in S$ and $x \in V \otimes_k S$, we then have

$$
\begin{aligned}
&\widehat{\phi}\big(s_{\mathcal{A}\otimes\mathcal{S}} \circ \partial^{(n)}_{\mathcal{A}\otimes\mathcal{S}}(x)\big) \\
&\qquad = \widehat{\phi}\Big(\sum_{i=0}^{n}(-1)^i \partial^{(n)}_{\mathcal{A}\otimes\mathcal{S}} \circ \delta_S(s)^{(n-i)}_{\mathcal{A}\otimes\mathcal{S}}(x)\big)\Big) && \text{(by 3.3)} \\
&\qquad = \sum_{i=0}^{n}(-1)^i \partial^{(n)}_{\mathcal{A}\otimes\mathcal{S}} \widehat{\phi}\Big(\delta_S(s)^{(n-i)}_{\mathcal{A}\otimes\mathcal{S}}(x)\big)\Big) && \text{(by definition of } \widehat{\phi}) \\
&\qquad = \sum_{i=0}^{n}(-1)^i \partial^{(n)}_{\mathcal{A}\otimes\mathcal{S}} \circ \delta_S(s)^{(n-i)}_{\mathcal{A}\otimes\mathcal{S}} \phi(x)\big) && (\mathcal{S}-\text{linearity on } V \otimes_k S) \\
&\qquad = s_{\mathcal{A}\otimes\mathcal{S}} \circ \partial^{(n)}_{\mathcal{A}\otimes\mathcal{S}} \phi(x) && \text{(by 3.3)} \\
&\qquad = s_{\mathcal{A}\otimes\mathcal{S}} \circ \widehat{\phi}\big(\partial^{(n)}_{\mathcal{A}\otimes\mathcal{S}}(x)\big), && \text{(by definition of } \widehat{\phi})
\end{aligned}
$$

so $\widehat{\phi}$ commutes with the operator $s_{\mathcal{A}\otimes\mathcal{S}}$, and $\widehat{\phi}$ is thus S-linear. □

3.1 Current conformal superalgebras

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a Lie superalgebra of arbitrary dimension over the field k. Let $\mathrm{Curr}\, V$ be the *current conformal superalgebra*

$$\mathrm{Curr}\,(V) := k[\partial] \otimes_k V$$

with n-products defined by the λ-bracket[7] $[v_\lambda w] = [v, w]$, where $[v, w]$ is the Lie superbracket for all $v, w \in V$. The derivation $\partial_{\mathrm{Curr}\,(V)}$ is given by the natural action of ∂ on the k-space $\mathrm{Curr}\,(V)$.

Theorem 3.4 *Let V be a semisimple Lie superalgebra over k, and let $\sigma \in \mathrm{Aut}_{k-conf}(Curr\, V)$. Then $\sigma(V) \subseteq V$.*

Proof Let $\mathcal{A} = \mathrm{Curr}\, V$, and let $\pi_V : \mathcal{A} \to V = k \otimes_k V$ be the projection of $\mathcal{A}$ onto the first component of the (vector space) direct sum

$$\mathcal{A} = (k \otimes_k V) \oplus (\partial k[\partial] \otimes_k V).$$

Let $\sigma_V = \pi_V \sigma : \ V \to V$, and extend σ_V to $\mathcal{A}$ by $k[\partial]$-linearity.

To verify that σ_V is a k-conformal homomorphism, we expand both sides of the following equation for all $x, y \in V$:

$$\sigma\,[x_\lambda y] = [\sigma(x)_\lambda \sigma(y)]\,. \tag{3.5}$$

[7] See Remark 1.8.

The left-hand side expands as

$$\sigma\,[x_\lambda y] = \sigma_V\,[x_\lambda y] + (\sigma - \sigma_V)\,[x_\lambda y]\,. \tag{3.6}$$

The right-hand side is

$$\begin{aligned} [\sigma(x)_\lambda \sigma(y)] &= [\sigma_V(x)_\lambda \sigma_V(y)] + [\sigma_V(x)_\lambda (\sigma - \sigma_V)(y)] \\ &+ \ [(\sigma - \sigma_V)(x)_\lambda \sigma_V(y)] + [(\sigma - \sigma_V)(x)_\lambda (\sigma - \sigma_V)(y)]\,. \end{aligned} \tag{3.7}$$

By (1.9),

$$[\sigma_V(x)_\lambda (\sigma - \sigma_V)(y)] + [(\sigma - \sigma_V)(x)_\lambda \sigma_V(y)] + [(\sigma - \sigma_V)(x)_\lambda (\sigma - \sigma_V)(y)]$$

is contained in the space

$$k[\lambda] \otimes \partial k[\partial] \otimes V + \lambda k[\lambda] \otimes k[\partial] \otimes V,$$

as is $(\sigma - \sigma_V)\,[x_\lambda y]$. Therefore, we can apply π_V to the right-hand sides of (3.6) and (3.7) and then evaluate at $\lambda = 0$ to obtain

$$\sigma_V\,[x_\lambda y] = [\sigma_V(x)_\lambda \sigma_V(y)]\,. \tag{3.8}$$

Thus $\sigma_V : \ \mathcal{A} \to \mathcal{A}$ is a homomorphism of k-conformal superalgebras.

In fact, σ_V is a k-conformal superalgebra *isomorphism*. By the $k[\partial]$-linearity of σ_V, it is sufficient to verify that its restriction $\sigma_V : \ V \to V$ is bijective. But this is straightforward: if $\sigma_V(x) = 0$, then $\sigma(x) = \partial a$ for some $a \in \mathcal{A}$, and $x = \partial\sigma^{-1}(a)$. But $x \in V$, so $x = 0$, and σ_V is injective. Likewise, if $y \in V$, then write $\sigma^{-1}(y) = z + \partial b$ for some $z \in V$ and $b \in \mathcal{A}$. Then

$$\begin{aligned} y &= \sigma_V \sigma^{-1}(y) \\ &= \sigma_V(z) + \sigma_V(\partial b) \\ &= \sigma_V(z) + \pi_V(\partial\sigma(b)) \\ &= \sigma_V(z), \end{aligned}$$

so σ_V is also surjective. Hence $\sigma_V : \ \mathcal{A} \to \mathcal{A}$ is a k-conformal automorphism.

Therefore the map

$$\tau = \sigma_V^{-1}\sigma : \ \mathcal{A} \to \mathcal{A} \tag{3.9}$$

is a k-conformal automorphism. Note that $\pi_V \tau(x) = x$ for all $x \in V$. Since $\sigma = \sigma_V \tau$ and $\sigma_V(V) \subseteq V$, Theorem 3.4 will be proven if we show that $\tau(V) \subseteq V$.

For nonzero $x \in V$ write

$$\tau(x) = \sum_{i=0}^{M(x)} \partial^{(i)} v_{ix} \tag{3.10}$$

for some $v_{ix} \in V$, with $v_{M(x),x} \neq 0$. Define v_{i0} to be zero for all i and $M(0) = -1$. Let

$$W = \mathrm{Span}_k\{v_{M(x),x} \mid x \in V\}. \tag{3.11}$$

We claim that $W \subseteq V$ is an ideal of the Lie algebra V. Indeed, for any $x, y \in V$,

$$\begin{aligned}
\tau[x,y] &= \tau\,[x_\lambda y] \\
&= [\tau(x)_\lambda \tau(y)] \\
&= \left[\sum_{i=0}^{M(x)} \partial^{(i)} v_{ix}\ {}_\lambda \sum_{j=0}^{M(y)} \partial^{(j)} v_{jy}\right] \\
&= \sum_{i=0}^{M(x)} \sum_{j=0}^{M(y)} (-\lambda)^{(i)} (\partial+\lambda)^{(j)} [v_{ix}, v_{jy}].
\end{aligned} \tag{3.12}$$

The highest power of ∂ in (3.12) is $\partial^{(M(y))}$. Since the indeterminate λ does not occur in the expression $\tau[x,y]$, we see that either

$$M([x,y]) = M(y) \tag{3.13}$$

or else

$$[v_{0x}, v_{M(y),y}] = 0. \tag{3.14}$$

If (3.13) holds, then $v_{M([x,y]),[x,y]} = [v_{0x}, v_{M(y),y}]$, so

$$[x, v_{M(y),y}] \in W \tag{3.15}$$

since $v_{0x} = \pi_V \tau(x) = x$. If (3.14) holds, then (3.15) holds trivially since $[x, v_{M(y),y}] = 0$. Therefore $[V, v_{M(y),y}] \subseteq W$ for all $y \in V$, so W is an ideal of the Lie superalgebra V.

Suppose that $M(x) > 0$ for some x. Comparing the highest powers of λ occurring on both sides of (3.12), we have

$$0 = (-\lambda)^{(M(x))} \lambda^{(M(y))} [v_{M(x),x}, v_{M(y),y}]$$

for all $y \in V$. That is, $[v_{M(x),x}, v_{M(y),y}] = 0$ for all $y \in V$, so $v_{M(x),x}$ is in the centre $Z(W)$ of the Lie superalgebra W. But $Z(W)$ is an ideal of V. (This follows easily from the Jacobi identity.) Since $Z(W)$ is solvable and V is semisimple, we see that $Z(W) = 0$ and $v_{M(x),x} = 0$, a contradiction. Hence $M(x) = 0$ for all nonzero $x \in V$. Therefore, $\tau(x) = v_{0x} = \pi_V \tau(x) = x$ for all $x \in V$, so the $k[\partial]$-linear map τ is the identity map on $\mathcal{A}$, and $\sigma = \sigma_V \tau = \sigma_V : V \to V$. $\square$

Corollary 3.16 *Let V be a simple Lie superalgebra over k. Then*

$$\mathrm{Aut}_{k-conf}(Curr(V)) = \mathrm{Aut}_{k-Lie}(V),$$

where $\mathrm{Aut}_{k-Lie}(V)$ is the group of Lie superalgebra automorphisms of V.

Proof Lie automorphisms of V extend uniquely to superconformal automorphisms of Curr (V) by $k[\partial]$-linearity. Conversely, superconformal automorphisms of Curr (V) restrict to automorphisms of the Lie superalgebra V by Theorem 3.4. These correspondences are clearly inverse to one another. $\square$

Corollary 3.17 *Let V be a Lie superalgebra over k, and let $\mathcal{S}$ be an extension in $k - \delta alg$ for which $V \otimes \mathcal{S}$ is semisimple as a Lie superalgebra.*[8] *Then*

$$\mathrm{Aut}_{\mathcal{S}-conf}(Curr(V) \otimes_k \mathcal{S}) = \mathrm{Aut}_{S-Lie}(V \otimes_k S).$$

Proof Every $\mathcal{S}$-conformal automorphism σ of Curr $(V) \otimes_k \mathcal{S}$ is also a k-conformal automorphism via the restriction functor. By Theorem 3.4,

$$\sigma(V \otimes S) \subseteq V \otimes S.$$

The λ-bracket in the $\mathcal{S}$-conformal superalgebra Curr $(V) \otimes_k \mathcal{S}$ is

$$[v \otimes r_\lambda w \otimes s] = [v, w] \otimes rs$$

for all $v, w \in V$ and $r, s \in S$. That is, Curr $(V) \otimes_k \mathcal{S} =$ Curr $(V \otimes_k S)$ as $\mathcal{S}$-conformal superalgebras. Then the argument of Corollary 3.16 holds in the $\mathcal{S}$-conformal context as well. $\square$

Remark 3.18 Let V be a finite-dimensional simple Lie superalgebra over k. By Theorem 2.16 and Proposition 2.29, the $\mathcal{R}$-isomorphism classes of $\widehat{\mathcal{S}}/\mathcal{R}$-forms of the $\mathcal{R}$-conformal superalgebra

$$\mathrm{Curr}\,(V) \otimes_k \mathcal{R} = \big(k[\partial] \otimes_k V\big) \otimes_k \mathcal{R}$$

[8] For example, any finite-dimensional simple Lie algebra V over k satisfies this condition with any extension $\mathcal{S} \in k - \delta alg$.

are parametrized by

$$H^1(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathrm{Curr}\,(V))) \simeq H^1_{\mathrm{ct}}\big(\pi_1(R), \mathbf{Aut}(\mathrm{Curr}\,(V))(\widehat{\mathcal{S}})\big),$$

By Corollary 3.17, $\mathbf{Aut}(\mathrm{Curr}\,(V))(\widehat{\mathcal{S}}) = \mathrm{Aut}_{\widehat{S}-Lie}(V \otimes \widehat{S})$, a group that is computed in [7, 13, 15]. For example, if $V = \mathfrak{sl}_2(k)$, then $\mathrm{Aut}_{\widehat{S}-Lie}(V \otimes \widehat{S}) = \mathbf{PGL}_2(\widehat{S})$, so

$$H^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathrm{Curr}\,(V)) = H^1_{\mathrm{ct}}\big(\pi_1(R), \mathbf{PGL}_2(\widehat{S})\big) = H^1_{\text{ét}}\big(R, \mathbf{PGL}_2\big) = \{1\}.$$

In particular, all $\widehat{\mathcal{S}}/\mathcal{R}$-forms of $\mathrm{Curr}\,(\mathfrak{sl}_2(k)) \otimes_k \mathcal{R}$ are trivial, that is, isomorphic to $\mathrm{Curr}\,(\mathfrak{sl}_2(k)) \otimes_k \mathcal{R}$ as an $\mathcal{R}$-conformal superalgebra.

3.2 The Virasoro conformal algebra

The (centreless) Virasoro conformal algebra $\mathcal{V}ir$ is the free $k[\partial]$-module

$$\mathcal{V}ir = k[\partial] \otimes kL,$$

with λ-bracket given by

$$[L_\lambda L] = (\partial + 2\lambda)L.$$

Proposition 3.19 *Let* $\widehat{\mathcal{V}ir} = \mathcal{V}ir \otimes_k \widehat{\mathcal{S}}$. *Then* $\mathrm{Aut}_{\widehat{\mathcal{S}}-conf}\widehat{\mathcal{V}ir} = 1$.

Proof Let $\sigma \in \mathrm{Aut}_{\widehat{\mathcal{S}}-\mathrm{conf}}\widehat{\mathcal{V}ir}$, and write $\sigma(L \otimes 1) = \sum_{j=0}^{N} \widehat{\partial}^{(j)}(L \otimes r_j)$ for some $r_j \in \widehat{\mathcal{S}}$, where $\widehat{\partial} = \partial_{\widehat{\mathcal{V}ir}}$ and $r_N \neq 0$. Comparing the coefficients of the highest powers of λ occurring in the relation

$$\sigma\,[L \otimes 1_\lambda L \otimes 1] = [\sigma(L \otimes 1)_\lambda \sigma(L \otimes 1)]\,, \tag{3.20}$$

we see that $\sigma(L \otimes 1) = L \otimes r$ for some $r \in \widehat{\mathcal{S}}$. Applying (3.20) again, we obtain $r = r^2$. Since σ is an automorphism, it is nonzero and $\widehat{\mathcal{S}}$-linear, so $r = 1$ and $\sigma = 1$. ☐

In particular, all $\widehat{\mathcal{S}}/\mathcal{R}$-forms of $\mathcal{V}ir \otimes_k \mathcal{R}$ are $\mathcal{R}$-isomorphic (and hence k-isomorphic), since

$$H^1(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{V}ir)) = H^1_{\mathrm{ct}}(\pi_1(R), 1) = \{1\}.$$

It is straightforward to check that not only is $\widehat{\mathcal{V}ir}$ centreless[9], but $\widehat{\mathcal{V}ir}$ has no *abelian* conformal subalgebras:

[9] Hence it satisfies the hypotheses of Cor 2.39, classifying $\widehat{\mathcal{S}}/\mathcal{R}$-forms of $\mathcal{V}ir \otimes_k \mathcal{R}$ up to k-isomorphism of universal central extensions.

Lemma 3.21 *If $\mathcal{A} \subseteq \widehat{\mathcal{V}ir}$ is a conformal subalgebra, then $[\mathcal{A}_\lambda \mathcal{A}] \neq 0$.*

Proof Let $u = \sum_{j=0}^{N} \widehat{\partial}^{(j)}(L \otimes r_j)$ be an element of a conformal subalgebra $\mathcal{A} \subseteq \widehat{\mathcal{V}ir}$, with $r_N \neq 0$. Then

$$[u_\lambda u] = \sum_{0 \le i,j \le N} (-\lambda)^{(i)}(\partial + \lambda)^{(j)} \left[L \otimes r_{i\lambda} L \otimes r_j\right],$$

so $u_{(2N+1)}u = \frac{2(-1)^N}{(N!)^2} L \otimes r_N^2 \neq 0$. □

The above lemma is in stark contrast to the fact that every current conformal algebra is linearly spanned by abelian conformal subalgebras:

$$\operatorname{Curr} \mathfrak{g} = \sum_{x \in \mathfrak{g}} k[\partial] \otimes kx$$

and the subalgebra $k[\partial] \otimes kx = \operatorname{Curr} kx$ has trivial λ-bracket. Since $\widehat{\mathcal{V}ir}$ has no abelian conformal subalgebras, any homomorphism $\operatorname{Curr} \mathfrak{g} \to \widehat{\mathcal{V}ir}$ must be trivial.

Corollary 3.22 *Let $\mathfrak{g}$ be a Lie algebra over k, and let $\sigma :$ Curr $\mathfrak{g} \to \widehat{\mathcal{V}ir}$ be a homomorphism of k-conformal algebras. Then $\sigma = 0$.*

□

3.3 Forms of the $N = 2$ conformal superalgebra

Recall that the classical $N = 2$ k-conformal superalgebra $\mathcal{A}$ is the free $k[\partial]$-module $\mathcal{A} = k[\partial] \otimes_k V$ where $V = V_{\bar{0}} \oplus V_{\bar{1}}$,

$$V_{\bar{0}} = kL \oplus kJ$$
$$V_{\bar{1}} = kG^+ \oplus kG^-,$$

with λ-bracket given by[10]

[10] These conditions say that J (respectively, $G^\pm$) is a primary eigenvector of conformal weight 1 (resp., $\frac{3}{2}$) with respect to the Virasoro element L.

$$
\begin{aligned}
[L_\lambda L] &= (\partial + 2\lambda)L && (3.23)\\
[L_\lambda J] &= (\partial + \lambda)J && (3.24)\\
\left[L_\lambda G^\pm\right] &= (\partial + \frac{3}{2}\lambda)G^\pm && (3.25)\\
[J_\lambda J] &= 0 && (3.26)\\
\left[J_\lambda G^\pm\right] &= \pm G^\pm && (3.27)\\
\left[G^+{}_\lambda G^+\right] &= \left[G^-{}_\lambda G^-\right] \;=\; 0 && (3.28)\\
\left[G^+{}_\lambda G^-\right] &= L + \frac{1}{2}(\partial + 2\lambda)J. && (3.29)
\end{aligned}
$$

Proposition 3.30 *Let $\widehat{\mathcal{A}} = A \otimes_k \widehat{\mathcal{S}}$. Then*

(1) *For each $s = \alpha t^q \in \widehat{\mathcal{S}}^\times$, with $\alpha \in k^\times$ and $q \in \mathbb{Q}$, there exists a unique automorphism $\theta_s \in \mathrm{Aut}_{\widehat{\mathcal{S}}}(\widehat{\mathcal{A}})$ such that*

$$
\begin{aligned}
\theta_s : \qquad & L \mapsto L + qJ \otimes t^{-1}\\
& J \mapsto J\\
& G^+ \mapsto G^+ \otimes s\\
& G^- \mapsto G^- \otimes s^{-1}.
\end{aligned}
$$

(2) *There exists a unique automorphism $\omega \in \mathrm{Aut}_{\widehat{\mathcal{S}}}(\widehat{\mathcal{A}})$ such that*

$$
\begin{aligned}
\omega : \qquad & L \mapsto L\\
& J \mapsto -J\\
& G^+ \mapsto G^-\\
& G^- \mapsto G^+.
\end{aligned}
$$

(3) *The map $s \mapsto \theta_s$ is a group isomorphism between $\widehat{S}^\times$ and the subgroup $\langle\theta_s\rangle_{s\in\widehat{S}^\times}$ of $\mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_k \widehat{\mathcal{S}})$ generated by the θ_s. This isomorphism is compatible with the action of the algebraic fundamental group $\pi_1(R)$.*

(4) *Let $\mathbb{Z}/2\mathbb{Z}$ act on $\widehat{S}^\times$ by ${}^{\overline{1}}t^{p/q} = t^{-p/q}$. There exists an isomorphism of $\pi_1(R)$-groups*

$$\psi : \widehat{S}^\times \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_k \widehat{\mathcal{S}})$$

such that

$$\psi(s, \overline{\varepsilon}) \mapsto \theta_s \omega^\varepsilon$$

for all $s \in \widehat{S}^\times$ and $\varepsilon = 0, 1$.

Proof The proofs of (1) and (2) are based on Lemma 3.1. One must check that θ_s and ω preserve the λ-product of any two elements of $V \otimes 1$. This is done by direct (tedious) calculations.

(3) This is a straightforward consequence of the various definitions.

(4) The delicate point is to show that the $(\theta_s)_{s\in\widehat{S}^\times}$ and ω generate $\mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A}\otimes_k\widehat{\mathcal{S}})$. To do this we start with an arbitrary element $\sigma \in \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A}\otimes_k \widehat{\mathcal{S}})$ and reason as follows:

Step 1: $\sigma(J\otimes 1) = J\otimes s$ for some $s\in\widehat{S}$.
Proof: Let $\pi: \widehat{\mathcal{A}}_{\overline{0}} \to \widehat{\mathcal{V}ir}$ be the projection onto the second component of

$$\widehat{\mathcal{A}}_{\overline{0}} = \Big((k[\partial]\otimes_k kJ)\otimes_k \widehat{\mathcal{S}}\Big) \oplus \Big((k[\partial]\otimes_k kL)\otimes_k \widehat{\mathcal{S}}\Big) = \mathrm{Curr}\,(kJ\otimes_k \widehat{\mathcal{S}}) \oplus \widehat{\mathcal{V}ir}.$$

The map $\pi\sigma: \widehat{\mathcal{A}}_{\overline{0}} \to \widehat{\mathcal{V}ir}$ restricts to a map

$$\mathrm{Curr}\,(kJ) = (k[\partial]\otimes kJ)\otimes 1 \to \widehat{\mathcal{V}ir}$$

of k-conformal algebras. By Corollary 3.22, this map is zero, and

$$\sigma(J\otimes 1) = \sum_{k=0}^{N} \partial^{(k)}_{\mathcal{A}\otimes\widehat{\mathcal{S}}}(J\otimes s_k),$$

for some $s_k\in\widehat{\mathcal{S}}$.

Comparing the coefficients of the highest powers of λ occurring in the relation

$$\lambda\sigma(J\otimes 1) = \sigma\,[J\otimes 1_\lambda L\otimes 1] = [\sigma(J\otimes 1)_\lambda \sigma(L\otimes 1)]$$

shows that $N=0$. Hence $\sigma(J\otimes 1) = J\otimes s$ for some $s\in\widehat{\mathcal{S}}$.

Step 2: There exist $c\in k^\times$ and $s\in\widehat{S}$ such that $\sigma(J\otimes 1) = J\otimes c$ and $\sigma(L\otimes 1) = L\otimes 1 + J\otimes s$.
Proof: Apply σ to the relation $[J\otimes 1_\lambda L\otimes 1] = \lambda J\otimes 1$. Comparing the coefficients of the top powers of λ shows that $\sigma(J\otimes 1) = J\otimes c$ for some $c\in k^\times$ and

$$\sigma(L\otimes 1) = L\otimes 1 + \sum_{j=0}^{N} \widehat{\partial}^{(j)}(J\otimes s_j),$$

for some $s_j \in \widehat{S}$, where $s_N \neq 0$ and $\widehat{\partial} = \partial_{\mathcal{A}\otimes\widehat{S}}$. We then consider the relation

$$[\sigma(L \otimes 1)_\lambda \sigma(L \otimes 1)] = (\widehat{\partial} + 2\lambda)\sigma(L \otimes 1). \tag{3.31}$$

Expanding and comparing the coefficients of $\lambda\widehat{\partial}^{(N)}(J \otimes s_N)$ shows that N is 0 or 1.

If $N = 1$, then (3.31) becomes $\lambda J \otimes s_0 = -(\widehat{\partial} + 2\lambda) J \otimes \delta_{\widehat{S}}(s_1)$. Since there are no $\widehat{\partial}$ terms in the left-hand side of this equation, we see that $\delta_{\widehat{S}}(s_1) = 0$. But then $s_0 = 0$. Thus $\sigma(L \otimes 1) = L \otimes 1 + \widehat{\partial}(J \otimes c_1)$ for some $c_1 \in k$ or else $\sigma(L \otimes 1) = L \otimes 1 + J \otimes s$ for some $s \in \widehat{S}$.

Suppose that $\sigma(L \otimes 1) = L \otimes 1 + \widehat{\partial}(J \otimes c_1)$. Write

$$\sigma(G^+ \otimes 1) = \sum_{k=0}^{M_+} \widehat{\partial}^{(k)}(G^+ \otimes r_k) + \sum_{\ell=0}^{M_-} \widehat{\partial}^{(\ell)}(G^- \otimes s_\ell),$$

with r_{M_+} and s_{M_-} both nonzero. Consider the relation

$$\sigma\left[L \otimes 1_\lambda G^\pm \otimes 1\right] = \left[\sigma(L \otimes 1_\lambda \sigma(G^\pm \otimes 1)\right]. \tag{3.32}$$

Expanding this relation and comparing the coefficients of terms involving $\lambda^{1+M_+}G^+$, we see that $\frac{3}{2}r_0\lambda = \frac{\lambda^{1+M_+}}{M_+!} r_{M_+}\left(\frac{3}{2} - c_1\right)$. If $r_0 \neq 0$, then $M_+ = c_1 = 0$. Similar arguments show that if we write

$$\sigma(G^- \otimes 1) = \sum_{k=0}^{N_+} \widehat{\partial}^{(k)}(G^+ \otimes a_k) + \sum_{\ell=0}^{N_-} \widehat{\partial}^{(\ell)}(G^- \otimes b_\ell),$$

then $c_1 = 0$ if $a_0 \neq 0$. But if $r_0 = a_0 = 0$, then the map σ is not surjective, so $c_1 = 0$ and $\sigma(L \otimes 1) = L \otimes 1 + J \otimes s$ for some $s \in \widehat{S}$.

Step 3: There exist s^+ and s^- in $\widehat{S}^\times$ such that either

3(a) $\sigma(J \otimes 1) = J \otimes 1$ *and* $\sigma(G^\pm \otimes 1) = G^\pm \otimes s^\pm$, *or*

3(b) $\sigma(J \otimes 1) = -J \otimes 1$ *and* $\sigma(G^\pm \otimes 1) = G^\mp \otimes s^\pm$.

Proof: From the proof of Step 2, we see that $M_+ = 0$. Likewise, $M_- = N_+ = N_- = 0$. Hence

$$\sigma(G^+ \otimes 1) = G^+ \otimes r_+ + G^- \otimes s_+$$
$$\sigma(G^- \otimes 1) = G^+ \otimes r_- + G^- \otimes s_-,$$

for some $r_\pm, s_\pm \in \widehat{S}$. The relation $\sigma\,[J \otimes 1_\lambda G^\pm \otimes 1] = [\sigma(J \otimes 1)_\lambda \sigma(G^\pm \otimes 1)]$ shows that

$$G^+ \otimes r_+ + G^- \otimes s_+ = cG^+ \otimes r_+ - cG^- \otimes s_+, \tag{3.33}$$
$$-(G^+ \otimes r_- + G^- \otimes s_-) = cG^+ \otimes r_- - cG^- \otimes s_-. \tag{3.34}$$

If $r_+ \neq 0$, then by (3.33), we see that $s_+ = 0$ and $c = 1$. Then by (3.34), $r_- = 0$ and $s_- \neq 0$. Hence we are in Case 3(a).

Similarly, if $r_+ = 0$, then $s_+ \neq 0$, $c = -1$, and we are in Case 3(b).

Step 4: If σ is as in Case 3(a) above, then s^+ and s^- are inverses of each other. Furthermore, if we write $s^+ = \alpha t^q$ for some $\alpha \in k^\times$ and $q \in \mathbb{Q}$, then $\sigma = \theta_{s^+}$.

Proof: Since we are in Case 3(a), the relation

$$\sigma\left[G^+ \otimes 1_\lambda G^- \otimes 1\right] = \left[\sigma(G^+ \otimes 1)_\lambda \sigma(G^- \otimes 1)\right]$$

says that

$$\begin{aligned}&L \otimes 1 + J \otimes s + \tfrac{1}{2}(\widehat{\partial} + 2\lambda)(J \otimes 1)\\ &\qquad = L \otimes s^+ s^- + J \otimes \delta_{\widehat{S}}(s^+) s^- + \tfrac{1}{2}\partial J \otimes s^+ s^- + \lambda J \otimes s^+ s^-.\end{aligned}$$

Thus s^+ and s^- are inverses of each other, and

$$s = \delta_{\widehat{S}}(s^+) s^- = q\alpha t^{q-1}(s^+)^{-1} = qt^{-1}.$$

Hence $\sigma = \theta_{s^+}$.

To finish the proof of Proposition 3.30, we have to consider the case when σ is as in 3(b). Replacing σ by $\sigma\omega$ yields an automorphism that satisfies 3(a), and we can conclude by Step 4. □

Theorem 3.35 *Let $\mathcal{A}$ be the classical $N = 2$ conformal superalgebra. Up to k-conformal isomorphism, there are exactly two twisted loop algebras of $\mathcal{A}$. These are $L(\mathcal{A}, \mathrm{id})$ and $L(\mathcal{A}, \omega)$. Furthermore, any $\widehat{\mathcal{S}}/\mathcal{R}$-form of $\mathcal{A}$ is isomorphic to one of these two loop algebras.*

Proof By Theorem 2.16, Proposition 2.29, and Proposition 3.30, the $\mathcal{R}$-isomorphism classes of $\widehat{\mathcal{S}}/\mathcal{R}$-forms of the $\mathcal{R}$-conformal superalgebra $\mathcal{A}$ are parametrized by

$$H^1(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{A})) \simeq H^1_{\mathrm{ct}}\big(\pi_1(R), \mathbf{Aut}(A)(\widehat{\mathcal{S}})\big) \simeq H^1_{\mathrm{ct}}\big(\pi_1(R), \widehat{S}^\times \rtimes \mathbb{Z}/2\mathbb{Z})\big).$$

Consider the split exact sequence of $\pi_1(R) = \widehat{\mathbb{Z}}$-groups

$$1 \to \widehat{S}^\times \to \widehat{S}^\times \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 1. \tag{3.36}$$

Passing to (continuous) cohomology yields

$$H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \widehat{S}^\times) \to H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \widehat{S}^\times \rtimes \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\psi} H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}). \tag{3.37}$$

The map ψ admits a section (hence is surjective) because the sequence (3.36) is split. Since $\widehat{\mathbb{Z}}$ acts trivially on the (abelian) group $\mathbb{Z}/2\mathbb{Z}$, we have

$$H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Hom}_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Note that ψ maps the cohomology classes of $H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbf{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_k \widehat{\mathcal{S}}))$ corresponding to the loop algebras $L(\mathcal{A}, \mathrm{id})$ and $L(\mathcal{A}, \omega)$ to the two distinct classes of $H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z})$. To prove Theorem 3.35, it is thus enough to show that ψ in (3.37) is bijective, and that $\widehat{\mathcal{S}}/\mathcal{R}$-forms of $\mathcal{A}$ are k-conformal isomorphic if and only if they are $\mathcal{R}$-conformal isomorphic.

Let $\mathbf{G_m} = \mathrm{Spec}(k[z^{\pm 1}])$ denote the multiplicative group. Recall that $\mathbf{Aut}(\mathbf{G}_m) \simeq \mathbb{Z}/2\mathbb{Z}$ where the generator $\overline{1}$ of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{Spec}(k[z^{\pm 1}])$ via $z \mapsto z^{-1}$. We now proceed by exploiting the considerations of Remark 2.30. Since $H^1_{\text{ét}}(R, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, the bijectivity of ψ translates into the bijectivity of the analoguous map (also denoted by ψ) at the étale level, namely

$$H^1_{\text{ét}}(R, \mathbf{G}_m) \to H^1_{\text{ét}}(R, \mathbf{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\psi} H^1_{\text{ét}}(R, \mathbb{Z}/2\mathbb{Z})$$

Since $H^1_{\text{ét}}(R, \mathbf{G}_m) = \mathrm{Pic}\,(R) = 1$ our map ψ has trivial kernel. The fibre of ψ over the non–trivial class of $H^1(R, \mathbb{Z}/2\mathbb{Z})$ is measured by the cohomology $H^1_{\text{ét}}(R, \mathcal{R}^1_{S_2/R}(\mathbf{G}_m))$ where $\mathcal{R}^1_{S_2/R}(\mathbf{G}_m)$ is the twisted form of the multiplicative R-group $\mathbf{G}_m$ that fits into the exact sequence

$$1 \longrightarrow \mathcal{R}^1_{S_2/R}(\mathbf{G}_m) \longrightarrow \mathcal{R}_{S_2/R}(\mathbf{G}_m) \xrightarrow{\widehat{N}} \mathbf{G}_m \longrightarrow \mathbf{1},$$

where $\widehat{N}$ comes from the reduced norm N of the quadratic extension S_2/R, and $\mathcal{R}$ is the Weil restriction. The functor of points of the R-group $\mathcal{R}^1_{S_2/R}(\mathbf{G}_m)$ is thus given by

$$\mathcal{R}^1_{S_2/R}(\mathbf{G}_m)(R') = \{x \in (S_2 \otimes_R R')^\times : N(x) = 1\}.$$

for all $R' \in R{-}alg$. Passing to cohomology on this last exact sequence yields

$$\mathcal{R}^1_{S_2/R}(\mathbf{G}_m)(R) \xrightarrow{N} \mathbf{G}_m(R) \to H^1(R, \mathcal{R}^1_{S_2/R}(\mathbf{G}_m)) \to H^1(R, \mathcal{R}_{S_2/S}(\mathbf{G}_m)).$$

By Shapiro's Lemma,

$$H^1(R, \mathcal{R}_{S_2/S}(\mathbf{G}_m)) = \text{ Pic}\,(S) = 0.$$

On the other hand, the norm map

$$\{x \in S_2^\times : N(x) = 1\} \xrightarrow{N} R^\times$$

is surjective. Thus $H^1(R, \mathcal{R}^1_{S_2/R}(\mathbf{G}_m)) = 1$ as desired.[11]

The above cohomological reasoning shows that $L(\mathcal{A}, \mathrm{id})$ and $L(\mathcal{A}, \omega)$ are nonisomorphic as $\mathcal{R}$-conformal superalgebras, and they represent the $\mathcal{R}$-isomorphism classes of $\widehat{S}/\mathcal{R}$-forms of $\mathcal{A}$. To finish the proof, it suffices to note that the hypotheses of Corollary 2.36 are satisfied, so (in this case) $\mathcal{R}$-isomorphism is the same as k-isomorphism. This follows easily from the same argument used in the $N = 4$ case in the proof of Theorem 3.71 below. $\square$

3.4 Forms of the $N = 4$ conformal superalgebra

We now consider the classical $N = 4$ conformal superalgebra $\mathcal{A}$. We compute the automorphism group $\mathbf{Aut}(\mathcal{A})(\widehat{\mathcal{S}})$, from which the existence of infinitely many non-isomorphic twisted loop algebras will follow easily from the theory developed in §2.2.

We begin by recalling the definition of the $N = 4$ conformal superalgebra $\mathcal{A}$. Let $\mathcal{A} = \mathcal{A}_{\overline{0}} \oplus \mathcal{A}_{\overline{1}}$ be the k-vector space with

$$\mathcal{A}_{\bar{\imath}} = k[\partial]V_{\bar{\imath}} \cong V_{\bar{\imath}} \otimes_k k[\partial] \tag{3.38}$$

for $\bar{\imath} = \overline{0}, \overline{1}$, and

$$\begin{aligned} V_{\overline{0}} &= kL \oplus \bigoplus_{s=1}^{3} kJ^s \\ V_{\overline{1}} &= \bigoplus_{a=1}^{2} kG^a \oplus \bigoplus_{b=1}^{2} k\overline{G}^b. \end{aligned}$$

[11] One can also see that $H^1(R, \mathcal{R}^1_{S_2/R}(\mathbf{G}_m)) = 1$ directly by applying Remark 2.30 (2) to the reductive R-group $\mathcal{R}^1_{S_2/R}(\mathbf{G}_m)$.

Let $J^s = \frac{1}{2}\sigma^s$, where σ^s are the Pauli spin matrices

$$\begin{aligned}
\sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

The space $\mathcal{A}$ is a (k, δ)-conformal superalgebra, with multiplication given by

$$[L_\lambda L] = (\partial + 2\lambda)L \tag{3.39}$$
$$[L_\lambda J^s] = (\partial + \lambda)J^s \tag{3.40}$$
$$[J^m{}_\lambda J^n] = [J^m, J^n] := J^m J^n - J^n J^m \tag{3.41}$$
$$[L_\lambda G^a] = (\partial + \frac{3}{2}\lambda)G^a \tag{3.42}$$
$$\left[L_\lambda \overline{G}^a\right] = (\partial + \frac{3}{2}\lambda)\overline{G}^a \tag{3.43}$$
$$[J^s{}_\lambda G^a] = -\frac{1}{2}\sum_{b=1}^{3} \sigma^s_{ab} G^b \tag{3.44}$$
$$\left[J^s{}_\lambda \overline{G}^a\right] = \frac{1}{2}\sum_{b=1}^{3} \sigma^s_{ba} \overline{G}^b \tag{3.45}$$
$$\left[G^a{}_\lambda G^b\right] = \left[\overline{G}^a{}_\lambda \overline{G}^b\right] = 0 \tag{3.46}$$
$$\left[G^a{}_\lambda \overline{G}^b\right] = 2\delta_{ab}L - 2(\partial + 2\lambda)\sum_{s=1}^{3} \sigma^s_{ab} J^s \tag{3.47}$$

for all $m, n, s \in \{1, 2, 3\}$ and $a, b \in \{1, 2\}$, where δ_{ab} is the Kronecker delta and σ^s_{ab} is the (a, b)-entry of the matrix σ^s. The algebra $\mathcal{A}$ is the $N = 4$ conformal superalgebra described in [9].

To compute $\mathbf{Aut}(\mathcal{A})(\widehat{\mathcal{S}}) = \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_k \widehat{\mathcal{S}})$, it is enough (by Lemma 3.1) to compute the action of each automorphism in $\mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_k \widehat{\mathcal{S}})$ on the subspace $V \otimes_k 1 \subseteq \mathcal{A} \otimes_k \widehat{\mathcal{S}}$. Fix $\sigma \in \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes_k \widehat{\mathcal{S}})$, and choose $d > 0$ sufficiently large so that $\sigma(V \otimes 1) \subseteq \mathcal{A}_d := \mathcal{A} \otimes k[t^{1/d}, t^{-1/d}]$. (Such a d exists since V is finite-dimensional.) Let $z = t^{1/d}$ and $\widehat{\delta} = \frac{d}{dt} : S_d \to S_d$ with $S_d = k[t^{1/d}, t^{-1/d}] = k[z, z^{-1}]$.

As in the $N = 2$ case, we consider the projection

$$\pi : \ (\mathcal{A} \otimes \widehat{\mathcal{S}})_{\bar{0}} = \operatorname{Curr} \mathfrak{sl}_2(\widehat{\mathcal{S}}) \oplus \widehat{\mathcal{V}ir} \longrightarrow \widehat{\mathcal{V}ir}$$

and the composite map

$$\pi\sigma : \ (\mathcal{A} \otimes \widehat{\mathcal{S}})_{\bar{0}} \longrightarrow \widehat{\mathcal{V}ir}$$

restricted to $\operatorname{Curr} \mathfrak{sl}_2(k) = (k[\partial] \otimes \bigoplus_{s=1}^{3} kJ^s) \otimes 1$.

By Corollary 3.22, the map $\pi\sigma$ is zero, and thus σ restricts to an $\widehat{\mathcal{S}}$-conformal automorphism of the conformal subalgebra $\operatorname{Curr} \mathfrak{sl}_2(\widehat{\mathcal{S}}) = (k[\partial] \otimes \mathfrak{sl}_2(k) \otimes_k \widehat{\mathcal{S}} \subseteq \mathcal{A} \otimes \widehat{\mathcal{S}}$. By Corollary 3.17, σ is the $k[\partial]$-linear extension of an $\widehat{S}$-Lie algebra automorphism of $\mathfrak{sl}_2(\widehat{S})$, so by [13], there is some $Y \in \mathbf{GL}_2(\widehat{S})$ such that

$$\sigma(J^s \otimes 1) = Y J^s Y^{-1} \tag{3.48}$$

for $s = 1, 2, 3$. The units of $\widehat{S}$ are the monomials, so $\det Y = ct^q$ for some $q \in \mathbb{Q}$ and nonzero $c \in k$. Since k is assumed to be algebraically closed, c has a square root $\sqrt{c} \in k$. Let $\widehat{Y} = \frac{1}{\sqrt{c}} t^{-q/2} Y$. Then $\widehat{Y}$ has determinant 1, and conjugation by $\widehat{Y}$ has the same effect on J^s as conjugation by Y, so we can assume without loss of generality that $Y \in \mathbf{SL}_2(\widehat{S})$.

Next we consider the image of $L \otimes 1$. Write

$$\sigma(L \otimes 1) = \sum_{i \in \mathbb{Z}} P_i(\widehat{\partial})(L \otimes z^i) + w$$

for some polynomials $P_i(\widehat{\partial})$ in the polynomial ring $k[\widehat{\partial}]$ and $w \in \operatorname{Curr} \mathfrak{sl}_2(\widehat{S})$. Note that $\{i \in \mathbb{Z} \mid P_i \neq 0\}$ is nonempty (or else σ would not be surjective). Set $N = \max\{i \in \mathbb{Z} \mid P_i \neq 0\}$ and $M = \min\{i \in \mathbb{Z} \mid P_i \neq 0\}$. If $N > 0$, then comparing the coefficients of $L \otimes z^{2N}$ on both sides of

$$[\sigma(L \otimes 1)_\lambda \sigma(L \otimes 1)] = \sigma\,[L \otimes 1_\lambda L \otimes 1] \tag{3.49}$$

gives

$$P_N(-\lambda) P_N(\widehat{\partial} + \lambda)(\widehat{\partial} + 2\lambda)(L \otimes z^{2N}) = 0.$$

That is, $P_N = 0$, a contradiction. Hence $N \leq 0$.

If $N < 0$, then comparing the coefficients of $L \otimes z^{2M-d}$ in (3.49) gives

$$P_M(-\lambda) P_M(\widehat{\partial} + \lambda)(-M/d)(L \otimes z^{2M-d}) = 0,$$

keeping in mind that $\partial L \otimes z^{NM} = \widehat{\partial}(L \otimes z^M) - L \otimes \widehat{\delta}(z^M)$ with $\widehat{\delta} = \frac{d}{dt}$ and $z = t^{1/d}$. Hence $P_M = 0$, another contradiction. Thus $N = M = 0$.

Let $P(\partial) := P_0(\partial)$. Then comparing the coefficients of $L \otimes 1$ in (3.49) gives

$$P(-\lambda)P(\widehat{\partial}+\lambda)(\widehat{\partial}+2\lambda)(L \otimes 1) = P(\widehat{\partial})(\widehat{\partial}+2\lambda)(L \otimes 1),$$

so $P(\partial)$ is a constant (i.e. a member of k) and $P^2 = P$. Since $\{i \in \mathbb{Z} \mid P_i \neq 0\}$ is nonempty, $P \neq 0$, so $P = 1$. Hence

$$\sigma(L \otimes 1) = L \otimes 1 + w$$

for some $w \in \operatorname{Curr} \mathfrak{sl}_2(\widehat{S})$.

Write $w = \sum_{j=0}^{N'} \widehat{\partial}^{(j)}(w_j)$ for some $w_j \in \mathfrak{sl}_2(\widehat{S})$, with $w_{N'} \neq 0$. Suppose $N' > 0$. For $u \in \mathfrak{sl}_2(\widehat{S}) = \bigoplus_{s=1}^{3} kJ^s \otimes \widehat{S}$,

$$[u_\lambda L \otimes 1] = (1 \otimes \widehat{\delta})u + \lambda u,$$

so

$$\begin{aligned} \sigma\big((1 \otimes \widehat{\delta})u\big) + \lambda\sigma(u) &= [\sigma(u)_\lambda \sigma(L \otimes 1)] \\ &= (1 \otimes \widehat{\delta})\sigma(u) + \lambda\sigma(u) + \sum_{j=0}^{N'} (\widehat{\partial}+\lambda)^{(j)}[gs(u), w_j], \end{aligned}$$

using the fact that $\sigma(u) \in \mathfrak{sl}_2(\widehat{S})$ (Corollary 3.17). Comparing powers of $\widehat{\partial}$ gives $[\sigma(u), w_{N'}] = 0$. This holds for all $u \in \mathfrak{sl}_2(\widehat{S})$, so $w_{N'}$ is in the centre $Z(\mathfrak{sl}_2(\widehat{S})) = 0$ of the Lie algebra $\mathfrak{sl}_2(\widehat{S})$. Therefore $w_{N'} = 0$, a contradiction. Hence $w \in \mathfrak{sl}_2(\widehat{S})$.

Hence we have now proven the following lemma:

Lemma 3.50 *Let $\mathcal{A}$ be the $N = 4$ conformal superalgebra defined above. Then for any $\sigma \in \operatorname{Aut}_{\widehat{S}-conf}(\mathcal{A} \otimes \widehat{S})$, there is some $Y \in \mathbf{SL}_2(\widehat{S})$ and some $w \in \mathfrak{sl}_2(\widehat{S})$ so that*

$$\begin{aligned} \sigma(J^s \otimes 1) &= Y J^s Y^{-1} \\ \sigma(L \otimes 1) &= L \otimes 1 + w \end{aligned}$$

for all $s \in \{1, 2, 3\}$. □

Our next task is to find the value of w in Lemma 3.50. For all $u \in \mathfrak{sl}_2(\widehat{S})$, we have

$$(\widehat{\partial}+\lambda)\sigma^{-1}(u) - \frac{d}{dt}\sigma^{-1}(u) = \left[L \otimes 1_\lambda \sigma^{-1}(u)\right],$$

so

$$\begin{aligned}(\widehat{\partial}+\lambda)u-\sigma\big(\frac{d}{dt}\sigma^{-1}(u)\big) &= [\sigma(L\otimes 1)_\lambda u]\\ &= [L\otimes 1+w_\lambda u]\\ &= (\widehat{\partial}+\lambda)u-\frac{d}{dt}u+[w,u],\end{aligned}$$

and

$$\begin{aligned}[w,u] &= \frac{d}{dt}u-\sigma\big(\frac{d}{dt}\sigma^{-1}(u)\big)\\ &= u'-Y(Y^{-1}uY)'Y^{-1},\end{aligned}$$

where prime ($'$) denotes the derivative taken with respect to the variable t. But

$$Y'Y^{-1}+Y(Y^{-1})'=(YY^{-1})'=0, \tag{3.51}$$

so

$$\begin{aligned}[w,u] &= u'-Y(Y^{-1}uY)'Y^{-1}\\ &= -Y(Y^{-1})'u-uY'Y^{-1}\\ &= [Y'Y^{-1},u].\end{aligned}$$

Thus $w-Y'Y^{-1}$ is in the centralizer of $\mathfrak{sl}_2(\widehat{S})$ in $\mathfrak{gl}_2(\widehat{S})$. But writing

$$Y=\begin{pmatrix} e & f\\ g & h\end{pmatrix},$$

with $e,f,g,h\in\widehat{S}$, a quick computation shows that the trace $tr(Y'Y^{-1})=(eh-fg)'$. But $(eh-fg)'=0$ since $Y\in\mathbf{SL}_2(\widehat{S})$. Hence, $w-Y'Y^{-1}\in Z(\mathfrak{sl}_2(\widehat{S}))=0$, the centre of $\mathfrak{sl}_2(\widehat{S})$, so we have the following proposition.

Proposition 3.52 *Let $\mathcal{A}$ be the $N=4$ conformal superalgebra defined above. Then for any $\sigma\in\mathrm{Aut}_{\widehat{\mathcal{S}}-conf}(\mathcal{A}\otimes\widehat{\mathcal{S}})$, there is some $Y\in\mathbf{SL}_2(\widehat{S})$ so that*

$$\begin{aligned}\sigma(J^s\otimes 1) &= YJ^sY^{-1}\\ \sigma(L\otimes 1) &= L\otimes 1+Y'Y^{-1}\end{aligned}$$

for all $s\in\{1,2,3\}$. □

Next we consider the action of σ on the odd part of $\mathcal{A} \otimes \widehat{\mathcal{S}}$. Let $v \in V_{\bar{1}}$. Write

$$\sigma(v \otimes 1) = \sum_{i=0}^{M'} \sum_{s=1}^{4} \widehat{\partial}^{(i)}(v_s \otimes r_{si}),$$

where $v_1 = G^1$, $v_2 = G^2$, $v_3 = \overline{G}^1$, $v_4 = \overline{G}^2$, and $r_{sM'} \neq 0$ for some s. Then writing w for $Y'Y^{-1}$, we have

$$\begin{aligned}
(\widehat{\partial} + \frac{3}{2}\lambda)\sigma(v \otimes 1) &= \sigma\left[L \otimes 1_\lambda v \otimes 1\right] \\
&= \left[\sigma(L \otimes 1)_\lambda \sigma(v \otimes 1)\right] \\
&= \sum_{i=0}^{M'} \sum_{s=1}^{4} (\widehat{\partial} + \lambda)^{(i)} \Big((\widehat{\partial} + \frac{3}{2}\lambda)(v_s \otimes r_{si}) \\
&\qquad\qquad - v_s \otimes \widehat{\delta}(r_{si}) + [w_\lambda v_s \otimes r_{si}] \Big).
\end{aligned}$$

From the definition of the relevant products, (3.42), (3.43), and (1.13), it is clear that $[w_\lambda v_s \otimes r_{si}]$ contains no nonzero powers of λ. If $M' > 0$, then comparing the coefficients of $\lambda^{M'+1}$ gives

$$0 = \sum_{s=1}^{4} \frac{3}{2}(M'+1)\lambda^{(M'+1)} v_s \otimes r_{sM'}.$$

Thus $r_{sM'} = 0$ for all s, a contradiction. Hence $M' \leq 0$, and $\sigma(v \otimes 1) \in V \otimes S$. Therefore, by the $\widehat{S}$-linearity of σ and Proposition 3.52, we have proven the following lemma.

Lemma 3.53 *Let $\sigma \in \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes \widehat{\mathcal{S}})$. Then $\sigma(V \otimes \widehat{S}) \subseteq V \otimes \widehat{S}$.* □

For the computations that follow, it will be helpful to use the following notation:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} := G^1 \otimes a + G^2 \otimes b$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} := \overline{G}^1 \otimes a + \overline{G}^2 \otimes b$$

for all $a, b \in \widehat{S}$. In this notation, the relations (3.44), (3.45), and (3.47) become

$$\left[J^s{}_\lambda \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = -(J^s)^T \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.54}$$

$$\left[J^s{}_\lambda \begin{pmatrix} g \\ h \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = J^s \begin{pmatrix} g \\ h \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.55}$$

$$\left[\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} {}_\lambda \begin{pmatrix} g \\ h \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$
$$= 2 \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} L - \ 2(\partial + 2\lambda) \sum_{s=1}^{3} \begin{pmatrix} a & b \end{pmatrix} \sigma^s \begin{pmatrix} g \\ h \end{pmatrix} J^s. \tag{3.56}$$

for all $a, b, g, h \in k$ and $s \in \{1, 2, 3\}$, where $(J^s)^T$ is the transpose of the matrix J^s.

Using the fact that σ preserves the relation (3.54), we see that for any $a, b \in k$,

$$-(YJ^sY^{-1})^T \sigma_1 \left(\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + YJ^sY^{-1} \sigma_2 \left(\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$
$$= -\sigma_1 \left((J^s)^T \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) - \sigma_2 \left((J^s)^T \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right),$$

where $\sigma_i := \pi_i \sigma$ and π_1 (resp., π_2) is the projection of $\mathcal{A}_{\bar{1}} \otimes \widehat{S}$ onto

$$W_1 := \bigoplus_{j=1}^{2} kG^j \otimes \widehat{S} \quad \left(\text{resp., } W_2 := \bigoplus_{j=1}^{2} k\overline{G}^j \otimes \widehat{S} \right)$$

in the direct sum
$$\mathcal{A}_{\bar{1}} = W_1 \oplus W_2.$$

Then
$$-(YJ^sY^{-1})^T \sigma_1 = -\sigma_1 (J^s)^T \tag{3.57}$$
as $\widehat{S}$-linear maps on the vector space W_1. Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be in the kernel $\ker \sigma_1$ of the restriction $\sigma_1 : \ W_1 \to W_1$. Then by (3.57),

$$Mv := \left(M \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \ker \sigma_1$$

for all $M \in \mathfrak{sl}_2(\widehat{S})$. Thus $\ker \sigma_1 = 0$ or $\ker \sigma_1 = W_1$. If $\ker \sigma_1 = 0$, we see by (3.57) that conjugation by σ_1 has the same effect on $\mathfrak{sl}_2(k)$ as conjugation by $(Y^{-1})^T$. The centralizer of $\mathfrak{sl}_2(k)$ in $\mathbf{GL}_2(\widehat{S})$ consists of matrices of the form $cz^m I$, where $c \in k^\times$, $m \in \mathbb{Z}$, and I is the 2×2 identity matrix. Thus

$$\sigma_1 = cz^m (Y^{-1})^T : \ W_1 \to W_1 \tag{3.58}$$

for some $c \in k^{\times}$ and $m \in \mathbb{Z}$. If $\ker \sigma_1 = W_1$, then obviously (3.58) also holds, with $c = 0$.

By a similar argument,

$$\sigma_2 = dz^n Y \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : W_1 \to W_2$$

for some $d \in k$ and $n \in \mathbb{Z}$. Thus

$$\begin{aligned} &\sigma\left(\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ &= cz^m (Y^{-1})^T \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + dz^n Y \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{3.59}$$

Repeating this argument on W_2 using relation (3.55), we see that for some fixed $e, f \in k$ and $k, \ell \in \mathbb{Q}$,

$$\begin{aligned} &\sigma\left(\begin{pmatrix} g \\ h \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &\qquad = ez^k (Y^{-1})^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + fz^\ell Y \begin{pmatrix} g \\ h \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{3.60}$$

for all $g, h \in k$.

From (3.42), we see that

$$\left[L \otimes 1_\lambda \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = (\widehat{\partial} + \frac{3}{2}\lambda) \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.61}$$

for all $a, b \in k$. Apply σ_1 to both sides of (3.61) and compute using (3.59). Equating the terms on both sides of the equation which are constant with respect to λ and $\widehat{\partial}$ gives

$$(cz^m (Y^{-1})^T)' = -(Y' Y^{-1})^T cz^m (Y^{-1})^T,$$

where prime ($'$) denotes element-by-element differentiation with respect to $t = z^d$. Taking the transpose of both sides and simplifying using (3.51) gives $c = 0$ or $mz^{m-1} Y^{-1} = 0$. That is, $c = 0$ or $m = 0$. If $c = 0$, then we can obviously assume that $m = 0$. Similarly, $n = k = \ell = 0$.

By (3.56), the following equation holds for all $a, b, g, h \in k$:

$$\left[\sigma\left(\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)_\lambda \sigma\left(\begin{pmatrix} g \\ h \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right]$$
$$= 2 \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} \sigma(L \otimes 1) - 2(\widehat{\partial} + 2\lambda)\sigma\left(\sum_{s=1}^{3} \begin{pmatrix} a & b \end{pmatrix} \sigma^s \begin{pmatrix} g \\ h \end{pmatrix}\right) J^s. \tag{3.62}$$

Then comparing the coefficients of $L \otimes 1$ on both sides of (3.62) gives

$$2cf \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} - 2de \begin{pmatrix} g & h \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix},$$

so $cf - de = 1$. Thus we have the following proposition:

Proposition 3.63 *Let $\sigma \in \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes \widehat{\mathcal{S}})$. Then for some $Y \in \mathbf{SL}_2(\widehat{S})$ and $\begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathbf{SL}_2(k)$, σ satisfies the following formulas*

$$\sigma(L \otimes 1) = L \otimes 1 + Y'Y^{-1} \tag{3.64}$$
$$\sigma(J^s \otimes 1) = YJ^sY^{-1} \tag{3.65}$$
$$\sigma\left(\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = c(Y^{-1})^T \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + dY \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.66}$$
$$\sigma\left(\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = e(Y^{-1})^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + fY \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.67}$$

for all $a, b \in k$ and $s = 1, 2, 3$.

□

The converse to Proposition 3.63 is that given any $Y \in \mathbf{SL}_2(\widehat{S})$ and $\begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathbf{SL}_2(k)$, (3.64)–(3.67) defines an automorphism $\sigma \in \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes \widehat{\mathcal{S}})$. This follows (using Lemma 3.1) from the long and tedious verification that the $\widehat{S}$-linear map $\sigma : \mathcal{A} \otimes \widehat{\mathcal{S}} \to \mathcal{A} \otimes \widehat{\mathcal{S}}$ defined by (3.64)–(3.67) preserves the λ-bracket on the following relation:

$$\sigma\left[w_1 \otimes 1_\lambda w_2 \otimes 1\right] = \left[\sigma(w_1 \otimes 1)_\lambda \sigma(w_2 \otimes 1)\right] \tag{3.68}$$

for $w_1, w_2 \in \{L, J^s, G^i, \overline{G}^i \mid s = 1,2,3,\ i = 1,2\}$.

To determine the group structure on the set of automorphisms, fix $Y \in \mathbf{SL}_2(\widehat{S})$ and $X \in \mathbf{SL}_2(k)$. Let η_1 (resp., η_2) denote the automorphism determined by (Y, I) (resp., (I, X)), where I is the 2×2 identity matrix. Then it is straightforward to verify that

$$\eta_2 \eta_1 \eta_2^{-1} = \eta_1,$$

so there is a group epimorphism

$$\phi : \mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(k) \to \mathrm{Aut}_{\widehat{\mathcal{S}}}(\mathcal{A} \otimes \widehat{\mathcal{S}}).$$

Finally, suppose that the automorphism determined by the pair $(Y, X) \in \mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(k)$ is in the kernel $\ker \phi$. Writing $X = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$,

$$c(Y^{-1})^T \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for all $a, b \in k$ by (3.66). Thus $c(Y^{-1})^T = I$, so $Y = cI$ and $c = \pm 1$. By (3.66) and (3.67), we also see that $d = e = 0$ and $Y = fI$. Since $(-I, -I)$ determines the identity map on $\mathcal{A} \otimes \widehat{\mathcal{S}}$ by (3.64)–(3.67), the kernel of ϕ is the subgroup of $\mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(k)$ generated by $(-I, -I)$:

$$\ker(\phi) = \langle (-I, -I) \rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

We have now proven the following:

Proposition 3.69 *Let $\mathcal{A}$ be the $N = 4$ conformal superalgebra defined above, and let $\widehat{S} = k[t^q \mid q \in \mathbb{Q}]$. Then*

$$\mathrm{Aut}_{\widehat{\mathcal{S}}-conf}(\mathcal{A} \otimes_k \widehat{\mathcal{S}}) = \frac{\mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(k)}{\langle (-I, -I) \rangle}. \tag{3.70}$$

□

Applying our theory of forms (§2) now allows us to classify twisted loop algebras of the $N = 4$ conformal superalgebra $\mathcal{A}$.

Theorem 3.71 *Let $\mathcal{A}$ be the $N = 4$ conformal superalgebra defined above. Then there are canonical bijections between the following sets:*

(i) *$\mathcal{R}$-isomorphism classes of twisted loop algebras of $\mathcal{A}$,*

(ii) *k-isomorphism classes of twisted loop algebras of $\mathcal{A}$,*

(iii) *$\mathcal{R}$-isomorphism classes of $\widehat{\mathcal{S}}/\mathcal{R}$-forms of $\mathcal{A}\otimes_k\mathcal{R}$,*

(iv) *conjugacy classes of elements of finite order in $\mathbf{PGL}_2(k)$,*

Proof Let $\mathcal{A}_m$ be the $\mathcal{S}_m$-conformal superalgebra $\mathcal{A}\otimes_k\mathcal{S}_m$ where $\mathcal{S}_m=(S_m,\frac{d}{dt})$, $S_m=k[t^{\pm\frac{1}{m}}]$, and $m\geq 1$. Let $V=V_{\bar{0}}\oplus V_{\bar{1}}$, where $V_{\bar{0}}$ and $V_{\bar{1}}$ are defined as in (3.38). We divide our proof that (i) and (ii) are equivalent into several steps.

Step 1: $\mathcal{A}_m=\mathrm{Span}_k\{v_{(1)}\partial_{\mathcal{A}}^{(\ell)}L\otimes 1\mid v\in V\otimes S_m,\ \ell\geq 0\}$
Proof: We show that $\partial_{\mathcal{A}}^{\ell}V\otimes S_m\subseteq\sum_{j=0}^{\ell}V\otimes S_{m(1)}\partial_{\mathcal{A}}^{(j)}L\otimes 1$ using induction on ℓ. For $\ell=0$, we see that $V\otimes S_{m(1)}L\otimes 1=\big(V_{(1)}L\big)\otimes S_m=V\otimes S_m$, since $L,J^s,G^i,\overline{G}^i$ are *primary eigenvectors* of L with conformal weight greater than 1 for $s=1,2,3$ and $i=1,2$. That is,

$$\begin{aligned}a_{(0)}L&=(\Delta-1)\partial_{\mathcal{A}}a\\ a_{(1)}L&=\Delta a\\ a_{(m)}L&=0\end{aligned}$$

for all $m>1$, $a=L,J^s,G^i,\overline{G}^i$, and some $\Delta=\Delta(a)\geq 1$.

It is straightforward to verify that for $\ell\geq 1$ and $s\in S_m$,

$$a\otimes s_{(1)}\partial_{\mathcal{A}}^{(\ell+1)}L\otimes 1=\big((\ell+2)\Delta-(\ell+1)\big)\partial_{\mathcal{A}}^{(\ell+1)}a\otimes s+\Delta\partial_{\mathcal{A}}^{(\ell)}a\otimes\frac{ds}{dt}.$$

Since $\Delta\geq 1$, we see that $\Delta\partial_{\mathcal{A}}^{(\ell+1)}a\otimes s\in\sum_{j=0}^{\ell+1}V\otimes S_m,\ \ell\geq 0\}$.

Step 2: Let $\mathcal{B}=\mathcal{L}(\mathcal{A},\sigma)\subseteq\widehat{\mathcal{A}}$ for some finite order automorphism $\sigma:\ \mathcal{A}\to\mathcal{A}$. Then $\mathcal{B}=\mathrm{Span}_k\{a_{(1)}\partial_{\mathcal{A}}^{(\ell)}L\otimes 1\mid a\in\mathcal{B},\ \ell\geq 0\}$.
Proof: Let $\Gamma\subseteq\mathrm{Aut}_{\mathcal{S}_m}\mathcal{A}_m$ be the cyclic subgroup of order $m:=|\sigma|$ generated by $\sigma\otimes\psi$, where ψ is the $\mathcal{R}$-automorphism of $\mathcal{S}_m$ given by sending $t^{\frac{1}{m}}$ to $\xi_m^{-1}t^{\frac{1}{m}}$ and ξ_m is the primitive mth root of 1 fixed in §0. Let

$$\begin{array}{rccl}\pi: & \mathcal{A}_m & \to & \mathcal{A}_m\\ & a & \mapsto & \displaystyle\frac{1}{m}\sum_{i=0}^{m-1}(\sigma\otimes\psi)^i(a).\end{array}$$

Then $\mathcal{B}=\mathcal{A}_m^{\Gamma}$, the set of Γ-fixed points in $\mathcal{A}_m$, and π is a surjection from A_m to $\mathcal{B}$.

Since $\mathcal{A}_m = \mathrm{Span}_k\{v_{(1)}\partial_{\mathcal{A}}^{(\ell)}L \otimes 1 \mid v \in V \otimes S_m,\ \ell \geq 0\}$ and $\sigma \otimes \psi \in \mathrm{Aut}_{\mathcal{S}_m}(\mathcal{A}_m)$ by Lemma 2.11, we have

$$\begin{aligned}\mathcal{B} = \pi(\mathcal{A}_m) &= \mathrm{Span}_k\left\{\pi\left(v_{(1)}\partial_{\mathcal{A}}^{(\ell)}L \otimes 1\right) \mid v \in V \otimes S_m,\ \ell \geq 0\right\} \\ &= \mathrm{Span}_k\left\{\sum_{i=0}^{m-1} \pi(v)_{(1)}\partial_{\mathcal{A}}^{(\ell)}\sigma^i(L) \otimes 1 \mid v \in V \otimes S_m,\ \ell \geq 0\right\}.\end{aligned}$$

By 3.66, there is a $Y \in \mathbf{SL}_2(S_m)$ such that

$$\begin{aligned}\sigma(G^1) \otimes 1 &= (\sigma \otimes 1)(G^1 \otimes 1) \\ &= c(Y^{-1})^T \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + dY \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},\end{aligned}$$

so $Y \in \mathbf{SL}_2(k)$. Then $Y' = 0$, so $\sigma(L) \otimes 1 = (\sigma \otimes 1)(L \otimes 1) = L \otimes 1$ by (3.64). Hence $\sigma(L) = L$.

Therefore,

$$\begin{aligned}\mathcal{B} &= \mathrm{Span}_k\left\{\sum_{i=0}^{m-1} \pi(v)_{(1)}\partial_{\mathcal{A}}^{(\ell)}\sigma^i(L) \otimes 1 \mid v \in V \otimes S_m,\ \ell \geq 0\right\} \\ &= \mathrm{Span}_k\left\{\pi(v)_{(1)}\partial_{\mathcal{A}}^{(\ell)}L \otimes 1 \mid v \in V \otimes S_m,\ \ell \geq 0\right\} \\ &\subseteq \mathrm{Span}_k\{a_{(1)}\partial_{\mathcal{A}}^{(\ell)}L \otimes 1 \mid a \in \mathcal{B},\ \ell \geq 0\}.\end{aligned}$$

Step 3: Let $\chi \in \mathrm{Ctd}_k(\mathcal{B})$, where $\mathcal{B}$ is as above. Then $\chi(L \otimes 1) = L \otimes r$ for some $r \in R$.

Proof: By the argument in Step 2, every automorphism $\sigma \in \mathrm{Aut}_{k-\mathrm{conf}}\mathcal{A}$ fixes L, so $L \otimes 1 \in \mathcal{B}$. Then

$$L \otimes 1_{(1)}\chi(L \otimes 1) = \chi(L \otimes 1_{(1)}L \otimes 1) = 2\chi(L \otimes 1).$$

Taking an eigenspace decomposition of $\widehat{\mathcal{A}}$ with respect to the operator

$$L \otimes 1_{(1)} :\ a \otimes s \mapsto L \otimes 1_{(1)}a \otimes s = \left(L_{(1)}a\right) \otimes s,$$

we have

$$\widehat{\mathcal{A}} = \left(\bigoplus_{k=1}^{\infty} \widehat{\mathcal{A}}_k\right) \oplus \left(\bigoplus_{\ell=1}^{\infty} \widehat{\mathcal{A}}_{\frac{1}{2}+\ell}\right), \tag{3.72}$$

where

$$\begin{aligned}\widehat{\mathcal{A}}_k &= \operatorname{Span}\left\{\partial_{\mathcal{A}}^{(k-2)} L\otimes r, \partial_{\mathcal{A}}^{(k-1)} J\otimes r \;\middle|\; J\in\{J^1,J^2,J^3\} \text{ and } r\in\widehat{S}\right\}\\ \widehat{\mathcal{A}}_{\frac{1}{2}+\ell} &= \operatorname{Span}\left\{\partial_{\mathcal{A}}^{(\ell-1)} G\otimes r \;\middle|\; G\in\{G^1,G^2,\overline{G}^1,\overline{G}^2\} \text{ and } r\in\widehat{S}\right\}\end{aligned}$$

are the eigenspaces with eigenvalues k and $\frac{1}{2}+\ell$, respectively.

Thus $\chi(L\otimes 1)\in\widehat{\mathcal{A}}_2$, so $\chi(L\otimes 1) = L\otimes r + \sum_{i=1}^3 \partial_{\mathcal{A}} J^i\otimes r_i$ for some $r, r_i\in\widehat{S}$. But also

$$\begin{aligned}0 &= \chi(L\otimes 1_{(2)} L\otimes 1)\\ &= L\otimes 1_{(2)}\chi(L\otimes 1)\\ &= L\otimes 1_{(2)}\left(L\otimes r + \sum_{i=1}^3 \partial_{\mathcal{A}} J^i\otimes r_i\right)\\ &= 2\sum_{i=1}^3 J^i\otimes r_i.\end{aligned}$$

Hence $r_i = 0$ for $i=1,2,3$, so $\chi(L\otimes 1) = L\otimes r$. Since $\chi\in \mathrm{Ctd}_k(\mathcal{B})$ and $\sigma(L)=L$, we see that $r\in R$.

Step 4: Let χ and r be as in Step 3. Then $\chi\left(\partial_{\mathcal{A}}^{(k)} L\otimes 1\right) = \partial_{\mathcal{A}}^{(k)} L\otimes r$ for all $k\geq 0$.

Proof: If $b\in\mathcal{B}$ is a primary eigenvector[12] of itself with conformal weight Δ, then it is straightforward to verify that $b_{(k+1)}\partial_{\mathcal{B}}^{(k)} b = (\Delta+k-1)\partial_{\mathcal{B}}^{(k)} b$. By (CS4), we have

$$\begin{aligned}(\Delta+k-1)\chi\left(\partial_{\mathcal{B}}^{(k)} b\right) &= \chi\left(b_{(k+1)}\partial_{\mathcal{B}}^{(k)} b\right)\\ &= (-1)^k\chi\left(\partial_{\mathcal{B}}^{(k)} b_{(k+1)} b\right)\\ &= (-1)^k\partial_{\mathcal{B}}^{(k)} b_{(k+1)}\chi(b).\end{aligned}$$

If $\chi(b) = rb$, this gives

$$\begin{aligned}(\Delta+k-1)\chi\left(\partial_{\mathcal{B}}^{(k)} b\right) &= r_{\mathcal{B}}(-1)^k\partial_{\mathcal{B}}^{(k)} b_{(k+1)} b\\ &= r_{\mathcal{B}} b_{(k+1)}\partial_{\mathcal{B}}^{(k)} b\\ &= (\Delta+k-1) r\partial_{\mathcal{B}}^{(k)} b,\end{aligned}$$

[12] See definition in proof of Step 1.

so $\chi\left(\partial_{\mathcal{B}}^{(k)} b\right) = r_{\mathcal{B}} \circ \partial_{\mathcal{B}}^{(k)} b$ if $\Delta > 1$. Applying this result to $b = L \otimes 1$, we see that

$$\begin{aligned}
\chi\left(\partial_{\mathcal{A}}^{(k)} L \otimes 1\right) &= \chi\left(\partial_{\mathcal{B}}^{(k)} (L \otimes 1)\right) \\
&= r_{\mathcal{B}} \circ \partial_{\mathcal{B}}^{(k)} (L \otimes 1) \\
&= \partial_{\mathcal{A}}^{(k)} L \otimes r.
\end{aligned}$$

Step 5: In the notation of Step 3, $\chi = r_{\mathcal{B}}$. That is, $\mathrm{Ctd}_k(\mathcal{B}) \subseteq R_{\mathcal{B}}$.
Proof: For any $a \in \mathcal{B}$, Step 4 shows that

$$\begin{aligned}
\chi\left(a_{(1)} \partial_{\mathcal{A}}^{(\ell)} L \otimes 1\right) &= a_{(1)} \chi\left(\partial_{\mathcal{A}}^{(\ell)} L \otimes 1\right) \\
&= a_{(1)} \partial_{\mathcal{A}}^{(\ell)} L \otimes r \\
&= r_{\mathcal{B}}\left(a_{(1)} \partial_{\mathcal{A}}^{(\ell)} L \otimes 1\right).
\end{aligned}$$

Since $\mathcal{B}$ is spanned by elements of the form $a_{(1)} \partial_{\mathcal{A}}^{(\ell)} L \otimes 1$ (Step 2), we see that $\chi = r_{\mathcal{B}}$.

Step 6: Two twisted loop algebras of $\mathcal{A}$ are $\mathcal{R}$-isomorphic if and only if they are k-isomorphic. That is, (i) and (ii) are equivalent.
Proof: Clearly $R_{\mathcal{B}} \subseteq \mathrm{Ctd}_R(\mathcal{B}) \subseteq \mathrm{Ctd}_k(\mathcal{B})$. By Step 5, these inclusions are equalities. Thus (i) and (ii) are equivalent by Corollary 2.36.

From the eigenspace decomposition of $\widehat{\mathcal{A}}$, we see that the left multiplication operator $L \otimes 1_{(1)} : \ \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ is injective, and it follows easily that $Z(\mathcal{B}) = 0$ for any twisted loop algebra $\mathcal{B}$ of $\mathcal{A}$. Thus by Corollary 2.39, (ii) is equivalent to (iv).

It remains to check that (i), (iii), and (iv) are equivalent. By Theorem 2.16, Proposition 2.29, and Proposition 3.69, the $\mathcal{R}$-isomorphism classes of $\widehat{\mathcal{S}}/\mathcal{R}$-forms of the $\mathcal{R}$-conformal superalgebra $\mathcal{A} \otimes \mathcal{R}$ are parametrized by $H^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(k)\big)/\pm(I, I)\big)$.

To simplify the notation, we write $G = \mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(k)$, $\overline{G} = \big(\mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(k)\big)/\pm(I, I)$, and consider the exact sequence of (continuous) $\widehat{\mathbb{Z}}$-groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \to G \to \overline{G} \to 1.$$

where $\mathbb{Z}/2\mathbb{Z}$ is identified with the subgroup of G generated by $(-I, -I)$. This yields the exact sequence of pointed sets

$$H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\alpha} H^1_{ct}(\widehat{\mathbb{Z}}, G) \xrightarrow{\beta} H^1_{ct}(\widehat{\mathbb{Z}}, \overline{G}) \xrightarrow{\partial} H^2_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}). \tag{3.73}$$

whose individual terms can be understood as follows:

(a) $H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.
A representative cocycle of the nonzero element of this H^1 is the (unique and continuous) map $\widehat{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$ such that $1 \mapsto (-I, -I)$.

(b) $H^2_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) = 0$. This could be checked by brute force in terms of cocycles. An abstract argument is as follows. Since k is of characteristic 0 we have $\mathbb{Z}/2\mathbb{Z} \simeq \mu_2$. Now $H^2_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mu_2)$ is part (in fact all of) the 2-torsion of the (algebraic) Brauer group $H^2_{\text{ét}}(R, \mathbf{G}_m)$. Because R is of cohomological dimension 1, this Brauer group vanishes.

(c) $H^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{SL}_2(\widehat{S})\big) = 1$. Indeed this H^1_{ct} is the part of $H^1_{\text{ét}}(R, \mathbf{SL}_2)$ corresponding to the isomorphism classes of R–torsors under $\mathbf{SL}_2$ that become trivial over $\widehat{S}$.[13] As observed in Remark 2.30 $H^1_{\text{ét}}(R, \mathbf{SL}_2)$ vanishes.[14]

From (c), we immediately obtain

(d) *The canonical map* $H^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{SL}_2(k)\big) \to H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, G)$ *is bijective.*

Finally, since $\widehat{\mathbb{Z}}$ acts trivially on $\mathbf{SL}_2(k)$ we have

(e) $H^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{SL}_2(k)\big) \simeq \{$conjugacy classes of elements of finite order in $\mathbf{SL}_2(k)\}$.

[13] In fact all of $H^1_{\text{ét}}(R, \mathbf{SL}_2)$: Every R-torsor under $\mathbf{SL}_2$ is isotrivial. This follows from [16] in the case of R. More general isotriviality results can be found in [5].

[14] One can avoid the general considerations of [16] in the present case by the following direct argument. The exact sequence of R-groups $1 \to \mathbf{SL}_2 \to \mathbf{GL}_2 \xrightarrow{\det} \mathbf{G}_m \to 1$ yields

$$\mathbf{GL}_2(R) \xrightarrow{\det} R^\times \to H^1_{\text{ét}}(R, \mathbf{SL}_2) \to H^1_{\text{ét}}(R, \mathbf{GL}_2).$$

Since the map det is surjective, the map $H^1_{\text{ét}}(R, \mathbf{SL}_2) \to H^1_{\text{ét}}(R, \mathbf{GL}_2)$ has trivial kernel. On the other hand $H^1_{\text{ét}}(R, \mathbf{GL}_2) \simeq H^1_{\mathrm{Zar}}(R, \mathbf{GL}_2) = 1$ (the first equality by Grothendieck-Hilbert 90, and the last since all rank 2 projective modules over R are free; because R is a principal ideal domain).

By (b) and (e), we have a surjective map

$$\beta : \{ \text{ conjugacy classes of elements of finite order in } \mathbf{SL}_2(k)\} \longrightarrow H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \overline{G}).$$

Tracing through the various definitions, we see that the explicit nature of the map β is as follows: let $\theta \in \mathbf{SL}_2(k)$ be of finite order. Define a cocycle $u_\theta \in Z^1(\widehat{\mathbb{Z}}, \overline{G})$ by $u_\theta(n) = \overline{(1, \theta^n)}$ where $^- : G \to \overline{G}$ is the canonical map. Then β maps the conjugacy class of θ to the class of u_θ in $H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \overline{G})$.

It remains to show that u_θ and u_σ are cohomologous if and only if θ is conjugate to $\pm\sigma$. If $[u_\theta] = [u_\sigma]$ there exists $(x, \tau) \in G = \mathbf{SL}_2(\widehat{S}) \times \mathbf{SL}_2(k)$ such that

$$\overline{(x, \tau)}^{\,-1} u_\theta(n)\, {}^n\, \overline{(x, \tau)} = u_\sigma(n)$$

for all $n \in \mathbb{Z} \subseteq \widehat{\mathbb{Z}}$. In particular, for $n = 1$ we get

$$\overline{(x^{-1}\,{}^1x, \tau^{-1}\theta\tau)} = \overline{(1, \sigma)}$$

which forces either

$$\begin{array}{rlll} & x^{-1}\,{}^1x & = 1 & \text{and} \quad \tau^{-1}\theta\tau = \sigma \\ \text{or} & x^{-1}\,{}^1x & = -1 & \text{and} \quad \tau^{-1}\theta\tau = -\sigma. \end{array}$$

Thus θ is conjugate to either σ or $-\sigma$. The converse is obvious given that the element $\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{1/2} \end{pmatrix} \in \mathbf{SL}_2(\widehat{S}^\times)$ satisfies $x^{-1}\,{}^1x = -1$.

We have thus shown that $H^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \overline{G})$ is in bijective correspondence with the conjugacy classes of elements of finite order in $\mathbf{PGL}_2(k)$. This completes the proof of the theorem. □

The grading operator L is stable under all automorphisms of the $N = 2$ and $N = 4$ conformal superalgebras, so $L \otimes 1$ is an element of every twisted loop algebra of these conformal superalgebras. By considering the n-products of elements with $L \otimes 1$, it is easy to verify that every twisted loop algebra of the $N = 2$ and $N = 4$ conformal superalgebras is centreless. They each admit a (unique) one-dimensional universal central extension, as was previously shown by one of the authors [11]. Using Corollary 2.39, we see that Theorems 3.35 and 3.71 actually give a parametrization of the k-isomorphism classes of universal central extensions of twisted loop algebras of the $N = 2$ and $N = 4$ conformal superalgebras. Summarizing:

Corollary 3.74 *There are exactly two $\mathbb{C}$-isomorphism classes of twisted loop algebras based on the $N = 2$ conformal superalgebra, and infinitely many $\mathbb{C}$-isomorphism classes of twisted loop algebras based on the $N = 4$ conformal superalgebra. The explicit automorphisms giving the distinct isomorphism classes are the identity map and the automorphism ω in the $N = 2$ case; and representatives of the conjugacy classes of elements of finite order in $\mathbf{PGL}_2(\mathbb{C})$ in the $N = 4$ case.*

Furthermore, two of these twisted loop algebras are isomorphic if and only if their universal central extensions are isomorphic. □

Remark 3.75 As explained in the introduction, the superconformal algebras in Schwimmer and Seiberg's original work are obtained as formal distribution algebras of the twisted loop algebras of Lie conformal superalgebras. Since isomorphic twisted loop algebras lead to isomorphic superconformal algebras, our work shows that in the $N = 2, 4$ case there can be no more superconformal algebras than those listed by Schwimmer and Seiberg.

Remark 3.76 Our methods work equally well for the other N-conformal superalgebras: The isomorphism classes of loop algebras based on an N-conformal superalgebra $\mathcal{A}$ are parametrized by

$$H^1\big(\widehat{\mathcal{S}}/\mathcal{R},\, \mathbf{Aut}(\mathcal{A})\big) \simeq H^1_{\mathrm{ct}}\big(\pi_1(R),\, \mathbf{Aut}\,(\mathcal{A})(\widehat{\mathcal{S}})\big) \simeq H^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}},\, \mathbf{Aut}\,(\mathcal{A})(\widehat{\mathcal{S}})\big)$$

For $N = 0$, the conformal superalgebra is $\mathcal{V}ir$, and the group $\mathbf{Aut}\,(\mathcal{A})(\widehat{\mathcal{S}})$ is trivial. Thus the only twisted loop algebra is the affinization of $\mathcal{A}$. For $N = 1$, we have $\mathbf{Aut}\,(\mathcal{A})(\widehat{\mathcal{S}}) \simeq \mathbb{Z}/2\mathbb{Z}$. There are thus two non-isomorphic twisted loop algebras (Ramond and Neveu-Schwarz). For $N = 3$, the group $\mathbf{Aut}\,(\mathcal{A})(\widehat{\mathcal{S}})$ would appear to be $\mathbf{O}_3(\widehat{\mathcal{S}})$.[15] Under this assumption, the cohomology will yield two non-isomorphic twisted loop algebras–again Ramond and Neveu-Schwarz.

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