Laplace Transforms of Busy Periods in Queues

by

Bingsen Yan

A Major Paper

Submitted to the Faculty of Graduate Studies

through the Department of Mathematics and Statistics

in Partial Fulfillment of the Requirements for

the Degree of Master of Science at the

University of Windsor

Windsor, Ontario, Canada

2013

Laplace Transforms of Busy Periods in Queues

by

Bingsen Yan

APPROVED BY:

Dr. A.Hussein
Department of Mathematics and Statistics

Dr.Myron Hlynka, Supervisor
Department of Mathematics and Statistics

Sep 12, 2013

Author's Declaration of Originality

I hereby certify that I am the sole author of this major paper and that no part of this major paper has been published or submitted for publication.

I certify that to the best of my knowledge, my major paper does not infringe upon anyone's copyright nor violate any proprietary rights and that any ideas, techniques, quotations, or any other material from the work of other people included in my major paper, published or otherwise, are fully acknowledged in accordance with the standard referencing practices. Furthermore, to the extent that I have included copyrighted materials that surpasses the bounds of fair dealing within the meaning of the Canada Copyright Act, I certify that I have obtained a written permission of the copyright owner(s) to include such materials in my major paper and have included copies of such copyright clearances to my appendix.

I declare that this is a true copy of my major paper, including any final revisions, as approved by committee and the Graduate Studies Office, and that this major paper has not been submitted for a higher degree to any other University or Institution.

Abstract

In this paper, we apply the probabilistic interpretation of Laplace transform approach to queueing models and busy periods. We first introduce background knowledge about Laplace transforms, catastrophe Processes and busy periods. We also give some properties which will be used in this paper. Then we present 10 models and the Laplace transforms of their busy periods, including $M/M/C$, $M/M/1/C$, $M/M/C/C$, $M/E_2/1$ and $M/E_2/2$ (together with equations and graphs). In the introduction, the proof of the busy period of the $M/M/1$ model is from K.Roy,M.Hlynka,and R.Caron (1999), the other proofs are new. Finally, we present the inverse Laplace transform of the busy period, and give the graph of the busy period pdf for each model.

Acknowledgements

I would like to express my gratitude to my supervisor Dr. Myron Hlynka for giving me the opportunity to work on this great project, for his sincerity, encouragement and guidance which enabled me to develop an understanding of the subject. He has always provided the balance of assistance, pressure and encouragement – helping me when I needed it and leaving me on my own when I needed to figure things out by myself. Moreover, in completing this major paper, he spent lots of time helping me to correct it.

I want to thank my parents for their love, support and teaching me to think outside the box. I would not be in Canada, if it were not for them.

Last but not least, I offer my regards and blessings to all of those who supported me in any respect during the completion of the major paper.

TABLE OF CONTENTS

LIST OF FIGURES

Page

Chapter 1

Introduction

There is much work done on finding Laplace transforms of various measures (such as waiting times, busy periods, etc) using a probabilistic interpretation.

Van Dantzig (1949) introduced catastrophes and used probabilities as a method for finding Laplace transforms. Runnenberg (1965) revived and popularized the method, and gave numerous applications. Rade (1972) continued the use of the method with applications in queueing. Kleinrock's (1975) classic book discussed the method and extended its popularity. Roy (1997) used the method to give a probabilistic interpretation of the expression for the Laplace transforms of the busy period of an M/G/1 queue. Horn (1999) used the probabilistic interpretation to find distributions of order statistics of Erlang random variables. Hlynka, Brill and Horn (2010) extended these results. Jahan (2008) used the interpretation to study a queueing control model. Hussain (2011) used the probabilistic interpretation to evaluate the Laplace Transform of the time to ruin for loss reserves.

In Chapter 2, we recall the definition of a Laplace transform for probability density functions of continuous random variables with nonnegative support.

In this paper, we consider the problem of finding Laplace transforms for the busy periods of $M/M/1$, $M/M/2$, $M/M/3$, $M/M/c$, $M/M/1/1$, $M/M/2/2$, $M/M/3/3$, $M/M/c/c$, $M/M/1/2$, $M/M/1/3$, $M/M/1/c$, $M/E_2/1$ and $M/E_2/2$ models. These will all be done using the probabilistic interpretation. Other than the $M/M/1$ case, these are all new results. If we know the Laplace transform, then we can easily find all of the moments (such as the expected length of a busy period). This can be done by differentiation or by examining the graph of the Laplace transform near zero. In some of the models, we numerically invert the Laplace transforms to find the probability density functions of the length the busy period.

Chapter 2

Laplace Transform

This is taken from Hlynka (2001) "The purpose of transforms is to move a problem from its original setting where it is difficult to a new setting where the problem is easy. Once solved in the new setting, the solution must be moved back to the original setting. For example, consider likelihood functions in statistics. Maximization of a product may be difficult. By taking logs, we move the problem to a setting where we need only maximize a sum, which can be much easier. The maximum of the two functions occurs at the same place, so the solution is automatically moved back to the original setting. Besides logarithms, other transforms include moment generating functions, probability generating functions, characteristic functions, Fourier transforms and Laplace transforms."

The following description is taken from Roy (1997)

"The Laplace transform is an often used integral transform that is employed in many diverse fields of mathematics. It is particularly well known for its use in solving linear differential equations with constants coefficients. The study of stochastic processes also utilizes Laplace transform in areas such as risk theory, renewal theory and queueing theory. In fact, many well-known results for M/G/1 queues are stated in terms of Laplace transforms. We will restrict out study of Laplace transform to queueing applications."

The following description is taken from Rifat Ara Jahan (2008)

"We are interested in transforms of probability density functions (p.d.f.'s) of waiting times in queues. In this case, there is a probabilistic interpretation of the Laplace transform. The Laplace transform of a p.d.f. is the probability that the corresponding random variable is smaller than an exponential random variable with a particular rate where the random variables are independent. This interpretation can be employed to compute transforms of certain p.d.f.'s and prove relationships between quantities of interest in queueing theory without the standard computational and integration techniques."

"The probabilistic interpretation of the Laplace transform was first introduced in 1949 by van Dantzig whose original purpose was to give an interpretation of the z-transform (probability generating function). van Dantzig's (1949) interpretation (which he called "the theory of collective marks") and its associated techniques were described by Runnenburg (1965). In these papers, applications to queueing theory were emphasized. Rade (1972) also utilized these interpretations to solve problems in applied probability from a practical point of view, that would be understandable by both the technician and the theoretician."

The following description is taken from Roy (1997).

"Cong published several articles (Cong 1994a, Cong 1994b, Cong 1995) on queueing theory and collective marks. In these papers, Cong derives results for queueing systems with complicated restrictions. Cong's results are more general and have shorter, more efficient proofs than previous results regarding the same queueing models.

It is worth noting that van Dantzig (1949), Runnenburg (1965), Rade (1972) and Cong are all associated with the University of Amsterdam. While the probabilistic interpretation of Laplace transforms is known outside of the Netherlands, it does not seem to be well known and is definitely under-utilized as a tool in the analysis of queues. For instance, Lipsky (1992) mentions the interpretation of Laplace transforms as does Haight (1981), but they do not use these insights to prove any results. Kleinrock (1975) also notes the interpretation and derives some renewal theory results, but fails to utilize it in situations where the proofs could be made more efficient and intuitive. Most standard queueing texts ignore this subject completely."

Most of the following are taken from the class notes of Hlynka (2012).

Definition 2.0.1. If $F(x)$ is the cdf of a continuous random variable X with support on the positive real numbers, then define the Laplace Stieljes transform as

$$L(s) = f^*(s) = \int_0^\infty e^{-sx} dF(x).$$

If $f(x)$ $(x > 0)$ exists everywhere, define the Laplace transform of $f(x)$ to be

$$L(s) = f^*(s) = \int_0^\infty e^{-sx} f(x)dx,$$

where $s > 0$.

Property 2.0.2. *Let $m_i = E(X^i)$ denote the i^{th} moment of X where the p.d.f. of X is $f(x)$, $x > 0$. Then*

$$m_i = (-1)^i \frac{d^i}{ds^i} f^*(s)|_{s=0}$$

This result is related to fact that the Laplace transform of a p.d.f. $f(x)$ is almost the moment generating function of X, namely

$$M(t) = \int_0^\infty e^{xt} f(x)dx,$$

evaluated at t=-s.

Property 2.0.3. *Let X be a random variable with positive support and with first two moments $E(X)$ and $E(X^2)$. Then*

$$E(X) = -L'(s)|_{s=0}$$

$$E(X^2) = L''(s)|_{s=0}$$

Property 2.0.4. *The exponential distribution is "memoryless", that is, if Y os an* *exponential random variable,then*

$$P(Y > t + s|Y > s) = P(Y > t).$$

Proof:

Suppose $Y \sim ex(\lambda)$.Then

$$\begin{aligned} P(Y > t + s|Y > s) &= \frac{P(Y > t + s)}{P(Y > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= P(Y > t) \end{aligned}$$

This memoryless property is not confined to specific values of s and t as above, but can be extended to random variables.

Property 2.0.5. *If Y is an exponential random variable, X_1 and X_2 are random variables with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$, respectively, where Y, X_1 and X_2 are all mutually independent, then*

$$P(Y > X_1 + X_2|Y > X_1) = P(Y > X_2).$$

Proof:

Suppose $Y \sim ex(\lambda)$. Then

$$
\begin{aligned}
&P(Y > X_1 + X_2 | Y > X_1) \\
&\qquad = \int_0^\infty \int_0^\infty P(Y > x_1 + x_2 | Y > x_1) f_1(x_1) f_2(x_2) dx_1 dx_2 \\
&\qquad = \int_0^\infty \int_0^\infty P(Y > x_2) f_1(x_1) f_2(x_2) dx_1 dx_2 \\
&\qquad = \int_0^\infty P(Y > x_2) f_2(x_2) dx_2 \\
&\qquad = P(Y > X_2)
\end{aligned}
$$

It can be shown that the only continuous density with this property is the exponential distribution.

Property 2.0.6. *If $X_1 \sim ex(\lambda_1)$, $X_2 \sim ex(\lambda_2)$ are independent, then*

$$P(X_1 < X_2) = P(X_1 \text{ occurs first}) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Proof:

$$
\begin{aligned}
P(X_1 < X_2) &= \int_0^\infty (\int_0^\infty f(x_1) dx_1) f(x_2) dx_2 \\
&= \int_0^\infty F_{X_1}(x_2) f(x_2) dx_2 \\
&= \int_0^\infty (1 - e^{-\lambda_1 x_2}) \lambda_2 e^{-\lambda_2 x_2} dx_2
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2} dx_2 - \frac{\lambda_1}{\lambda_1+\lambda_2} \int_0^\infty (\lambda_1+\lambda_2) e^{(-\lambda_1+\lambda_2)x_2} dx_2 \\
&= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}
$$

Property 2.0.7. *Let*$X_1, X_2, ..., X_n$ *be independent exponential random variables with rates*$\lambda_1, \lambda_2, ..., \lambda_n$. *Then*

$$
P(X_j \text{ occurs first}) = \frac{\lambda_j}{\lambda_1+\lambda_2+...+\lambda_n}, \text{ for all } j = 1, 2, ..., n.
$$

Chapter 3

Catastrophe Process

The following property is taken from Hlynka (2012).

Theorem 3.0.8. *Let X and Y be independent random variables with positive support. Further, suppose that $Y \sim ex(s)$ and the p.d.f. of X is $f(x)$. Then,*

$$f^*(s) = L_X(s) = P(X < Y).$$

Proof:

$$\begin{aligned} P(X < Y) &= \int_0^\infty \int_x^\infty se^{-sy} f(x) dy dx \\ &= \int_0^\infty f(x) e^{-sx} dx \\ &= f^*(s) \end{aligned}$$

The following description is taken from Jahan (2008).

"The exponential random variable Y is called the catastrophe.The Laplace transform of a random variable X is the probability that X occurs before the catastrophe. More precisely, the Laplace transform of a probability density function $f(x)$ of a random variable X can be interpreted as the probability that X precedes a catastrophe where the time to the catastrophe is an exponentially distributed random variable Y with rate s, independent of X.

Another way of describing the process is in terms of a race. The Laplace transform of a random variable X is the probability that X wins a race against an exponential opponent Y."

Property 3.0.9. *Let X be an exponential random variable with rate λ. Then,*

$$L_X(s) = \frac{\lambda}{\lambda + s}.$$

Proof:

Let Y be exponential with rate s and independent of X. Then, we can get

$$L_X(s) = P(X < Y) = \frac{\lambda}{\lambda + s}.$$

"In this section, we interpret the Laplace transform of probability density functions as the probability that the corresponding random variable "wins a race" against an exponentially distributed catastrophe. We also use this interpretation to give intuitive explanations of some of the properties of the Laplace transform.

Our interest is in a process which generates events where the time until the next event has p.d.f. $f(x)$. To calculate the Laplace transform of $f(x)$, consider a process that generates "catastrophes" (a catastrophe is simply another type of event). If the time between catastrophes is distributed as an independent exponential random variable with rate s then Theorem 1 states that the Laplace transform of the distribution of the time until the next event, $f^*(s)$, is simply the long-term proportion of time that the event occurs before the catastrophe, for a large number of situations with the same initial position."

Theorem 3.0.10. *Let $X_1, X_2, ..., X_n$ be a sequence of n independent random variables where each X_i has p.d.f., $f^*(x_i)$. If $X = \sum_{i=1}^n X_i$ and the p.d.f. of X is $f(x)$, then*

$$f^*(s) = \prod_{i=1}^{n} f_i^*(s).$$

"Since the catastrophe process is memoryless, if we are given that k events have occurred before the catastrophe, we simply reset the "race" to be between the length of time for the remaining $n - k$ events to occur and the catastrophe.

Our probabilistic interpretation also allows us to numerically compute a Laplace transform using simulation (or real data), provided that we are able to simulate (or obtain) random values from the density function in question. To compare $f^*(s)$ for particular values for s, we can simulate a series of exponential values $y_1, y_2, ..., y_n$, at rate s and a series of values from the density in question, $f(x)$, $x_1, x_2, ..., x_n$ and count the proportion of pairs (x_i, y_i) such that $x_i < y_i$. As $n \to \infty$, this is exactly the value of the Laplace transform at point s, namely $f^*(s)$."

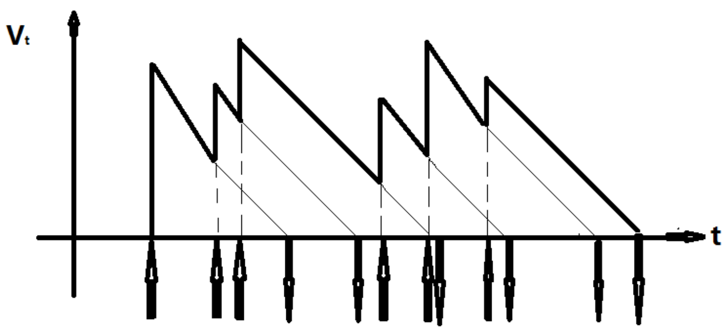
Chapter 4

Busy Period

The following is taken from Robert B. Cooper(1981).

"A notion that will prove useful and instructive is that of the busy period, defined as the length of time from the instant a customer enters a previously empty system until the next instant at which the system is completely empty.

Figure 4.1



"A typical realization of a busy period is illustrated in Figure 4.1. The height of the graph at any time t is the duration of time from t until all those customers present in the system at time t are finished with service. Arrivals are represented by arrows pointing up, and departures by arrows pointing down. At each arrival epoch the graph jumps up by an amount equal to the service time of the customers who just arrived. The graph decreases linearly with time until either another customer arrives or the graph decreases to zero, signifying the end of the busy period. Observe that the height of the graph at any time t equals the virtual waiting time V_t, if service is in order of arrival. The realization pictured in Figure 4.1 is a busy period composed of six service times.

"We shall now derive an expression for the distribution function $B(t)$ of the busy period for the $M/G/1$ queue. Consider the system in which the server does not start to serve any customers until there are $j \geq 1$ customers in the system, and then serves these j customers one at a time, and all subsequent arrivals, until for the first time the system becomes empty. Call this length of time a $j - busy$ period. Hence a $j - busy$ period is the length of continuous busy time of a server that starts serving when j customers are in the system. (For the $1 - busy$ period, we henceforth just use the term "busy period.")

"Let $B(t)$ be the distribution function of the busy period, and $B_j(t)$ be the distribution function of the $j - busy$ period, where $B_1(t) = B(t)$. Following Takacs(1962a), we now show that the distribution function $B_j(t)$ of the $j - busy$ period is the $j - fold$ convolution with itself of the distribution function $B(t)$ of the ordinary busy period,

$$B_j(t) = B^{*j}(t)(j = 0, 1, 2, \ldots; B^{*0}(t) = 1) \tag{4.0.1}$$

.

(where the case $j = 0$ is included to simplify forthcoming calculations).

"A little thought should convince the reader that the length of a $j - busy$ period is independent of the order in which the waiting customers are served. This being so, we choose to consider the following particular order of service. Suppose the server is about to begin service on one of $j \geq 1$ waiting customers. When it finishes serving the first of these j customers, it then serves all those customers, if any, who arrived during the service time of the first customer, and then those customers, if any, who arrived during these service periods, and so on. That is, it serves first all those customers who arrived during the busy period generated by the first of the original j customers. When the busy period generated by the first of the original j customers terminates, the server starts on the next of the original j customers, and serves all those customers who arrive during the busy period generated by him. The server continues in this manner, so that the length of time during which it is continuously busy is the sum of j busy periods. These j busy periods are clearly mutually independent and identically distributed. Hence Equation is true.

Property 4.0.11. *Consider an $M/M/1$ system with arrival rate λ and service rate μ. If the system is nonempty, then the probability that the next event is an arrival is*

$u = \frac{\lambda}{\lambda+\mu}$ *and the probability that the next event is a service completion is* $d = \frac{\mu}{\lambda+\mu}$.

Property 4.0.12. *Consider an* $M/M/1$ *system with arrival rate* λ *and service rate* μ. *Let* $u = \frac{\lambda}{\lambda+\mu}$ *and* $d = \frac{\mu}{\lambda+\mu}$. *If* $\lambda \leq \mu$ *(* $\mu \leq d$*), then the probability that the busy period eventually ends is 1.*

Proof:

Let p be the probability that the busy period ends. We begin with one customer. We want the probability p that the system eventually drops by 1 from the current level. The next event can be an arrival with probability u or a service completion with probability d. If the next event is an arrival then we must ?rst drop from 2 customers to one customer (probability p) and then from one customer to zero customers (probability p).
Thus $p = d+up^2$. So $up^2-p+(1-u) = 0$. This is implies that $(p-1)[up-(1-u)] = 0$, where $p = 1$ or $p = \frac{d}{u} = \frac{\mu}{\lambda}$."

Chapter 5

Models

5.1 M/M/C model

$M/M/C$ represents a system where arrivals form a single queue and are governed by a Poisson process, there are c servers and job service times are exponentially distributed. This section presents the models.

5.1.1 M/M/1 model

The following is from K.Roy, M.Hlynka,and R.Caron(July 1999).

"We begin with a probabilistic interpretation of the Laplace transform of the busy period of an $M/M/1$ system. The result is well known. The method of proof we use may be known also, but we have not seen it in the queueing literature. The memoryless property of the exponential distribution is used in the proof.

Let $L_1(s)$ be the Laplace transform for the busy period of an M/M/1 queueing system.

Let λ and μ be the arrival and service rates. Then,

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s}(L_1(s))^2$$

Proof:

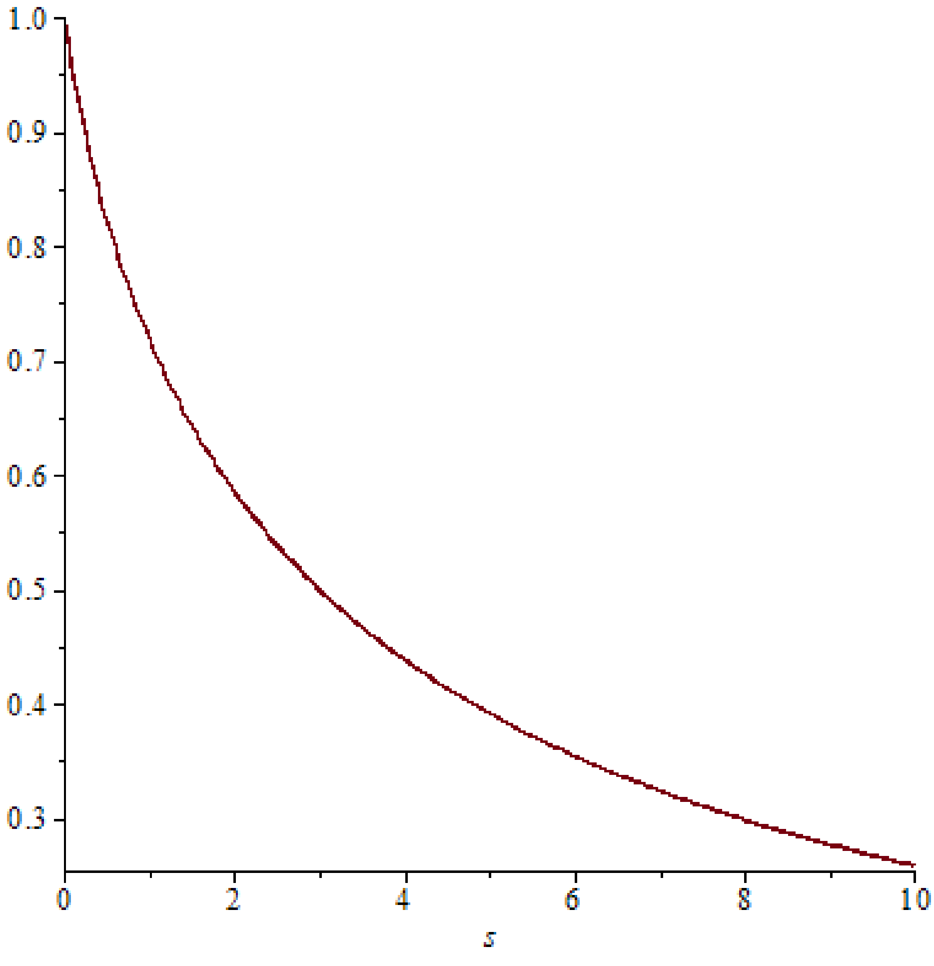
We begin with a single customer in service.There are three independent Poisson processes in effect-the arrival process at rate λ , the service process at rate μ and the catastrophe process at rate s. The Laplace transform of the busy period ends before the catastrophe. If the service completion is the first of the three types of event(with probability $\frac{\mu}{\lambda+\mu+s}$), then the busy period will end before the catastrophe. The busy period also may end before the catastrophe if the first of the three events is an arrival (with probability $\frac{\mu}{\lambda+\mu+s}$). This arrival would leave two customers in the system. We then would need both customers' busy periods and before the catastrophe. The probability that both customers' busy periods end before the catastrophe is $(L_1(s))^2$ and is the Laplace transform of the sum of two consecutive busy periods(convolution). The result follows."

After the calculation, we can get:

$$L_1(s) = \frac{\lambda + \mu + s - \sqrt{(\lambda + \mu + s)^2 - 4\lambda\mu}}{2\lambda}$$

We get the following graph of the busy period LT for $\lambda = 2, \mu = 4$:

Figure 5.1: LT of busy period of $M/M/1$ system



5.1.2 M/M/2 model

Let $L_1(s)$ be the Laplace transform for the busy period of an $M/M/2$ queueing system. Let $L_2(s)$ be the probability that a busy period of an $M/M/2$ system, which begins with two customers, will end before a catastrophe. Let $M_{2,1}(s)$ be the probability that the $M/M/2$ system drops from 2 customers to 1 customer before a catastrophe. Let $M_{3,2}(s)$ be the probability that the $M/M/3$ system drops from 3 customers to 2 customer before a catastrophe. Let λ and μ be the arrival and service rates. Then

$$L_1(s) = \frac{\mu}{\lambda+\mu+s} + \frac{\lambda}{\lambda+\mu+s} L_2(s) \tag{5.1.1}$$

$$L_2(s) = M_{2,1}(s) L_1(s) \tag{5.1.2}$$

$$M_{2,1}(s) = \frac{2\mu}{\lambda+2\mu+s} + \frac{\lambda}{\lambda+2\mu+s} M_{3,2}(s) M_{2,1}(s) \tag{5.1.3}$$

$$M_{3,2}(s) = M_{2,1}(s) \tag{5.1.4}$$

So we can get:

$$L_1(s) = \frac{\mu}{\lambda+\mu+s} + \frac{\lambda}{\lambda+\mu+s} M_{21}(s) L_1(s) \tag{5.1.5}$$

$$M_{2,1}(s) = \frac{2\mu}{\lambda+2\mu+s} + \frac{\lambda}{\lambda+2\mu+s} M_{2,1}(s)^2 \tag{5.1.6}$$

We explain the equation (5.1.1) as follows.

$L_1(s)$ is the Laplace transform of the busy period of an $M/M/2$ queueing system which begins with 1 customer. It is also the probability that the busy period finishes before any catastrophes (which occurs at rate s) occurs with 1 customers in the system, the next event will either be a service completion, an arrival, or a catastrophe. we are only in interested in the first two types of event. The first event will be a service completion with probability $\frac{\mu}{\lambda+\mu+s}$. In that case, the busy period ends before a catastrophe.The next event will be an arrival with probability $\frac{\lambda}{\lambda+\mu+s}$. In that case, we will have a system with two customers.

If the $M/M/2$ system begins will 2 customers, then the probability of dropping to 1 customer before a catastrophe can happen either by dropping to 1 customer first with the probability $\frac{2\mu}{\lambda+2\mu+s}$ or by the arrival of a customer (giving 3 customers) with the probability of $\frac{\lambda}{\lambda+2\mu+s}$ and then eventually moving from 3 to 2. This is proof the equation (5.1.3).

But the probability of moving from 2 to 1 before a catastrophe is the same as the probability of moving from 3 to 2 before a catastrophe. Hence, (5.1.4) holds.

Finally, we can get the equation (5.1.5),(5.1.6).

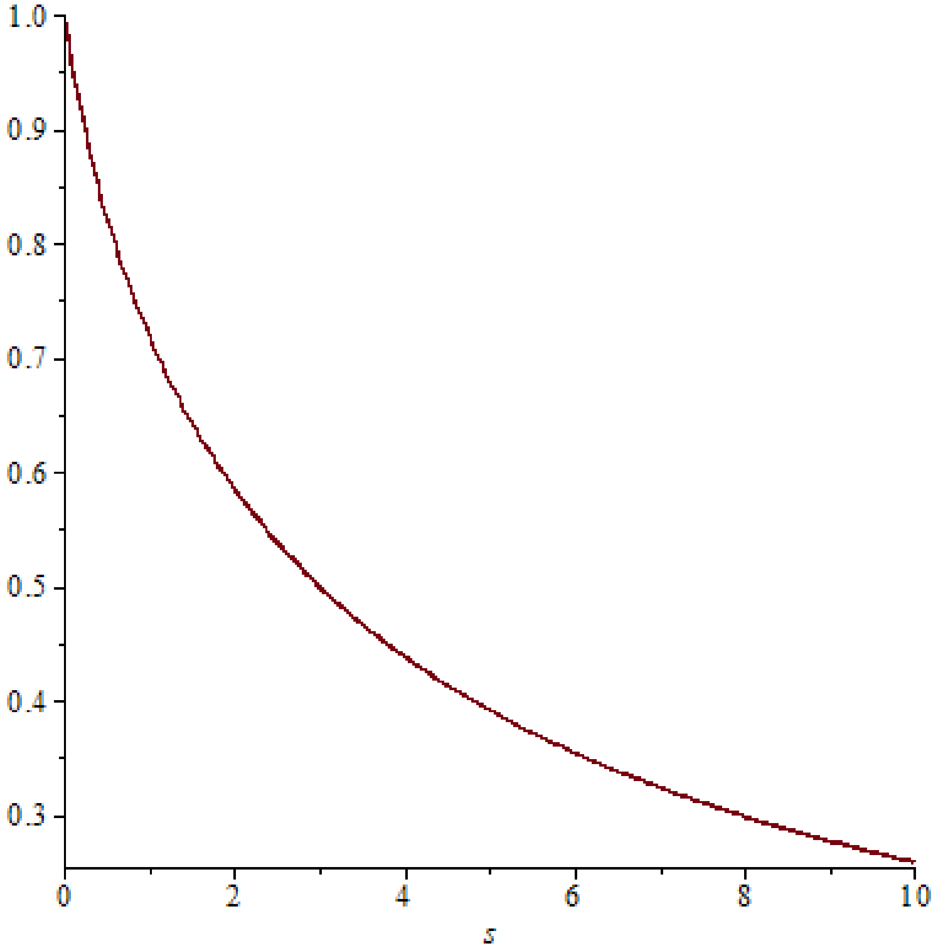
Solve the equations we can get the solution:

$$M_{2,1}(x) = \frac{(\lambda + 2\mu + s - \sqrt{\lambda + 2\mu + s)^2 - 8\lambda\mu}}{2\lambda}$$

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}}$$

We get the following graph of the busy period LT for $\lambda = 2, \mu = 4$:

Figure 5.2: $M/M/2$



5.1.3 M/M/3 model

Let $L_1(s)$ be the Laplace transform for the busy period of an M/M/2 queueing system. Let $L_2(s)$ be the probability that a busy period of an $M/M/2$ system, which begins with two customers, will end before a catastrophe. Let $M_{2,1}(s)$ be the probability that the $M/M/2$ system drops from 2 customers to 1 customer before a catastrophe. Let $M_{3,2}(s)$ be the probability that the $M/M/3$ system drops from 3 customers to 2 customer before a catastrophe. Let $M_{4,3}(s)$ be the probability that the $M/M/4$ system drops from 4 customers to 3 customer before a catastrophe. Let λ and μ be the arrival and service rates. Then

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_2(s) \tag{5.1.7}$$

$$L_2(s) = M_{2,1}(s) L_1(s) \tag{5.1.8}$$

$$M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{3,2}(s) M_{2,1}(s) \tag{5.1.9}$$

$$M_{3,2}(s) = \frac{3\mu}{\lambda + 3\mu + s} + \frac{\lambda}{\lambda + 3\mu + s} M_{4,3}(s) M_{3,2}(s) \tag{5.1.10}$$

$$M_{4,3}(s) = M_{3,2}(s) \tag{5.1.11}$$

We can get:

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{2,1}(s) L_1(s) \tag{5.1.12}$$

$$M_{21}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{3,2}(s) M_{2,1}(s) \tag{5.1.13}$$

$$M_{32}(s) = \frac{3\mu}{\lambda + 3\mu + s} + \frac{\lambda}{\lambda + 3\mu + s} M_{3,2}(s)^2 \tag{5.1.14}$$

Solve the equations we can get the solution:

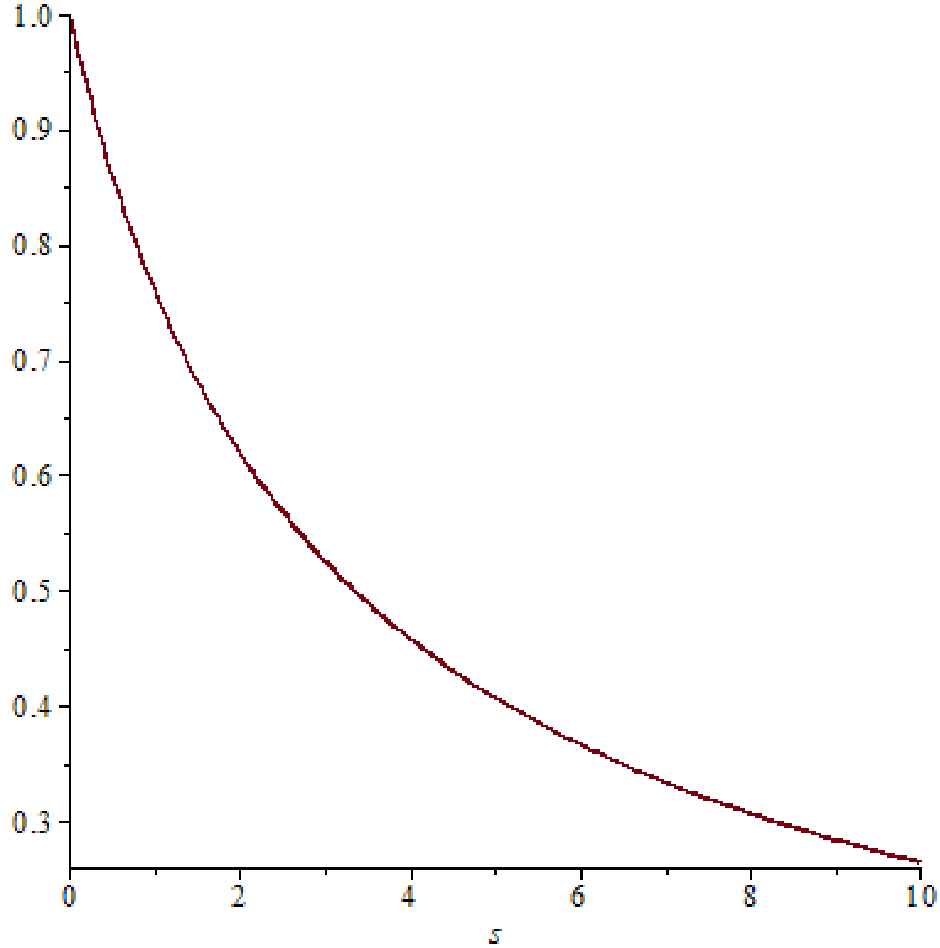
$$M_{3,2}(s) = \frac{\lambda + 3\mu + s - \sqrt{(\lambda + 3\mu + s)^2 - 12\lambda\mu}}{2\lambda}$$

$$M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s - \lambda M_{3,2}}$$

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}}$$

We get the following graph of the busy period LT for $\lambda = 2, \mu = 4$:

Figure 5.3: $M/M/3$



5.1.4 Conclusion

From above we can get a regular formula of the busy period of laplace of model $M/M/c$:

$$M_{c,c-1}(s) = \frac{\lambda + c\mu + s - \sqrt{(\lambda + c\mu + s)^2 - 4c\lambda\mu}}{2\lambda}$$

$$M_{c-1,c-2}(s) = \frac{(c-1)\mu}{\lambda + (c-1)\mu + s - \lambda M_{c,c-1}(s)}$$

$$M_{c-2,c-3}(s) = \frac{(c-2)\mu}{\lambda + (c-2)\mu + s - \lambda M_{c-1,c-2}(s)}$$

.

.

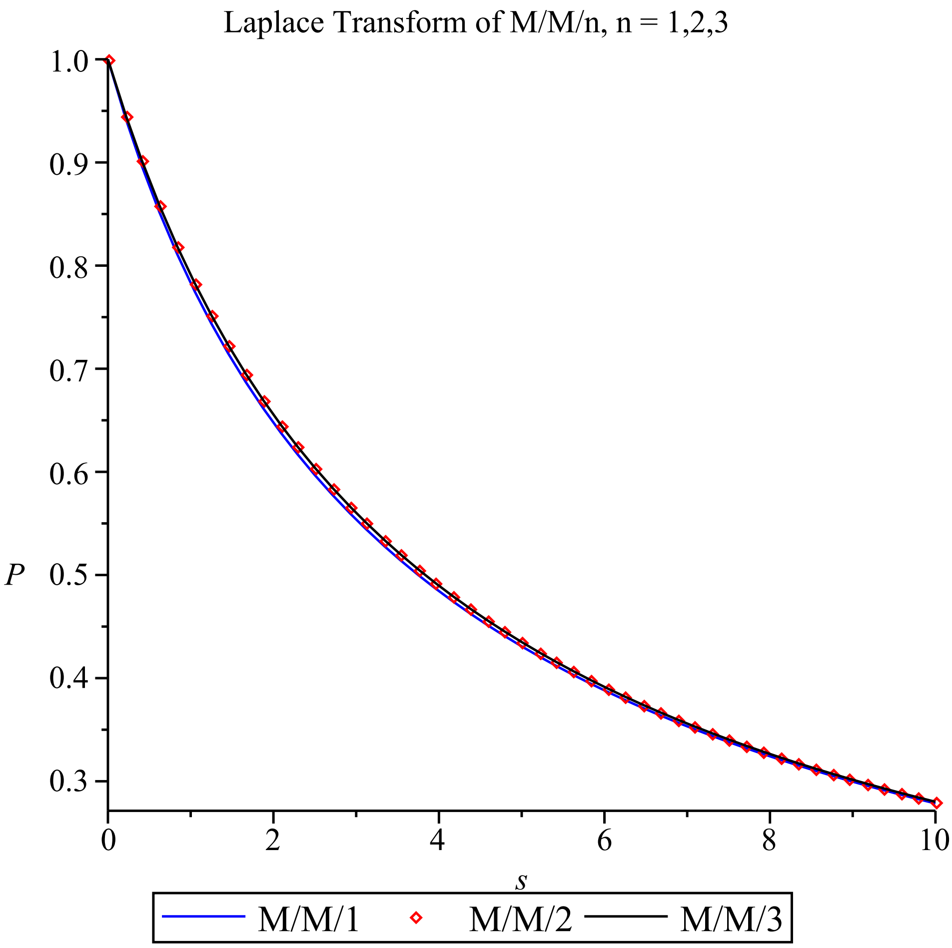
.

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}$$

The three graphs do not appear to be very different. This is because μ is considerably larger than λ. If we had chosen λ and μ closer to each other, the difference between the three graphs would have been more pronounced. The slope at 0 gives the negative of the expected busy period.

We put these three above-mentioned graphs into one graph as follows:

Figure 5.4: $M/M/C$



5.2 M/M/1/C model

$M/M/1/C$ represents a system where arrivals form a single queue and are governed by a Poisson process, there are 1 servers and we can think of C ai the size of the waiting area, or the buffer size, job service times are exponentially distributed. An example of such a system is a barber shop with one barber's chair and 3 waiting chairs. then we would have a $M/M/1/4$ queueing system.

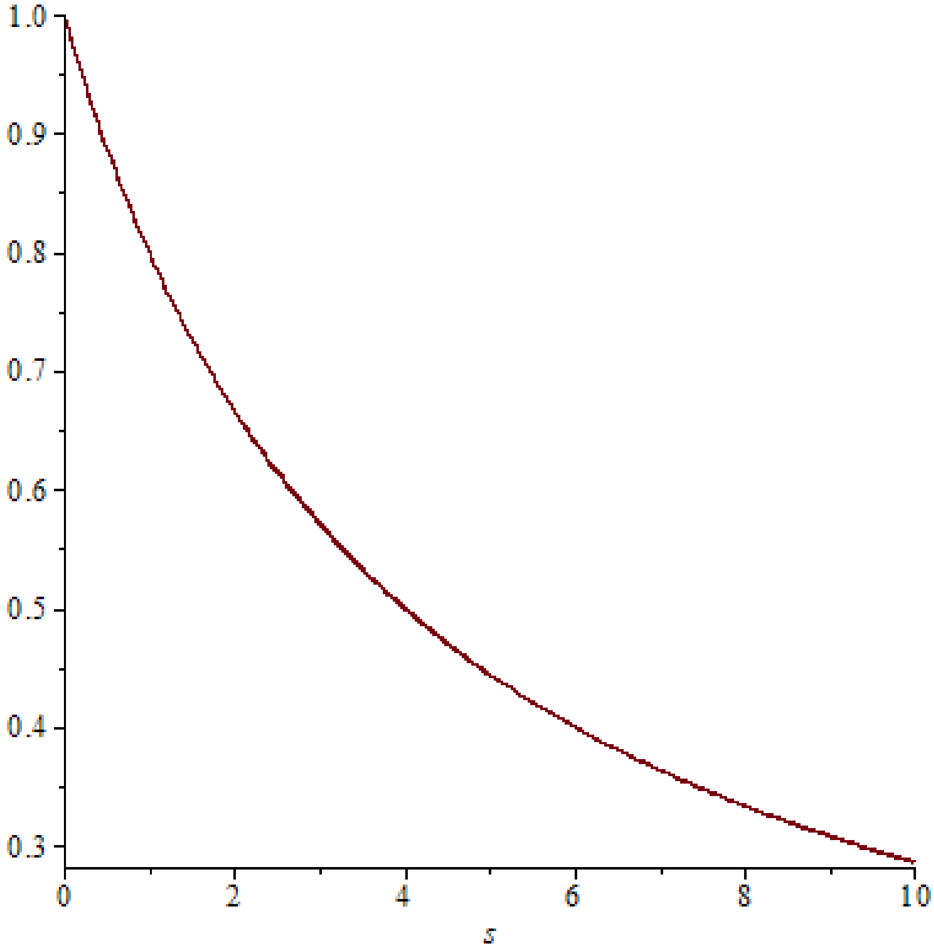
5.2.1 M/M/1/1 model

Let $L_1(s)$ be the Laplace transform for the busy period of an M/M/1 queueing system. Let λ and μ be the arrival and service rates. Then,

$$L_1(s) = \frac{\mu}{\mu + s} \tag{5.2.1}$$

We get the following graph of the busy period LT for $\lambda = 2, \mu = 4$:

Figure 5.5: $M/M/1/1$



5.2.2 M/M/1/2 model

Let $L_1(s)$ be the Laplace transform for the busy period of an $M/M/1/2$ queueing system. Let $L_2(s)$ be the probability that a busy period of an $M/M/2$ system, which begins with two customers, will end before a catastrophe. Let $M_{2,1}(s)$ be the probability that the $M/M/2$ system drops from 2 customers to 1 customer before a catastrophe. Let λ and μ be the arrival and service rates. Then

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_2(s) \tag{5.2.2}$$

$$L_2(s) = M_{2,1}(s) L_1(s) \tag{5.2.3}$$

$$M_{2,1}(s) = \frac{\mu}{\mu + s} \tag{5.2.4}$$

We explain the equation (5.2.2) as follows.

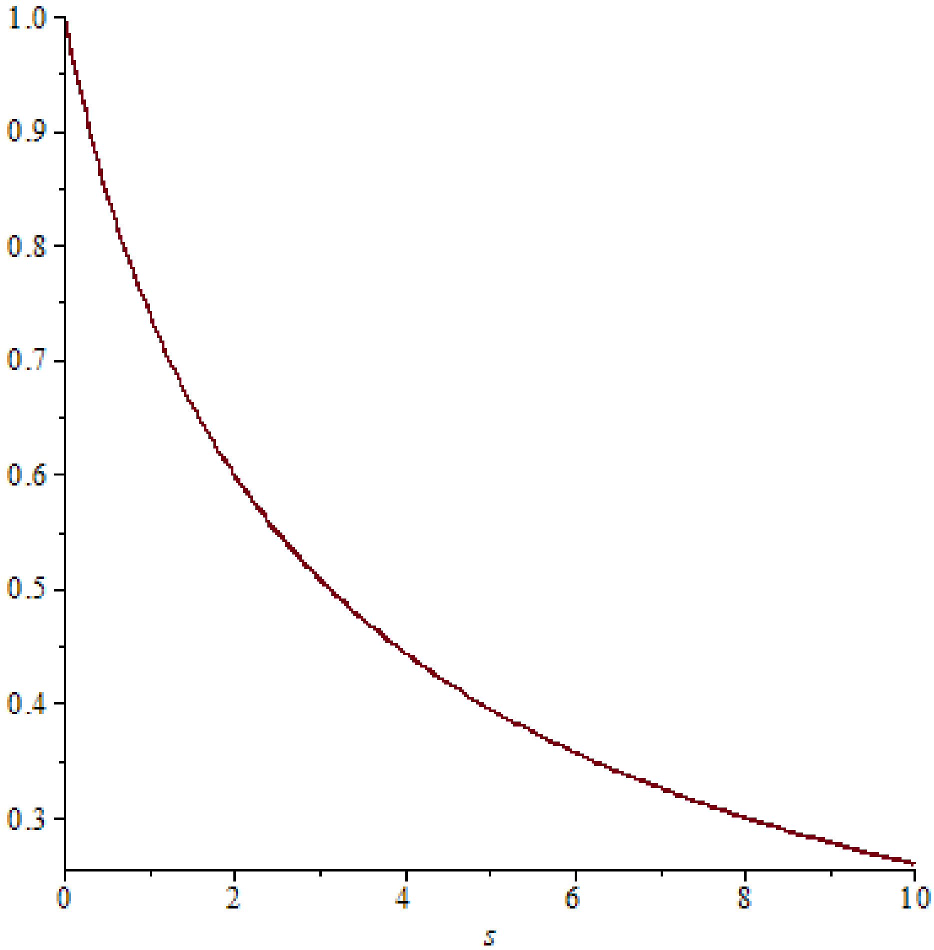
$L_1(s)$ is the Laplace transform of the busy period of an $M/M/1/2$ queueing system which begins with 1 customer. It is also the probability that the busy period finishes before any catastrophes (which occurs at rate s) occurs with 1 customers in the system, the next event will either be a service completion, an arrival, or a catastrophe. we are only in interested in the first two types of event. The first event will be a service completion with probability $\frac{\mu}{\lambda+\mu+s}$. In that case, the busy period ends before a catastrophe.The next event will be an arrival with probability $\frac{\lambda}{\lambda+\mu+s}$. In that case, we will have a system with two customers.

But in this model the size of the waiting area is 2. So if the $M/M/1/2$ system begin with 2 customers, then the probability of dropping to 1 customers before a catastrophe could only be $\frac{\mu}{\mu+s}$.Hence (5.2.4) holds. So we can get:

$$L_1(s) = \frac{\mu(\mu+s)}{\mu(\mu+s)+s(\lambda+\mu+s)}$$

We get the following graph of the busy period LT for $\lambda = 2, \mu = 4$:

Figure 5.6: $M/M/1/2$



5.2.3 M/M/1/3 model

Let $L_1(s)$ be the Laplace transform for the busy period of an M/M/2 queueing system. Let $L_2(s)$ be the probability that a busy period of an $M/M/2$ system, which begins with two customers, will end before a catastrophe. Let $M_{2,1}(s)$ be the probability that the $M/M/2$ system drops from 2 customers to 1 customer before a catastrophe. Let $M_{3,2}(s)$ be the probability that the $M/M/3$ system drops from 3 customers to 2 customer before a catastrophe. Let λ and μ be the arrival and service rates. Then

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_2(s) \tag{5.2.5}$$

$$L_2(s) = M_{2,1}(s) L_1(s) \tag{5.2.6}$$

$$M_{2,1}(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{3,2}(s) M_{2,1}(s) \tag{5.2.7}$$

$$M_{3,2}(s) = \frac{\mu}{\mu + s} \tag{5.2.8}$$

So we can get:

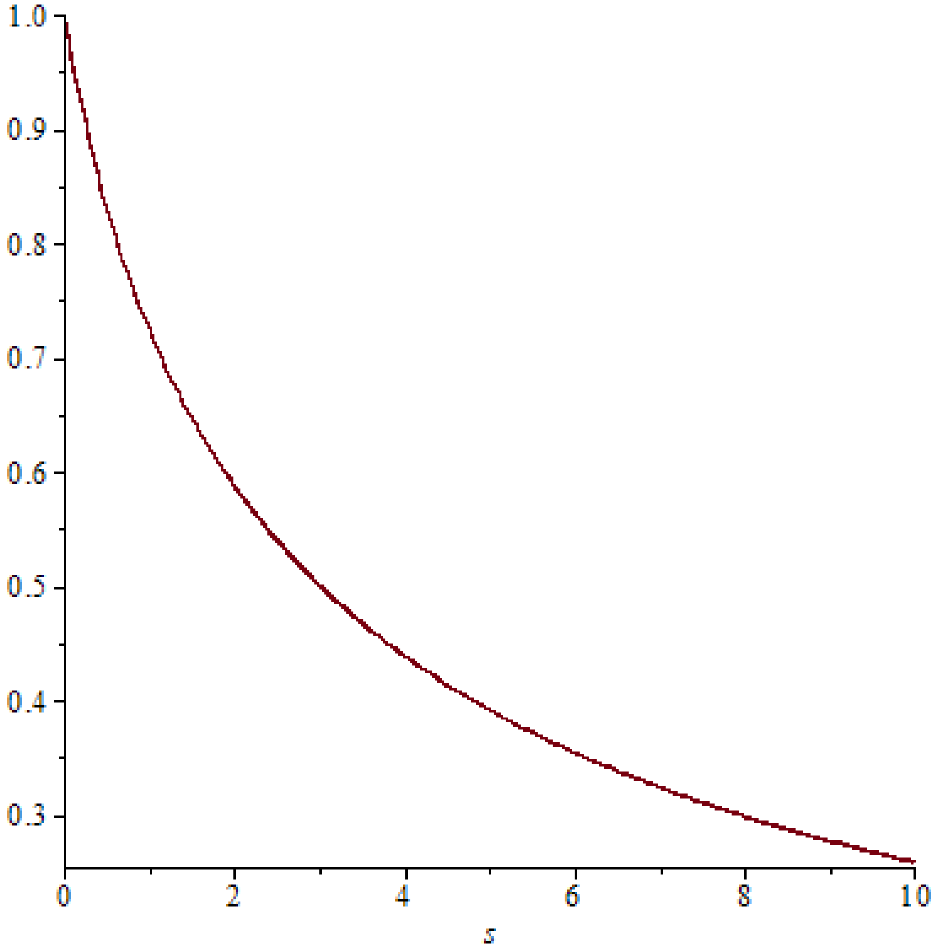
$$M_{3,2}(s) = \frac{\mu}{\mu + s}$$

$$M_{2,1}(s) = \frac{\mu(\mu + s)}{\mu(\mu + s) + s(\lambda + \mu + s)}$$

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}$$

We get the following graph of the busy period LT for $\lambda = 2, \mu = 4$:

Figure 5.7: $M/M/1/3$



5.2.4 Conclusion

From above we can get a regular formula of the busy period of laplace of model $M/M/1/c$:

$$M_{c,c-1}(s) = \frac{\mu}{\mu + s}$$

$$M_{c-1,c-2}(s) = \frac{\mu(\mu + s)}{\mu(\mu + s) + s(\lambda + \mu + s)}$$

$$M_{c-2,c-3}(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{c-1,c-2}(s)}$$

.

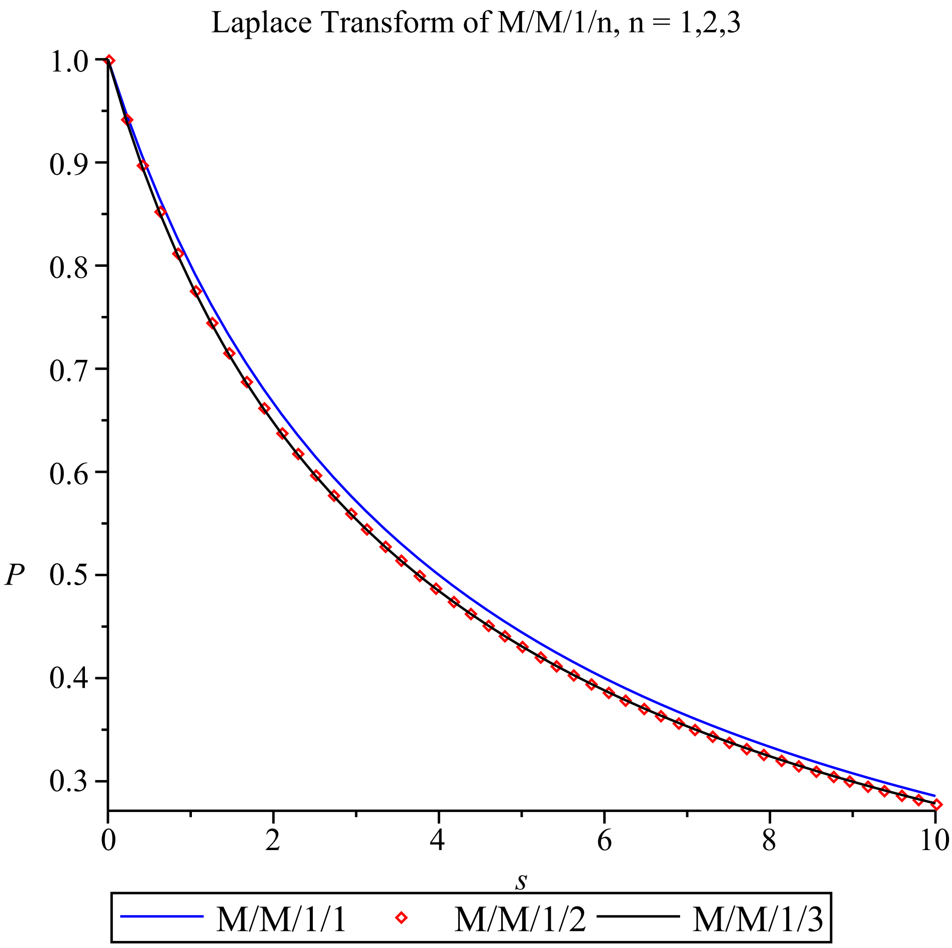
.

.

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}$$

We put these three above-mentioned graphs into one graph as follows:

Figure 5.8: $M/M/1/C$



5.3 M/M/C/C model

$M/M/1/C$ represents a system where arrivals form a single queue and are governed by a Poisson process, there are C servers and we can think of C ai the size of the waiting area, or the buffer size, job service times are exponentially distributed. One example of this type of queue would be a parking lot with C spaces. The states of the system are the number of spaces that have cars parked in them. Another example is a computer system with C telephone lines connected to do it. The states of the system are the number of lines being used.

5.3.1 M/M/2/2 model

Let $L_1(s)$ be the Laplace transform for the busy period of an $M/M/1/2$ queueing system. Let $L_2(s)$ be the probability that a busy period of an $M/M/2$ system, which begins with two customers, will end before a catastrophe. Let $M_{2,1}(s)$ be the probability that the $M/M/2$ system drops from 2 customers to 1 customer before a catastrophe. Let $M_{3,2}(s)$ be the probability that the $M/M/3$ system drops from 3 customers to 2 customer before a catastrophe. Let λ and μ be the arrival and service rates. Then

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_2(s) \tag{5.3.1}$$

$$L_2(s) = M_{2,1}(s) L_1(s) \tag{5.3.2}$$

$$M_{2,1}(s) = \frac{2\mu}{2\mu + s} \tag{5.3.3}$$

We explain the equation (5.3.1) as follows.

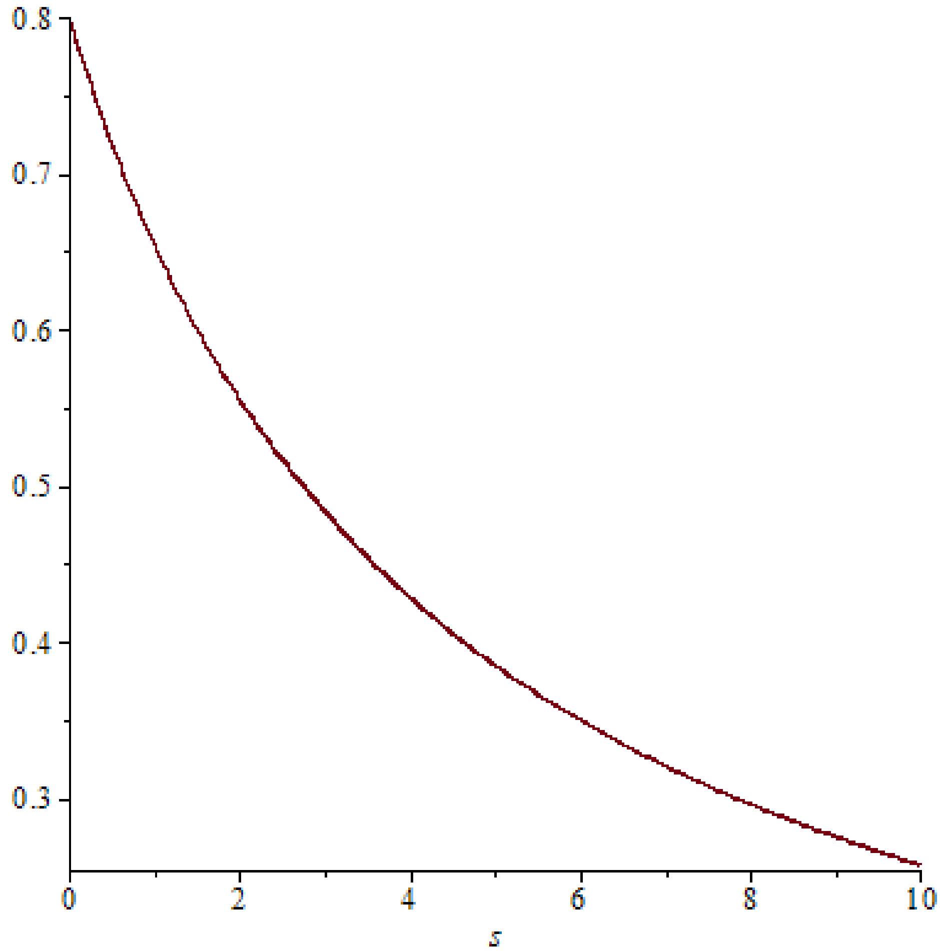
$L_1(s)$ is the Laplace transform of the busy period of an $M/M/2/2$ queueing system which begins with 1 customer. It is also the probability that the busy period finishes before any catastrophes (which occurs at rate s) occurs with 1 customers in the system, the next event will either be a service completion, an arrival, or a catastrophe. we are only in interested in the first two types of event. The first event will be a service completion with probability $\frac{\mu}{\lambda+\mu+s}$. In that case, the busy period ends before a catastrophe.The next event will be an arrival with probability $\frac{\lambda}{\lambda+\mu+s}$. In that case, we will have a system with two customers.

But in this model the size of the waiting area is 2. So if the $M/M/1/2$ system begin with 2 customers, then the probability of dropping to 1 customers before a catastrophe could only be $\frac{2\mu}{2\mu+s}$.Hence (5.3.3) holds. So we can get:

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}$$

We get the following graph of the busy period LT for $\lambda = 2, \mu = 4$:

Figure 5.9: $M/M/2/2$



5.3.2 M/M/3/3 model

Let $L_1(s)$ be the Laplace transform for the busy period of an M/M/2 queueing system. Let $L_2(s)$ be the probability that a busy period of an $M/M/2$ system, which begins with two customers, will end before a catastrophe. Let $M_{2,1}(s)$ be the probability that the $M/M/2$ system drops from 2 customers to 1 customer before a catastrophe. Let $M_{3,2}(s)$ be the probability that the $M/M/3$ system drops from 3 customers to 2 customer before a catastrophe. Let $M_{4,3}(s)$ be the probability that the $M/M/4$ system drops from 4 customers to 3 customer before a catastrophe. Let λ and μ be the arrival and service rates. Then

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_2(s) \tag{5.3.4}$$

$$L_2(s) = M_{2,1}(s) L_1(s) \tag{5.3.5}$$

$$M_{2,1}(s) = \frac{\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{3,2}(s) M_{2,1}(s) \tag{5.3.6}$$

$$M_{3,2}(s) = \frac{3\mu}{\lambda + 3\mu + s} \tag{5.3.7}$$

So we can get:

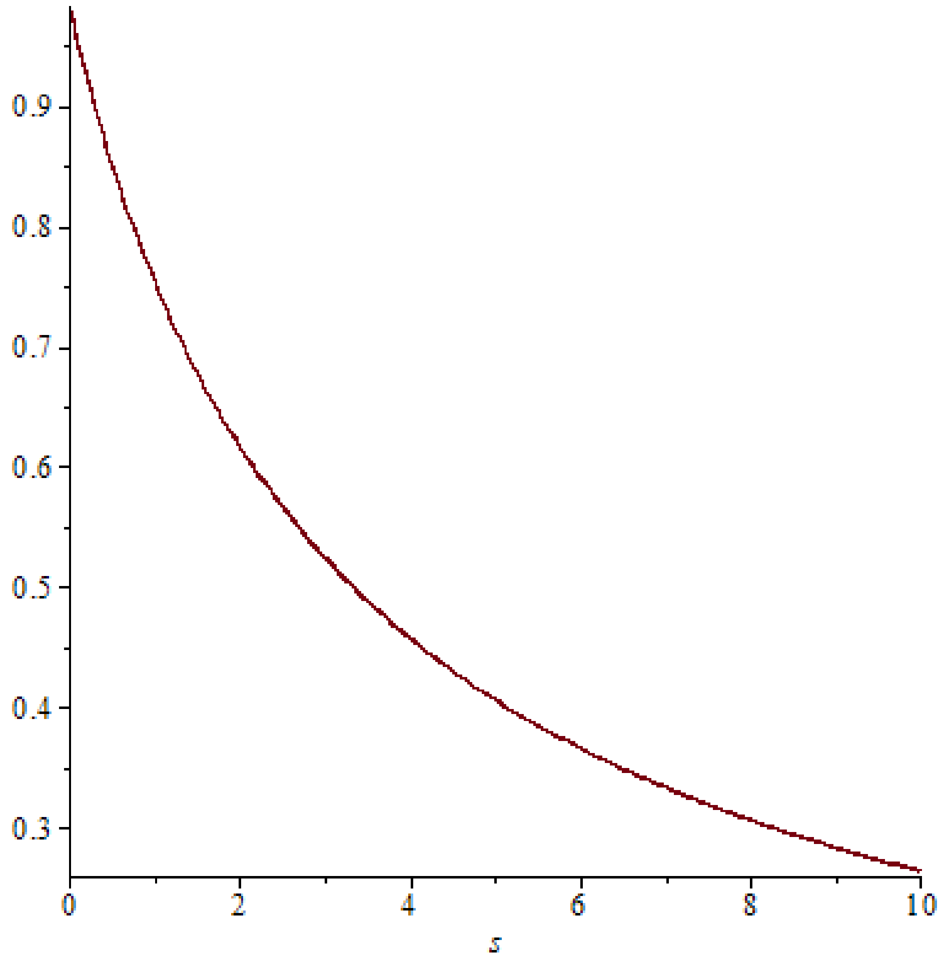
$$M_{3,2}(s) = \frac{3\mu}{\lambda + 3\mu + s}$$

$$M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s - \lambda M_{3,2}(s)}$$

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}$$

We get the following graph of the busy period LT for $\lambda = 2, \mu = 4$:

Figure 5.10: $M/M/3/3$



5.3.3 Conclusion

From above we can get a regular formula of the busy period of laplace of model $M/M/1/c$:

$$M_{c,c-1}(s) = \frac{n\mu}{\lambda + n\mu + s}$$

$$M_{c-1,c-2}(s) = \frac{(n-1)\mu}{\lambda + (n-1)\mu + s - \lambda M_{c,c-1}(s}$$

$$M_{c-2,c-3}(s) = \frac{(n-2)\mu}{\lambda + (n-2)\mu + s - \lambda M_{c-1,c-2}(s}$$

.

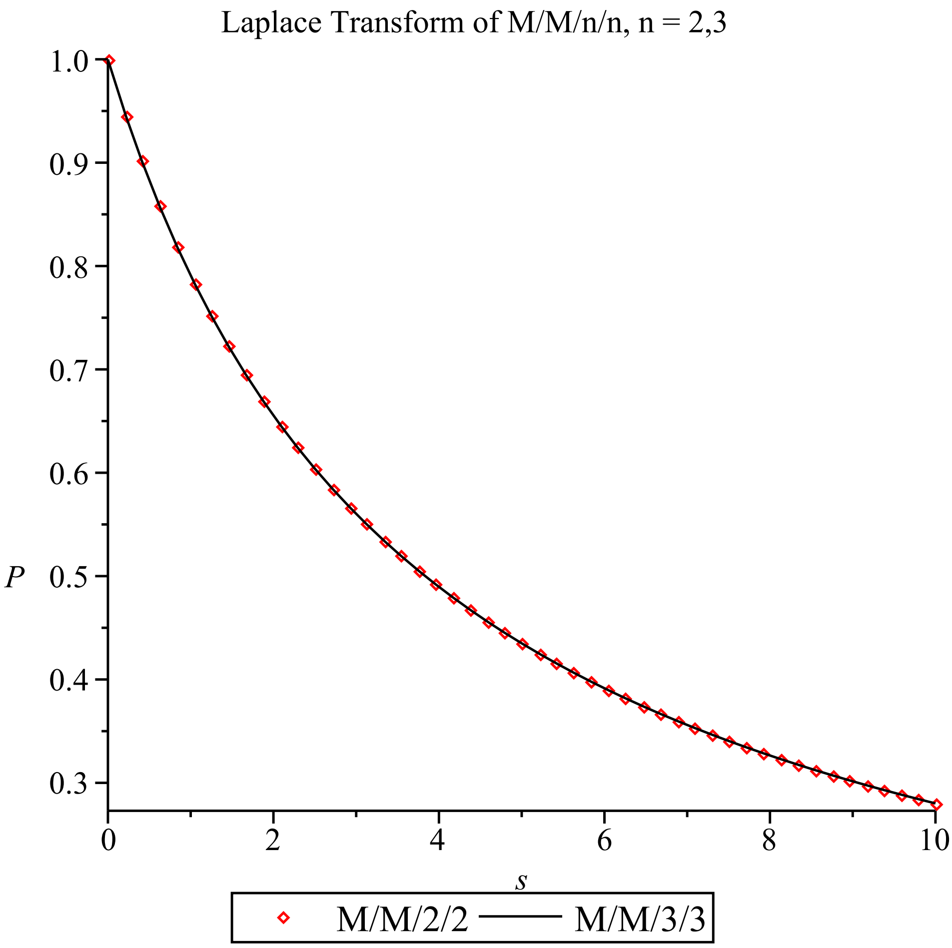
.

.

$$L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}$$

We put these two above-mentioned graphs into one graph as follows:

Figure 5.11: $M/M/C/C$



5.4 $M/E_2/1$ model

The following is from Hlynka (2001).

"Assume the p.d.f of the interarrival times is $a(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Assume that the service times have p.d.f., $b(x) = \frac{r\mu}{(r-1)!}(r\mu x)^{r-1}e^{-r\mu x}$, $x \geq 0$. (Here $r = 2$).

The method that we use is due to Erlang, and is referred to as the method of stages or the method of phases. This is a clever device that allows us to apply our exponential model analysis to non-exponential models.

The Erlang p.d.f. of the service time implies that there are r (here $r = 2$) stages of service for each customer. The state of the system at any time is the total number of stages which need to be completed for all the customers in the system."

Let $L_1(s)$ represent the Laplace transform of the time to complete a busy period if there is one full service period currently in system. $L_{0.5}(s)$ represents the Laplace transform of the time to complete a busy period if there is a 0.5 service period currently in system.

Let us build the model:

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} L_{0.5}(s) + \frac{\lambda}{\lambda + \mu + s} L_1(s)^2 \tag{5.4.1}$$

$$L_1(s) = L_{0.5}(s)^2 \tag{5.4.2}$$

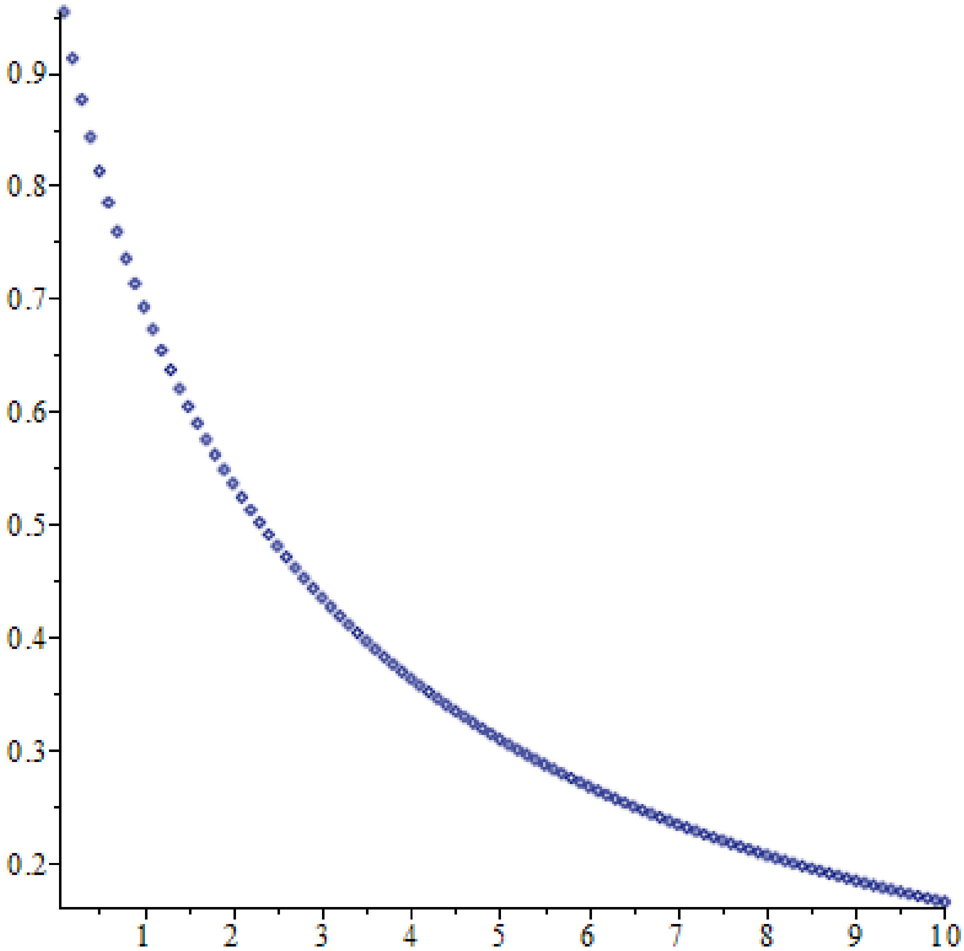
We explain the equation (5.4.1) as follows.

$L_1(s)$ is the Laplace transform of the busy period of an $M/E_2/1$ queueing system which begins with 1 customer. It is also the probability that the busy period finishes before any catastrophes (which occur at rate s). With 1 customer in the system, the first event will either be a half service, an arrival, or a catastrophe. We are only interested in the first two types of event. The next event will be a half service with probability $\frac{\mu}{\lambda+\mu+s}$. The next event will be an arrival with probability $\frac{\lambda}{\lambda+\mu+s}$. In that case, we will have a system with two customers. Here $L_1(s)^2$ is the probability that the busy period starting with two customers finishes before the catastrophe. Using the same reasoning, we can get $L_1(s) = L_{0.5}(s)^2$. Hence (5.4.2) holds.

Because this results in a cubic equation that is hard to solve, we can use MAPLE to help.

We get the following graph of the busy period LT for $\lambda = 2, \mu = 8$:

Figure 5.12: $M/E_2/1$



5.5 $M/E_2^*/2$ model

Here, we just consider the situation where both servers are busy if there is more than 1 customer in the system except in the case where there are exactly 2 customers each with a half service, in which case only 1 server is used.

Let $L_1(s)$ represent the Laplace transform of the time to complete a busy period if there is one full service period currently in system. $L_{0.5}(s)$ represents the Laplace transform of the time to complete a busy period if there is a 0.5 service period currently in system. $L_{1.5}(s)$ represents the Laplace transform of the time to complete a busy period if there are 1.5 service periods currently in system. Let $M_{2,1}(s)$ be the Laplace transform of the time from stage 2 to 1. Let $M_{1.5,1}(s)$ be the Laplace transform of the time from stage 1.5 to 1. Let $M_{3,2}(s)$ be the Laplace transform of the time from stage 3 to 2. Let $M_{2.5,1}(s)$ be the Laplace transform of the time from stage 2.5 to 1. Let $M_{2.5,2}(s)$ be the Laplace transform of the time from stage 2.5 to 2. Let $M_{1.5,1}(s)$ be the Laplace transform of the time from stage 1.5 to 1.

Let us build the model:

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} L_{0.5}(s) + \frac{\lambda}{\lambda + \mu + s} L_2(s) \tag{5.5.1}$$

$$L_2(s) = M_{2,1}(s) L_1(s) \tag{5.5.2}$$

$$L_{0.5}(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_{1.5}(s) \tag{5.5.3}$$

$$L_{1.5}(s) = M_{1.5,1}(s) L_1(s) \tag{5.5.4}$$

$$M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} M_{1.5,1}(s) + \frac{\lambda}{\lambda + 2\mu + s} M_{3,2}(s) M_{2,1}(s) \tag{5.5.5}$$

$$M_{3,2}(s) = M_{2,1}(s) \tag{5.5.6}$$

$$M_{1.5,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{2.5,1}(s) \tag{5.5.7}$$

$$M_{2.5,1}(s) = M_{2.5,2}(s) M_{2,1}(s) \tag{5.5.8}$$

$$M_{2.5,2}(s) = M_{1.5,1}(s) \tag{5.5.9}$$

So we get the final model:

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} L_{0.5}(s) + \frac{\lambda}{\lambda + \mu + s} M_{2,1}(s) L_1(s) \tag{5.5.10}$$

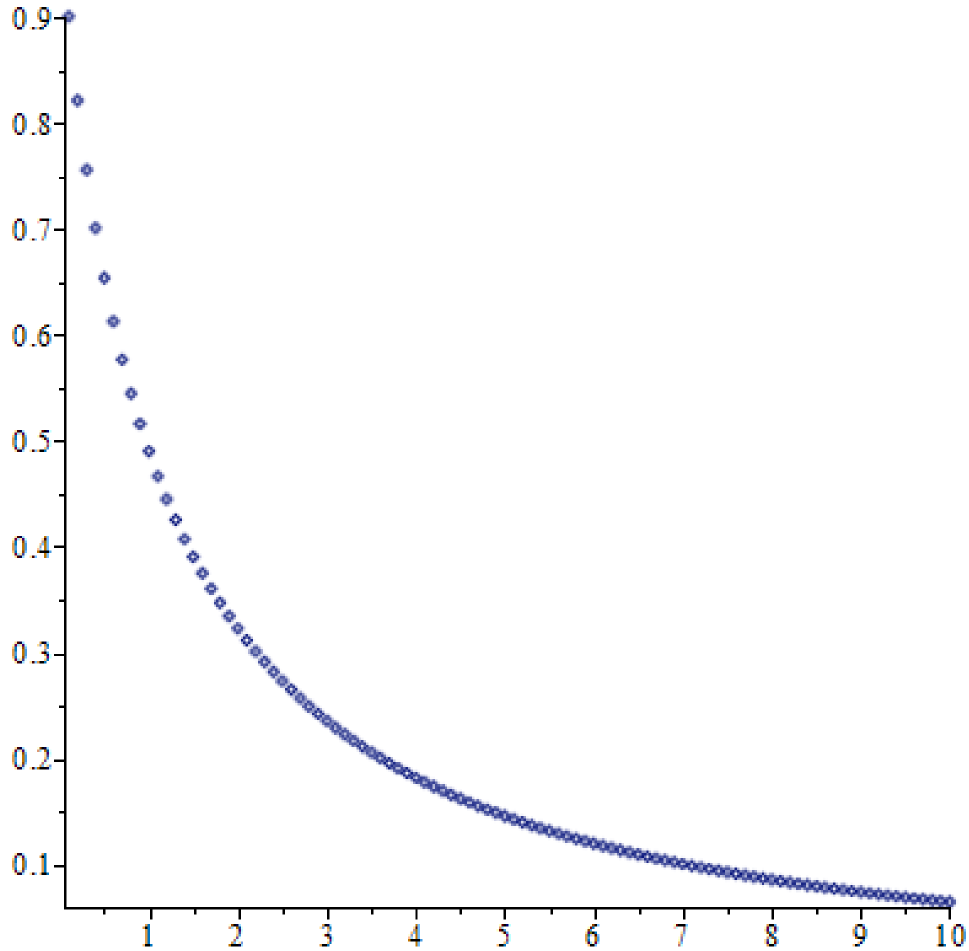
$$L_{0.5}(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{1.5,1}(s) L_1(s) \tag{5.5.11}$$

$$M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} M_{1.5,1}(s) + \frac{\lambda}{\lambda + 2\mu + s} M_{2,1}(s)^2 \tag{5.5.12}$$

$$M_{1.5,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{1.5,1}(s) M_{2,1}(s) \tag{5.5.13}$$

We can get the following graph:

Figure 5.13: $M/E_2/2$

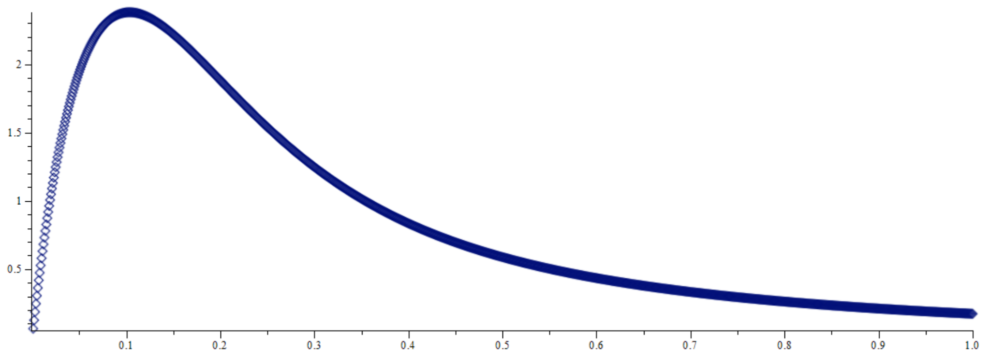


Chapter 6

Inverting the Laplace transform

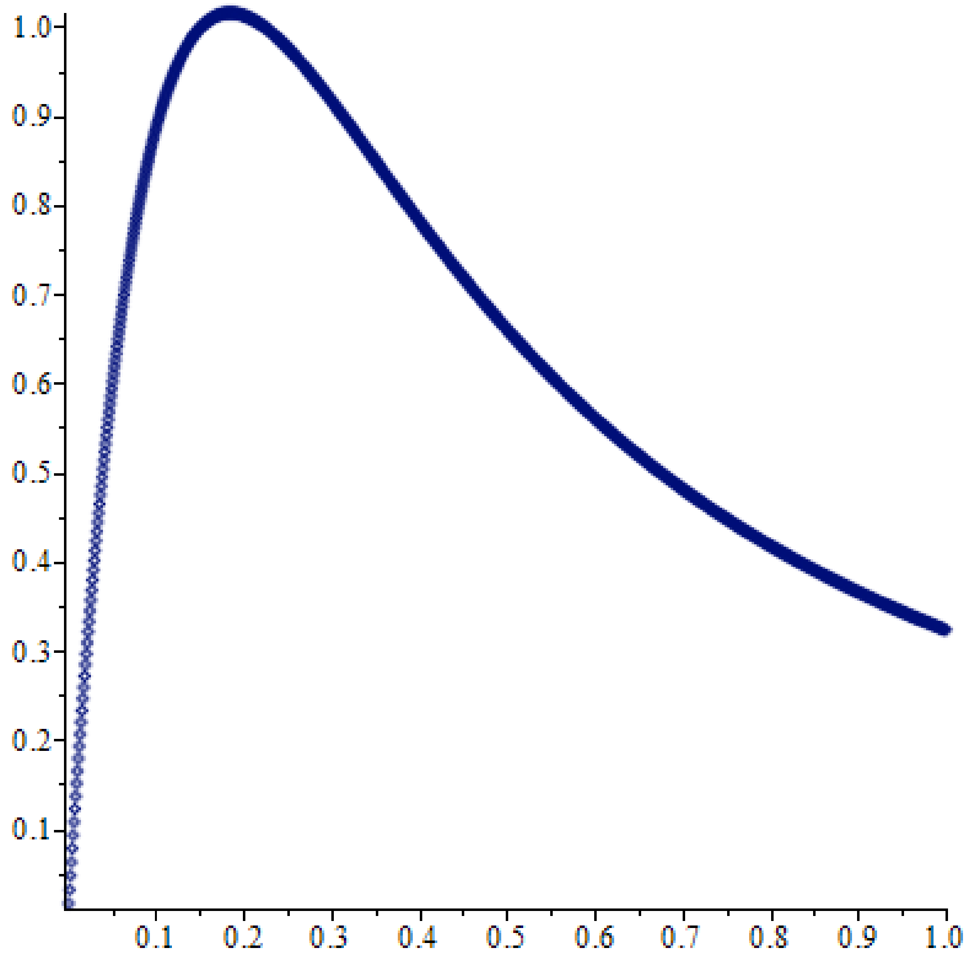
Numerical transform inversion has an odd place in computational probability. Historically, transforms were exploited extensively for solving queueing and related probability models.Now I have the busy period of Laplace transform, but the Laplace transform contains more information than the expected value. If we invert the Laplace transform, then we would have the pdf of busy period. Because it is really hard to solve the equation by hand, I use MAPLE to solve it. The method we used is the Gaver-Stehfest Method. To the $M/E_2/1$ model we can get the following graph:

Figure 6.1: $M/E_2/1$inversr



To the $M/E_2/2$ model we can get the following graph:

Figure 6.2: $M/E_2/2$inversr



Bibliography

[1] M. Hlynka, A Brief Introduction to Queueing Theory, unpublished book, University of Windsor (2001).

[2] M. Hlynka, Stochastic Processes class notes. University of Windsor (2012).

[3] R.B. Cooper, Introduction to Queueing Theory, North Holland Publishers (1981).

[4] K. Roy, M. Hlynka and R. Caron, Laplace Transform for the Busy Period of an M/G/1 system. WMSR 99-05, University of Windsor (1999).

[5] R.A. Jahan, Control of queues by Laplace transform, M.Sc. Major Paper, Dept. of Mathematics and Statistics. University of Windsor (2008).

[6] K. Roy, Laplace transforms, probability and queues, M.Sc. Thesis, Dept. of Mathematics and Statistics. University of Windsor (1997).

Appendices

A Inverse Laplace Transform in Maple $M/E_2/1$

restart:

$\lambda := 2; \mu := 8; m := \lambda + \mu + s;$

$L := \lambda * x^3 - m * x + \mu;$

$A := solve(L = 0, x)[3];$

plot$(A, s = 0..10);$

$[evalf(seq(A, s = 0..10))];$

plot$(A * A, s = 0..10);$

$F := A * A;$

Digit $s := 30; h := 16; aa := 1000$

for k from 1 to h do $w_k := (-1)^{h-k} * \frac{k_h}{k! * (h-k)!}$; end do

for ti from 1 to aa do

$t := \frac{ti}{1000};$

$temp[ti] := t;$

for m from 0 to $2 * h$ do

$f_{m,0} := Re(evalf(\frac{m*ln_2}{t} * subs(s = \frac{m*ln_2}{t}, F)));$

end do

for j from 1 to h do

for m from j to $2 * h - j$ do

$f_{m,j} := (1 + \frac{m}{j}) * f_{m,j-1} - (\frac{m}{j}) * f_{m+1,j-1};$

end do

end do

$G[ti] := \Sigma_{o=1}^{h} w_o * f_{o,o};$

end do

$X :=< seq(temp[i], i = 1..aa) >;$

$Y :=< seq(G[i], i = 1..aa) >;$

with(Statistics);

ScatterPlot(X, Y);

B Inverse Laplace Transform in Maple $M/E_2/2$

restart:

$\lambda := 2; \mu := 4; m := \lambda + 2 * \mu + s;$

$L := \lambda * x^3 - m * x + 2 * \mu;$

$v := \lambda + \mu + s;$

$A := solve(L = 0, x);$

$Oo := -\frac{1}{12} * (-432 + 6 * \sqrt{-816 - 1800 * s - 180 * s^2 - 6 * s^3})^{\frac{1}{3}}$

$+ \frac{3*(-\frac{5}{3}-\frac{1}{6}*s)}{(-432+6*\sqrt{-816-1800*s-180*s^2-6*s^3})^{\frac{1}{3}}}$

$- \frac{1}{2} * i * \sqrt{3} * [\frac{1}{6} * (-432 + 6 * \sqrt{-816 - 1800 * s - 180 * s^2 - 6 * s^3})^{\frac{1}{3}}$

$+ \frac{6*(-\frac{5}{3}-\frac{1}{6}*s)}{(-432+6*\sqrt{-816-1800*s-180*s^2-6*s^3})^{\frac{1}{3}}}];$

$P := \frac{(\lambda+2*\mu+s)*Oo-2*\mu}{\lambda*Oo};$

$S := Re(\frac{\mu^2}{v^2-\lambda*\mu*Oo-v*\lambda*P};$

Digit $s := 30; h := 16; aa := 1000$

for k from 1 to h do $w_k := (-1)^{h-k} * \frac{k_h}{k!*(h-k)!}$; end do

for ti from 1 to aa do

$t := \frac{ti}{1000};$

$temp[ti] := t;$

for m from 0 to $2 * h$ do

$f_{m,0} := Re(evalf(\frac{m*ln_2}{t} * subs(s = \frac{m*ln_2}{t}, F)));$

end do

for j from 1 to h do

for m from j to $2 * h - j$ do

$f_{m,j} := (1 + \frac{m}{j}) * f_{m,j-1} - (\frac{m}{j}) * f_{m+1,j-1};$

end do

end do

$G[ti] := \Sigma_{o=1}^{h} w_o * f_{o,o};$

end do

$X :=< seq(temp[i], i = 1..aa) >;$

$Y :=< seq(G[i], i = 1..aa) >;$

with(Statistics);

ScatterPlot(X, Y);

VITA AUCTORIS

Bingsen Yan was born in 1989 in Tianjin, China. After graduating from high school in 2008, he went on to the Hebei University of Technology where he obtained a B.A. in Mathematic in 2012. He is currently a candidate for the Master's degree in Statistics at the University of Windsor and hopes to graduate in autumn 2013.