Some results on Transient Queues

by

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A Major Paper

Submitted to the Faculty of Graduate Studies

through the Department of Mathematics and Statistics

in Partial Fulfillment of the Requirements for

the Degree of Master of Science at the

University of Windsor

Windsor, Ontario, Canada

2010

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Abstract

This paper studies two jobs which must be performed in tandem, when the order of service of the two jobs is unimportant to the customer. One of the jobs requires waiting in an M/M/1 queue while the other job requires a fixed amount of time with no queueing. The paper gives a graphical illustration as to when it is better (in the sense of minimizing total system time for an arriving customer) to choose the queue and when it is better to choose the fixed time job. The paper extends and expands on results given by van de Coevering (1995).

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CHAPTER 1

Literature Review

To obtain limiting probability results for M/M/1 queues is relatively easy. To obtain transient results is much harder.

Assume that the arrival rate is λ and the service rate is μ. Let $\rho = \dfrac{\lambda}{\mu}$ be the traffic intensity. The earliest known formulas for transient queueing probabilities are given by Clarke (1953) and by Ledermann and Reuter (1954). Gross and Harris (1985) and Kleinrock (1975) present the Ledermann and Reuter result in terms of modified Bessel functions. Their expression is:

$$p_{ij}(t) = e^{-(\lambda+\mu)t}[\rho^{(j-i)/2} I_{j-i}(2t\sqrt{\lambda\mu}) + \rho^{(j-i-1)/2} I_{j+i+1}(2t\sqrt{\lambda\mu})$$
$$+ (1-\rho)\rho^{j} \sum_{k=j+i+2}^{\infty} \rho^{-k/2} I_k(2t\sqrt{\lambda\mu})$$

where

$$I_k(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{k+2m}}{(k+m)!m!}$$

is the modified Bessel function of the first kind of order k.

(From Wikipedia, Daniel Bernoulli defined the Bessel functions, and they were then generalized by Friedrich Bessel. "[Modified] Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of the Bessel's differential equation that are finite at the origin ($x = 0$) for non-negative integer α, and diverge as x approaches zero for negative non-integer α. The solution type (e.g. integer or non-integer)and normalization of $J_\alpha(x)$ are defined by its properties .... It is possible to define the function by its Taylor series expansion around $x = 0$:

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(m+\alpha+1)}(x/2)^{(2m+\alpha)}$$

where $\Gamma(z)$ is the Gamma function.")

To obtain Clarke's expression, let q_t be the queue length at time t. Let $f_t(q)$ be the probability that $q_t = q$ when q_o is given. Then, for $q \geq q_0$, Clarke obtains

$$f_t(q) = F_t(q) - F_t(q+1) = e^{-(1+\lambda)t}\{\lambda^{\frac{1}{2}(q-q_0)} I_{q-q_0}(2\lambda^{\frac{1}{2}}t)$$
$$+ \lambda^{\frac{1}{2}(q-q_0-1)} I_{q+q_0+1}(2\lambda^{\frac{1}{2}}t) + (1-\lambda)\lambda^q \sum_{r=q+q_0+2}^{\infty} \lambda^{-\frac{1}{2}r} I_r(2\lambda^{\frac{1}{2}}t)\}$$

For $q < q_0$

$$f_t(q) = F_t(q) - F_t(q+1) = e^{-(1+\lambda)t}\{\lambda^{\frac{1}{2}(q-q_0)} I_{q-q_0}(2\lambda^{\frac{1}{2}}t)$$
$$+ \lambda^{\frac{1}{2}(q-q_0+1)} I_{q+q_0+1}(2\lambda^{\frac{1}{2}}t) + (1-\lambda)\lambda^q \sum_{r=q+q_0+2}^{\infty} \lambda^{-\frac{1}{2}r} I_r(2\lambda^{\frac{1}{2}}t)\}$$

where $I_r(x)$ is the modified Bessel function of the first kind of order r.

Champernowne (1956) gives a combinatorial argument to obtain the same expression as Clarke.

Sharma (1990) presents some new formulas for transient queueing probabilities.

$$p_i(t|\text{ n at time 0}) = (1-\rho)\rho^i + e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty}\left(\frac{(\lambda t)^k}{k!}\sum_{m=0}^{i+k+n}(k-m)\frac{(\mu t)^{m-1}}{m!}\right)$$
$$+ e^{-(\lambda+\mu)t}\sum_{k=0}^{\infty}(\lambda t)^{i+k-n}(\mu t)^k\left(\frac{1}{k!(i+k-n)!} - \frac{1}{(i+k)!(k-n)!}\right).$$

We present Conolly and Langaris (1993) version of Sharma's result. Let $\nu = \lambda+\mu$. The probability of having n customers in an M/M/1 system at time t, given that there are 0 customers at time 0, is given by

$$p_n^{(0)}(t) = (1-\rho)\rho^n + exp(-\nu t)\rho^n \sum_{m=0}^{\infty}\frac{(\lambda t)^m}{m!}\sum_{k=0}^{m+n}(m-k)\frac{(\mu t)^{k-1}}{k!},$$

where λ is the arrival rate, μ is the service rate, $\rho = \lambda/\mu$, $\nu = \lambda+\mu$.

Conolly and Langaris (1993) developed their own formula for which they claim computational improvements. It is given by

$$p_n^{(0)}(t) = (1 - \frac{(\lambda+\mu) - |(\lambda-\mu)|}{2\mu})\rho^n + exp(-\nu t)\sum_{m\geq 0} c_m^{(n)} t^m.$$

where

$$S(t) = \frac{\nu}{2\mu}\sum_{m\geq 1}\binom{1/2}{m}(-\frac{w^2}{\nu^2})m\sum_{k=0}^{2m-2}\frac{(\nu t)^k}{k!},$$

$c_m^{(0)}$ is the coefficient of t^m in $-S(t)$,

$c_m^{(1)} = \dfrac{m+1}{\mu}c_{m-1}^{(0)} - c_m^{(0)}$,

$c_m^{(n)} = \dfrac{m+1}{\mu}(c_{m+1}^{(n-1)} - \rho c_m^{(n-2)})$, for $n = 2, 3, \ldots$.

Karlin and McGregor (1958) and van Doorn (1980) give a result for the M/M/1 queue and more general birth-and-death processes, which is called the spectral representation of the probability transition function $P_{ij}(t)$. It is given by

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} q_i(x) q_j(x)\, d\psi(x)$$

where $\pi_0 = 1$, $\pi_j = (\lambda_0 \cdots \lambda_{j-1}/(\mu_1 \cdots \mu_j)$, where λ_j is the birth rate in state j, μ_j is the death rate in state j, $q_i(x)$ is a recursively defined system of polynomials in x satisfying orthogonality relations, $\psi(x)$ is a non decreasing real valued function on $[0, \infty)$. Details can be found in van Doorn (1980).

Abate and Whitt (1988) give a spectral representation for the M/M/1 queue, namely:

$$p_{in}(t) = (1-\rho)\rho^n + \int_{\tau_2^{-1}}^{\tau_1^{-1}} \rho^{n+1} q_i(x) q_n(x) \phi(x) e^{-xt} dx.$$

Here $q_i(x)$ are orthogonal polynomials for the M/M/1 system and the $q_i(x)$ form a recursively defined system of polynomials in x satisfying orthogonality relations

$$q_n(x) = \rho^{-n/2} U_n(\alpha(x)) - \rho^{-(n+1)/2} U_{n-1}(\alpha(x)),$$

where $U_n(\alpha)$ represents a polynomial in α of the form $c_{n0} + c_{n1}\alpha + \cdots + c_{nn}\alpha^n$ for $c_{00} = 1, c_{10} = 0, c_{11} = 2, c_{20} = -1, c_{21} = 0, c_{22} = 4, c_{n+1,j} = 2c_{n,j-1} - c_{n-1,j}$ for $0 \leq j \leq n+1$ and $n \geq 3$. $U_n(\alpha)$ are the Chebyshev polynomials of the second kind, with $U_{-1}(\alpha) = 0, U_0(\alpha) = 1, U_1(\alpha) = 2\alpha, U_2(\alpha) = 4\alpha^2 - 1, U_3(\alpha) = 8\alpha^3 - 4\alpha$ and

$$U_{n+1}(\alpha) = 2\alpha U_n(\alpha) - Un - 1(\alpha), n \geq 1;$$

$$\alpha(x) = \frac{1+\rho-2\theta^2 x}{2\sqrt{\rho}}.$$

Here

$$\tau_1 = \frac{(1-\sqrt{\rho})^2}{2}; \quad \tau_2 = \frac{(1+\sqrt{\rho})^2}{2},$$

where

$$\phi(x) = \frac{1-\rho}{2\pi\rho}\frac{\sqrt{(1-\tau_1 x)(\tau_2 x-1)}}{x}, \quad \tau_2^{-1} \le x \le \tau_1^{-1}.$$

Leguesdron, Pellaumail, Rubino, and Sericola (1993) give a transient analysis of the M/M/1 queueing system. They present a new method based on the uniformization technique and on generating functions. For $i \le j$, the transient probabilities are:

$$p_{i,j}(t) = e^{-(\lambda+\mu)t}\left(\frac{\lambda}{\mu}\right)^{(j-i)/2}[I_{j-i}(2t\sqrt{\lambda\mu}) - I_{i+j+2}(2t\sqrt{\lambda\mu})] + \left(\frac{\mu}{\lambda}\right)^{i+1} P_{0,i+j+1}(t)$$

for $I_k(x)$ as above, and the transient probabilities of the M/M/1 queue, given that the queue is empty at time $t = 0$, are:

$$p_{0,j}(t) = \frac{p^j}{q^j}\sum_{n=j}^{\infty} e^{-(\lambda+\mu)t}\frac{(\lambda+\mu)^n t^n}{n!}\sum_{k=0}^{\lfloor (n-j)/2\rfloor}\frac{n+1-2k}{n+1}\binom{n+1}{k}p^k q^{n-k}$$

Parthasarathy (1987) considers the M/M/1 queue system with Poisson arrivals and exponential service times by deriving the time-dependent solution in a direct way. It is given by:

$$p_n(t) = \frac{e^{-(\lambda+\mu)t}}{\mu}\sum_{k=1}^{n} q_k(t)\left(\frac{\lambda}{\mu}\right)^{n-k} + \left(\frac{\lambda}{\mu}\right)^n p_0(t)$$

for

$$p_0(t) = \int_0^t q_1(y) e^{-(\lambda+\mu)y} \, dy + \delta_{0a},$$

$$\alpha = 2\sqrt{(\lambda\mu)} \text{ and } \beta = \sqrt{(\lambda\mu)}$$

where

$$q_n(t) = \mu\beta(n-\alpha)(1-\delta_{0a})[I_{n-a}(\alpha t) - I_{n+a}(\alpha t)] + \lambda\beta^{n-\alpha-1}[I_{n+a+1}(\alpha t) - I_{n-a-1}(\alpha t)],$$

and δ is the Kronecker delta.

CHAPTER 2

van de Coevering's result and extensions

In this chapter, we consider results given in van de Covering (1995). We add additional explanations, correct one typo, add graphs, and extent his results.

For an M/M/1 queue, van de Coevering presents the result

$$p_{ij}(t) = \frac{2}{\pi}\rho^{(j-i)/2}\int_0^\pi \frac{e^{-\mu t\gamma(y)}}{\gamma(y)} a_i(y)a_j(y)\,dy + \begin{cases} (1-\rho)\rho^j & \rho < 1 \\ 0 & \rho \geq 1, \end{cases} \tag{1}$$

for

$$\gamma(y) = 1 + \rho - 2\sqrt{\rho}\cos(y) \text{ and } a_k(y) = \sin(ky) - \sqrt{\rho}\sin((k+1)y).$$

He credits Morse (1955) and Takacs (1962, page 23) for this trigonometric version. It is difficult to find illustrations of this result. The closest (but not the same) seems to be in Morse (1958, p.66).

For illustration, we fix $j = 2$. We present graphs of $p_{i2}(t)$ for $i = 0 \ldots 4$, $\lambda = 3$ or 1, $\mu = 4$, $0 < t < 10$ in Figures 2.1, 2.2. We stop at $i = 4$ arbitrarily.

We note that one of the $p_{i,2}(t)$ curves has $p_{i,2}(0) = 1$ while the others have $p_{i,2}(0) = 0$. Clearly $p_{2,2}(t) = 1$. See Figure 2.3

We eliminate that curve and consider the other curves $p_{i,2}(t)$, $i = 0, 1, 2, 3, 4$, and $0 < t < 3$ (for $\lambda = 3$) and $0 < t < 10$ (for $\lambda = 1$). This gives Figure 2.4, 2.5.

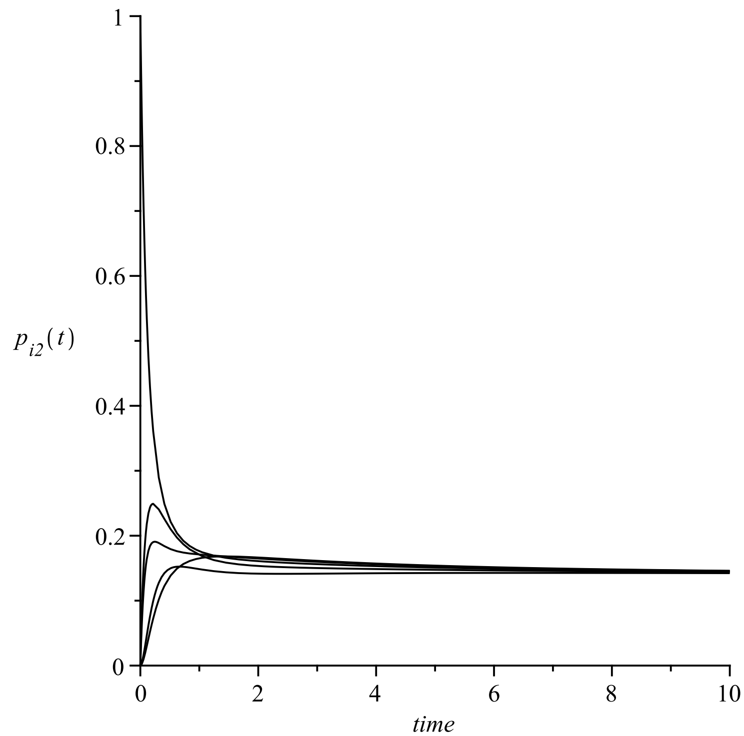

FIGURE 2.1. $p_{i2}(t)$ for $i = 0 \dots 4$, $\lambda = 3$, $\mu = 4$, $0 < t < 10$

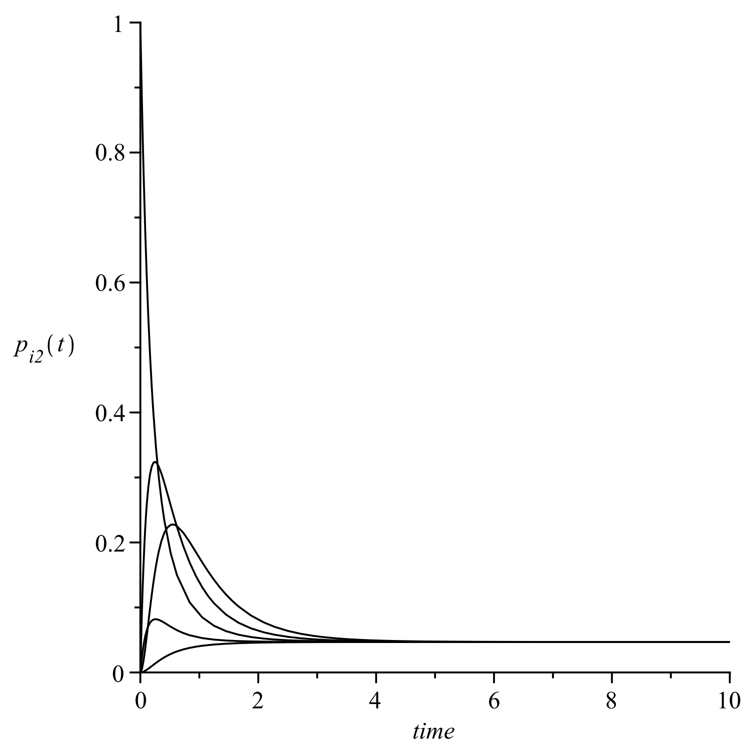

FIGURE 2.2. $p_{i2}(t)$ for $i = 0 \dots 4$, $\lambda = 1$, $\mu = 4$, $0 < t < 10$

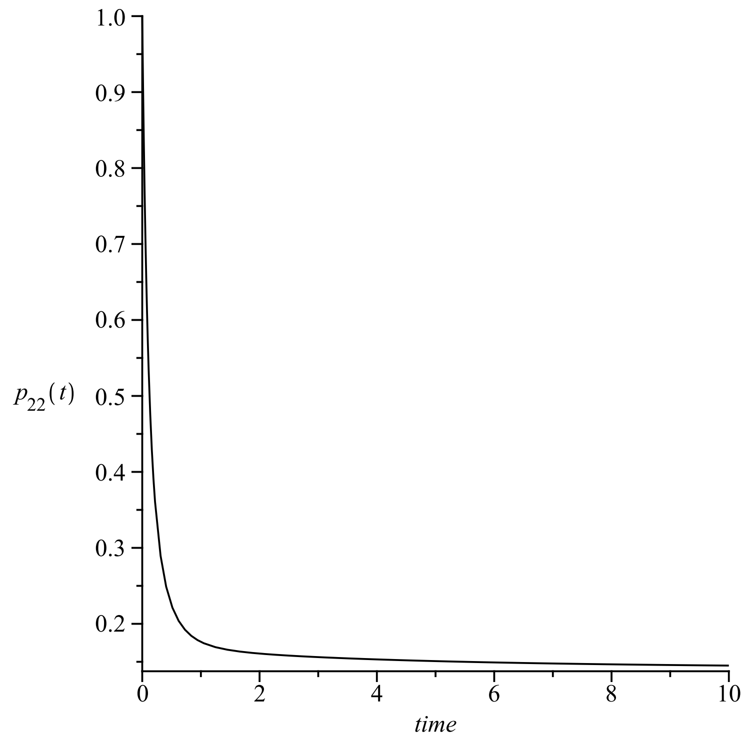

FIGURE 2.3. $p_{22}(t)$ for $i = 2$, $j = 2$, $\lambda = 3$, $\mu = 4$

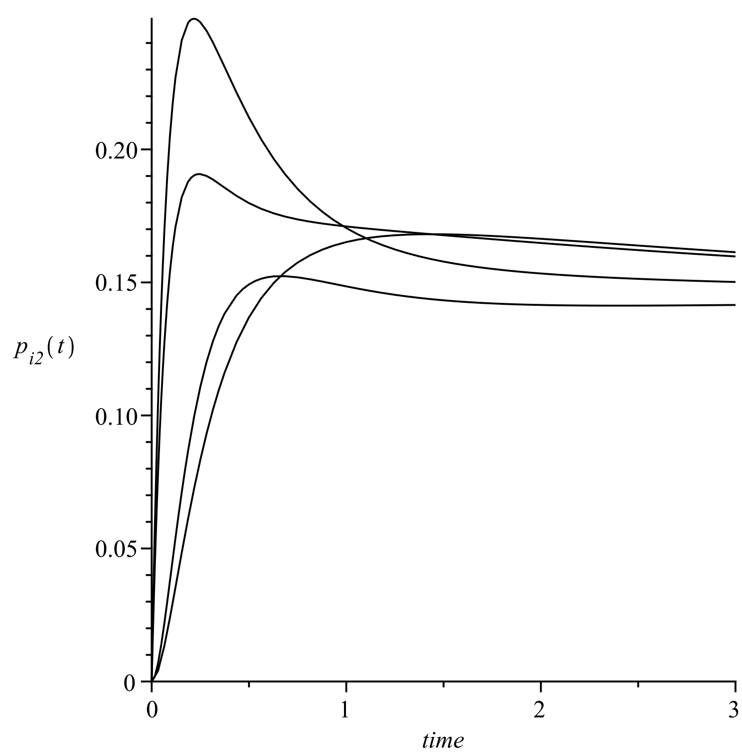

FIGURE 2.4. $p_{i2}(t)$ for $i = 0, 1, 3, 4$, $\lambda = 3$, $\mu = 4$, $0 < t < 3$

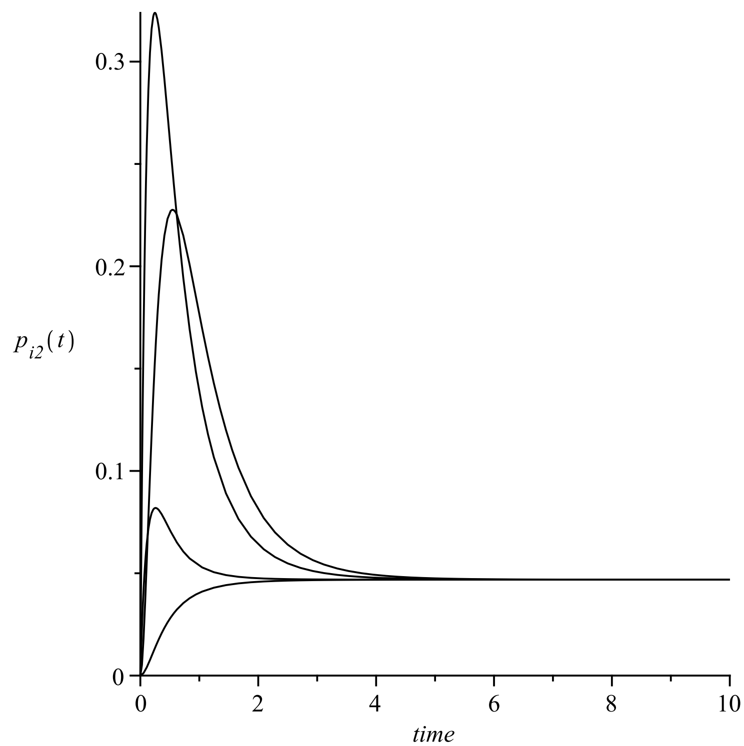

FIGURE 2.5. $p_{i2}(t)$ for $i = 0, 1, 3, 4$, $\lambda = 1$, $\mu = 4$, $0 < t < 10$

Can we determine which curve corresponds to $i = 0, 1, 3, 4$ from Figures 2.4 and 2.5? The answer is Yes but some additional derivation is required. We could simply draw the graphs individually, and compare them. For example, Figure 2.6 shows $p_{0,2}(t)$ in the $\lambda = 3$, $\mu = 4$ case.

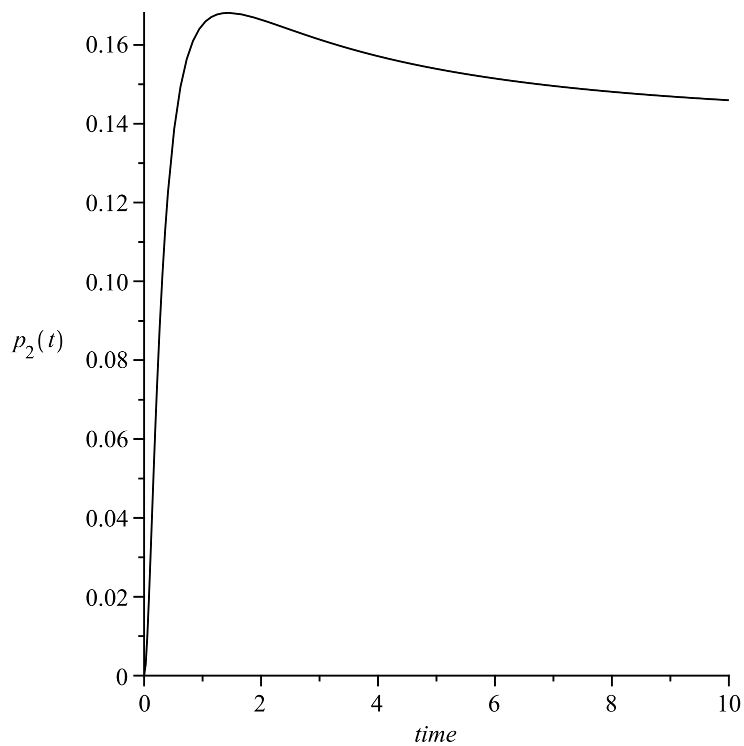

FIGURE 2.6. $p_{02}(t)$ for $i = 0$, $j = 2$, $\lambda = 3$, $\mu = 4$, $0 < t < 10$

However, we can get even more information without drawing the individual graphs.

PROPERTY 2.1. *(new result) For an $M/M/1$ queueing system with $\lambda < \mu$, for small Δt, and fixed $j > 0$, we have $p_{j+1,j}(\Delta t) > p_{j-1,j}(\Delta t) > p_{j+2,j}(\Delta t) > p_{j-2,j}(\Delta t)$.*

PROOF.

$$p_{j-1,j}(\Delta t) = \lambda \Delta t + o(\Delta t).$$

$$p_{j+1,j}(\Delta t) = \mu \Delta t + o(\Delta t).$$

$$p_{j-2,j}(\Delta t) = o(\Delta t).$$

Similarly,

$$p_{j+2,j}(\Delta t) = o(\Delta t).$$

Thus,

$$\begin{aligned}\frac{p_{j+1,j}(\Delta t)}{p_{j-1,j}(\Delta t)} &= \frac{\mu\Delta t + o(\Delta t)}{\lambda\Delta t + o(\Delta t)} \\ &= \frac{\mu + \frac{o(\Delta t)}{\Delta t}}{\lambda + \frac{o(\Delta t)}{\Delta t}} \\ &\longrightarrow \frac{\mu + 0}{\lambda + 0} > 1,\end{aligned}$$

so, $p_{j+1,j}(\Delta t) > p_{j-1,j}(\Delta t)$ for small Δt.

Similarly,

$\dfrac{p_{j+2,j}(\Delta t)}{p_{j-2,j}(\Delta t)} > 1$ for small Δt.

Also,

$$\begin{aligned}\frac{p_{j-2,j}(\Delta t)}{p_{j+2,j}(\Delta t)} &= \frac{\lambda\Delta t + o(\Delta t)}{\mu^2(\Delta t)^2 + o((\Delta t)^2)} \\ &= \frac{\frac{\lambda}{\Delta t} + \frac{o(\Delta t)}{(\Delta t)^2}}{\mu + \frac{o((\Delta t)^2)}{(\Delta t)^2}} \\ &\longrightarrow \infty \quad as \quad \Delta t \longrightarrow 0^+.\end{aligned}$$

So, $p_{j-2,j}(\Delta t) > p_{j+2,j}(\Delta t)$ for small Δt. □

Thus the graphs, in Figures 2.4 and 2.5, in order from largest to smallest, for small t, correspond to $i = 3, 1, 4, 0$.

Note in addition that all of the graphs in Figure 2.1 must have the same limit π_2 as $t \to \infty$. We compute $\pi_2 = (1-\rho)\rho^2 = (1-3/4)(3/4)^2 = 9/64$.

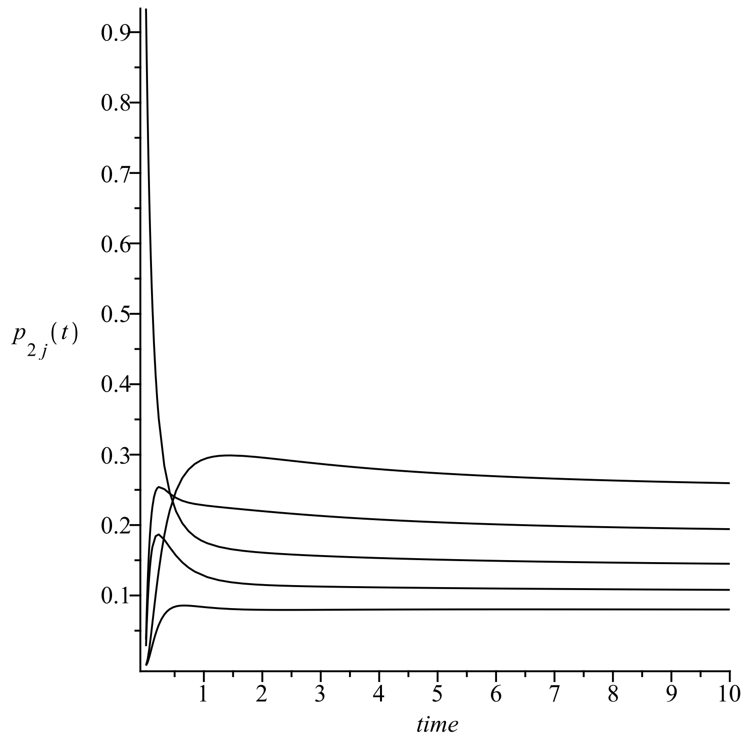

FIGURE 2.7. $p_{2j}(t)$ for $j = 0 \dots 4$, $\lambda = 3$, $\mu = 4$, $0 < t < 10$

Next we consider the graphs $p_{2j}(t)$ for $\lambda = 3$ or 1, $\mu = 4$, and $0 < t < 10$ in Figures 2.7, 2.8. We know that the limits of the different curves will be π_j for $j = 0, 1, 2, \dots$. We also know that $\pi_j = (1 - \rho)\rho^j$ for $j = 0, 1, 2, \dots$. Further $\pi_0 > \pi_1 > \pi_2 > \dots$. Thus, if we look at the right hand side of Figure 2.7, we see that the curves, from largest to smallest, correspond to $j = 0, 1, 2, 3, \dots$.

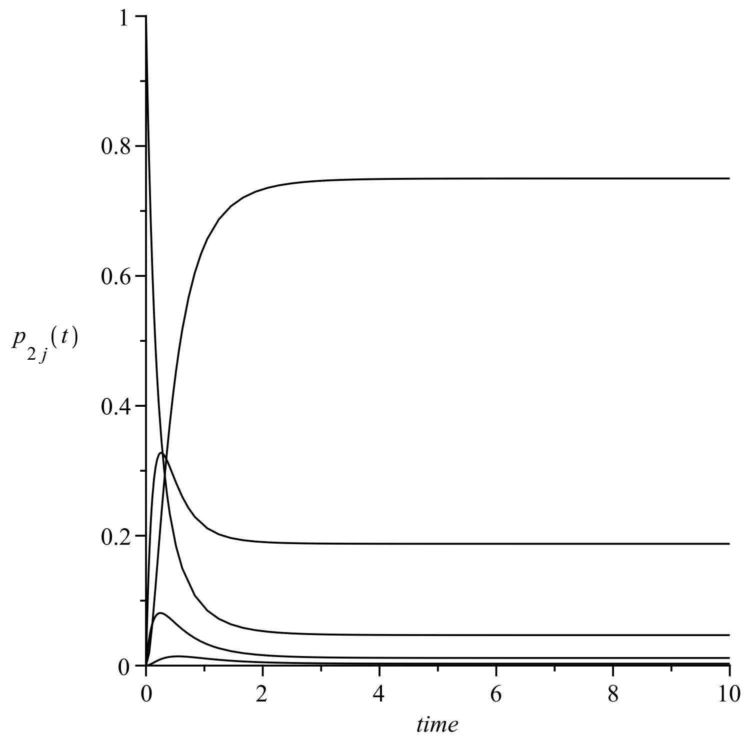

FIGURE 2.8. $p_{2j}(t)$ for $j = 0 \ldots 4$, $\lambda = 1$, $\mu = 4$, $0 < t < 10$

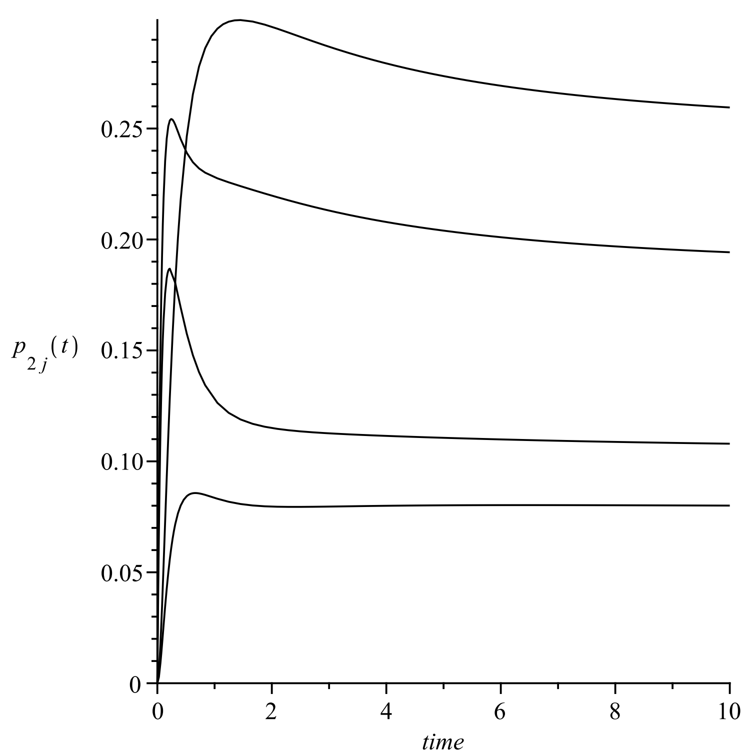

FIGURE 2.9. $p_{2j}(t)$ for $i = 2$, $j = 0, 1, 3, 4$,$\lambda = 3$, $\mu = 4$, $0 < t < 10$

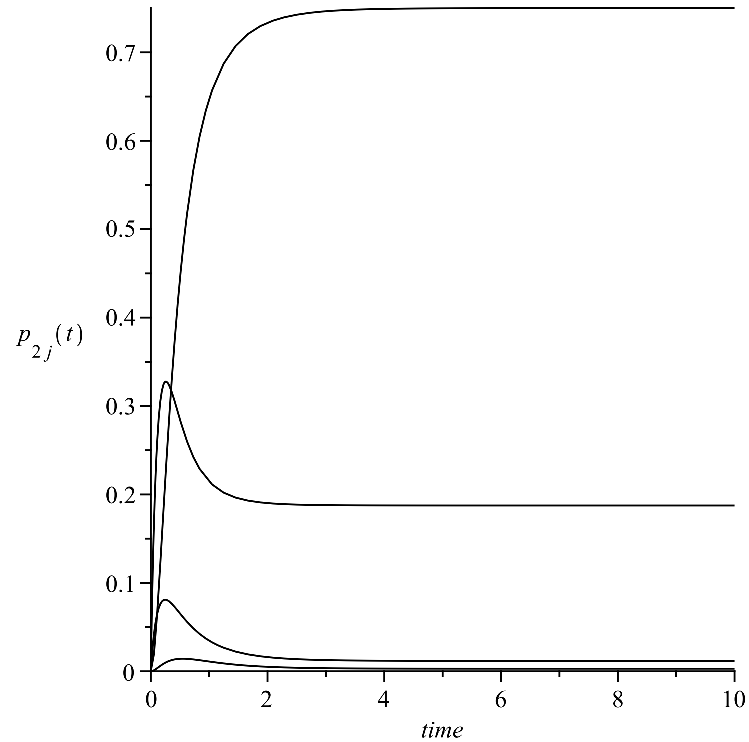

FIGURE 2.10. $p_{2j}(t)$ for $j = 0, 1, 3, 4$, $\lambda = 1$, $\mu = 4$, $0 < t < 10$

THEOREM 2.2. *(van de Coevering) For an $M/M/1$ queueing system, let $EL_i(t)$ be the expected number of customers at time t if there are i customers at time 0. Then*

$$EL_i(t) = \frac{2}{\pi}\rho^{(j-i)/2}\int_0^\pi \frac{e^{-\mu t\gamma(y)}}{\gamma(y)^2}a_i(y)\sin(y)\,dy + \begin{cases} \rho/(1-\rho) & \rho < 1 \\ i + (\lambda-\mu)t + \rho^{-1}/(\rho-1) & \rho > 1. \end{cases} \tag{2}$$

Rather than the above version, we use the following (also given in van de Coevering) for computational purposes.

THEOREM 2.3. *(van de Coevering) For an $M/M/1$ queueing system, for $\rho < 1$,*

$$EL_i(t) = (\lambda - \mu)t + \mu\int_0^t p_{i0}(x)\,dx + i. \tag{3}$$

PROOF.

$$EL_i(t) = \sum_{j=0}^{\infty} j p_{ij}(t).$$

So, using Kolmogorov's forward differential equations,

$$\begin{aligned}
\frac{d}{dt} EL_i(t) &= \sum_{j=0}^{\infty} j p'_{ij}(t) \\
&= \sum_{j=0}^{\infty} j(\mu p_{ij+1}(t) + \lambda p_{ij-1}(t) - (\lambda+\mu) p_{ij}(t)) \\
&= \mu \sum_{j=0}^{\infty} (j+1) p_{ij+1}(t) - \mu \sum_{j=0}^{\infty} p_{ij+1}(t) + \lambda \sum_{j=0}^{\infty} (j-1) p_{ij-1}(t) + \lambda \sum_{j=0}^{\infty} p_{ij-1}(t) \\
&\quad - (\lambda+\mu) \sum_{j=0}^{\infty} j p_{ij}(t) \\
&= \mu \sum_{j=0}^{\infty} j p_{ij}(t) - \mu(1 - p_{i0}(t)) + \lambda \sum_{j=0}^{\infty} j p_{ij}(t) + \lambda(1) - (\lambda+\mu) EL_i(t) \\
&= \mu EL_i(t) - \mu(1 - p_{i0}(t)) + \lambda EL_i(t) + \lambda - (\lambda+\mu) EL_i(t) \\
&= \lambda - \mu + \mu p_{i0}(t).
\end{aligned}$$

Integrate with respect to t.

$$EL_i(t) = (\lambda - \mu)t + \mu \int_0^t p_{i0}(x)\, dx + K.$$

Take $t = 0$. Then

$$\begin{aligned}
i = EL_i(0) &= (\lambda-\mu)0 + \mu \int_0^0 p_{i0}(x)\, dx + K = K \\
\implies EL_i(t) &= (\lambda - \mu)t + \mu \int_0^t p_{i0}(x)\, dx + i.
\end{aligned}$$

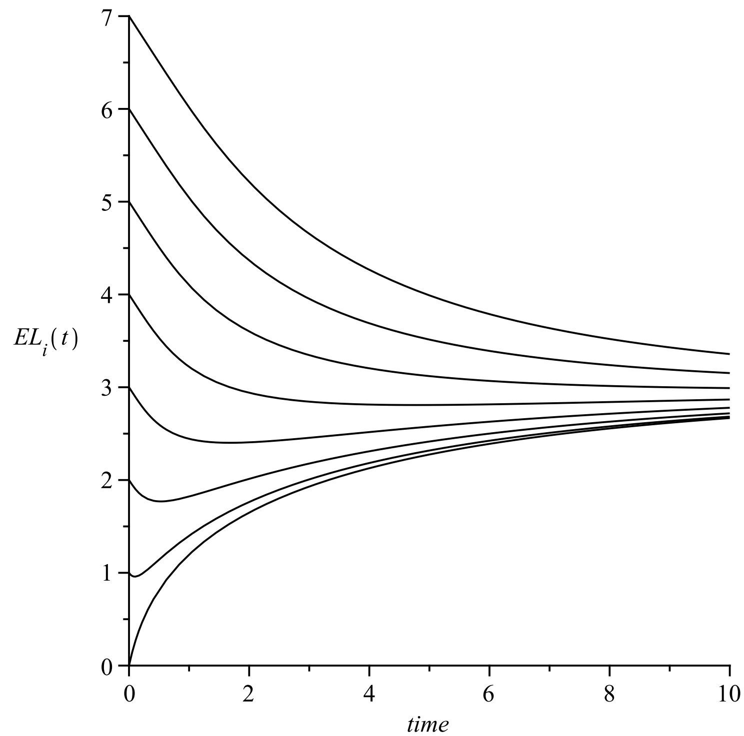

FIGURE 2.11. $EL_i(t)$ for $\lambda = 3$,$\mu = 4$, $i = 0, 1, 2, 3, 4$, $0 < t < 10$

□

Our results appear in Figure 2.11.

A graph of this type appears in Abate and Whitt (1988). They indicate that the only graphs of this type will be either (a) strictly increasing, (b) strictly decreasing, or (c) decreasing to a minimum and then increasing. Graphs of this type also appear in Yu et al. (2006).

Note that all of the curves $EL(i, t)$ for $i = 1, 2, 3, 4$ have negative slope for $t = 0$. This is not so clear for $i = 1$ so we magnify that curve in Figure 2.13.

We prove the general result as follows.

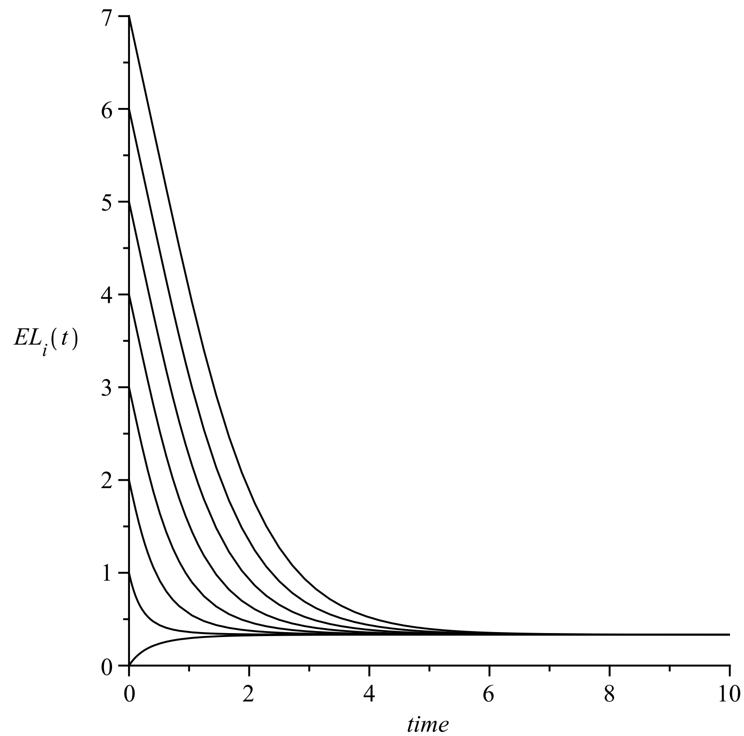

FIGURE 2.12. $EL_i(t)$ for $\lambda = 1$, $\mu = 4$, $0 < t < 10$

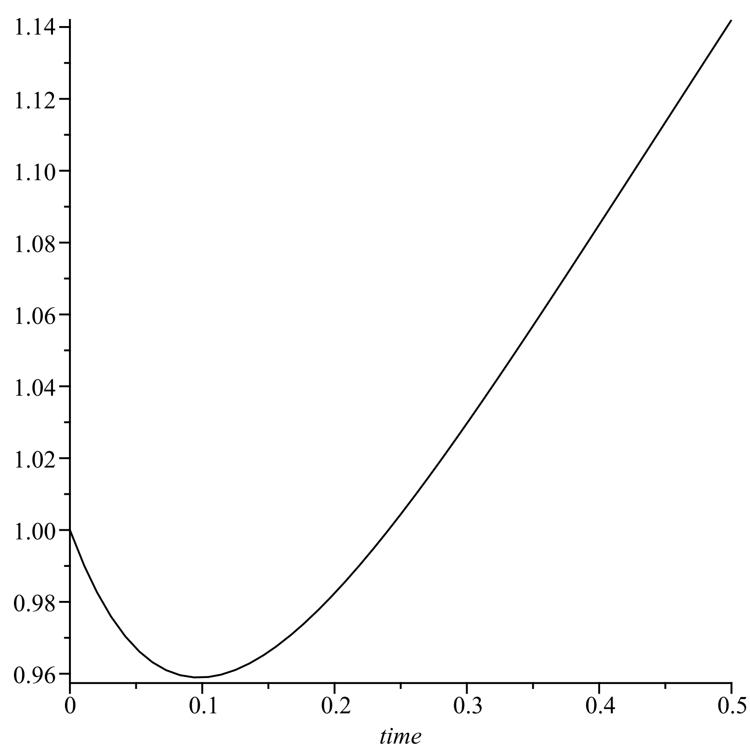

FIGURE 2.13. $EL_1(t)$ for $\lambda = 3$,$\mu = 4$, $0 < t < 0.5$

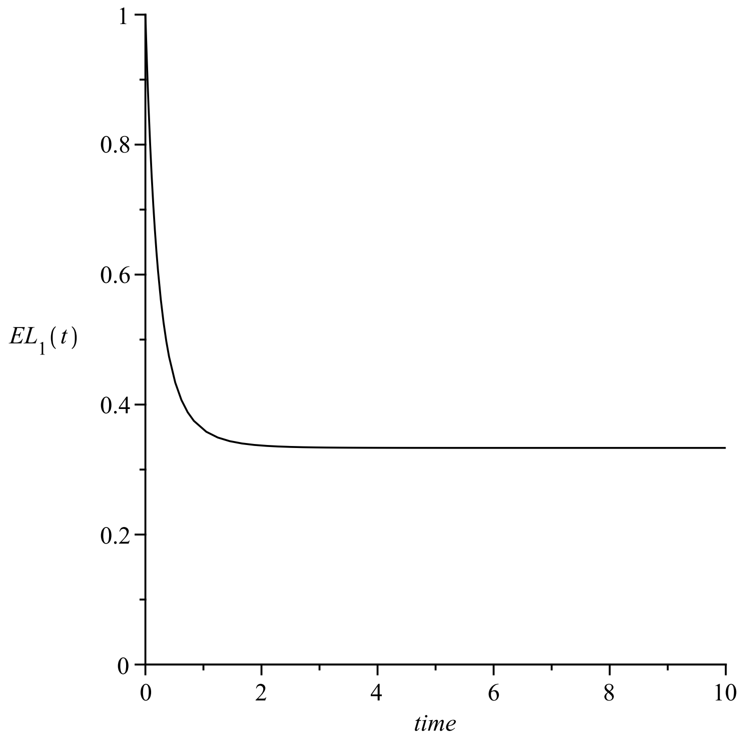

FIGURE 2.14. $EL_1(t)$ for $\lambda = 1$, $\mu = 4$, $0 < t < 10$

PROPERTY 2.4. *(new result) If $\lambda < \mu$, then*

$$\left.\frac{dEL_i(t)}{dt}\right|_{t=0} = \lambda - \mu < 0, \tag{4}$$

for $i = 1, 2 \ldots$

PROOF. $EL_i(0) = i$.

$$EL_i(\Delta t) = \sum_{\text{all j}} jp_{ij}(\Delta t) = (i+1)(\lambda\Delta t + o(\Delta t))$$

$$+ i(1 - \lambda\Delta t - \mu\Delta t + o(\Delta t))$$

$$+ (i-1)(\mu\Delta t + o(\Delta t)) + o(\Delta t)$$

$$= i + (\lambda - \mu)\Delta t + o(\Delta t)$$

$$\left.\frac{d(EL_i(t))}{dt}\right|_{t=0} = \lim_{\Delta t \to 0} \frac{EL_i(\Delta t) - EL_i(0)}{\Delta t}$$

$$= \lambda - \mu < 0.$$

□

Next we consider curves for $EL_i^2(t)$.

THEOREM 2.5. *(van de Coevering) For an $M/M/1$ queueing system,*

$$EL_i^2(t) = \frac{2}{\pi}\rho^{(j-i)/2}\int_0^\pi \frac{e^{-\mu t\gamma(y)}}{\gamma(y)^3}a_i(y)\sin(y)\,dy + \begin{cases} 2\rho(1-\rho)^{-2} - EL_i(t) & \rho < 1 \\ 2(\rho-1)it + (\rho-1)^2t^2 + 2\rho^{-i}t + 2\rho t - EL_i(t) + i + i^2 & \\ \quad -[\rho + 1 + (2i+1)(\rho-1)]\rho^{-i}(\rho-1)^{-2} & \rho \geq 1. \end{cases} \tag{5}$$

For computational purposes, we use the following.

THEOREM 2.6. *(corrected version of van de Coevering's result) For an $M/M/1$ queueing system, with $\rho < 1$,*

$$EL_i^2(t) = 2(\lambda - \mu)\int_0^t EL_i(t)\,dx + (\lambda + \mu)t - \mu\int_0^t p_{i0}(x)\,dx + i^2. \tag{6}$$

PROOF.

$$EL_i^2(t) = \sum_{j=0}^{\infty} j^2 p_{ij}(t)$$

so,

$$\begin{aligned}\frac{d}{dt}EL_i^2(t) &= \sum_{j=0}^{\infty} j^2 p'_{ij}(t)\\ &= \sum_{j=0}^{\infty} j^2(\mu p_{ij+1}(t) + \lambda p_{ij-1}(t) - (\lambda+\mu)p_{ij}(t)).\end{aligned}$$

Now if we simplify the summands $\sum_{j=0}^{\infty} j^2\mu p_{ij+1}(t)$ and $\sum_{j=0}^{\infty} j^2\lambda p_{ij-1}(t)$, then

$$\begin{aligned}\sum_{j=0}^{\infty} j^2\mu p_{ij+1}(t) &= \mu\sum_{j=0}^{\infty}(j+1)^2 p_{ij+1}(t) - 2\lambda\sum_{j=0}^{\infty} p_{ij+1}(t) + \mu\sum_{j=0}^{\infty} p_{ij+1}(t))\\ &= \mu EL_i^2(t) - 2\mu\sum_{j=0}^{\infty}(j+1)p_{ij+1}(t) + \mu\sum_{j=0}^{\infty} p_{ij+1}(t)\\ &= \mu EL_i^2(t) - 2\mu EL_i(t) + \mu(1 - p_{i0}(t))\end{aligned}$$

and

$$\begin{aligned}\sum_{j=0}^{\infty} j^2\lambda p_{ij-1}(t) &= \lambda\sum_{j=0}^{\infty}(j-1)^2 p_{ij-1}(t) + 2\lambda\sum_{j=0}^{\infty} jp_{ij-1}(t) - \lambda\sum_{j=0}^{\infty} p_{ij-1}(t))\\ &= \lambda EL_i^2(t) + 2\lambda EL_i(t) + \lambda\sum_{j=0}^{\infty} p_{ij-1}(t)\\ &= \lambda EL_i^2(t) + 2\lambda EL_i(t) + \lambda.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dt}EL_i^2(t) &= \mu EL_i^2(t) - 2\mu EL_i(t) + \mu - \mu p_{i0}(t) + \lambda EL_i(t) + \lambda - (\lambda+\mu)EL_i^2(t)\\ &= 2(\lambda-\mu)EL_i(t) + (\lambda+\mu) - \mu p_{i0}(t).\end{aligned}$$

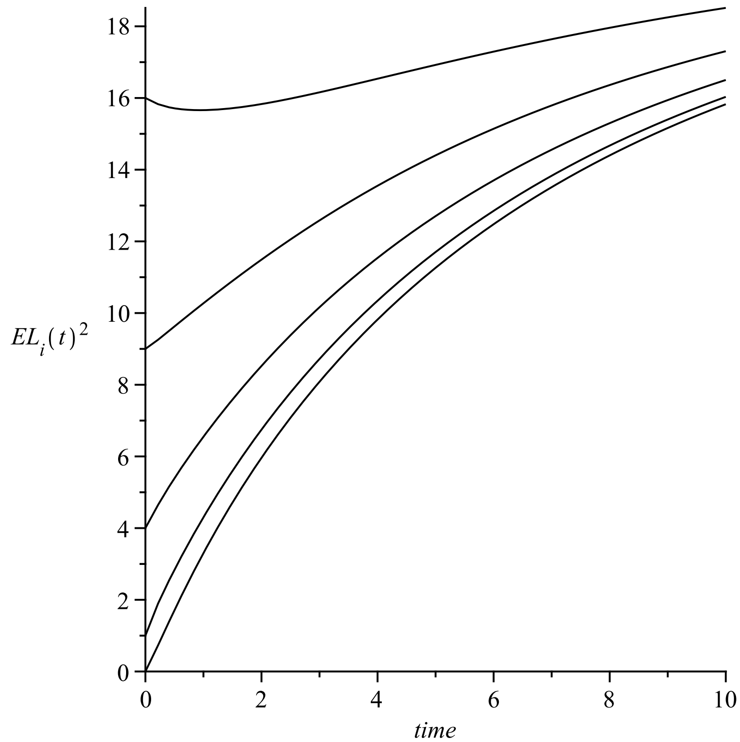

FIGURE 2.15. $EL_i^2(t)$ for $\lambda = 3$,$\mu = 4$, $0 < t < 10$

Integrate with respect to t to get

$$EL_i^2(t) = 2(\lambda - \mu) \int_0^t EL_i(t)\,dx + (\lambda + \mu)t - \mu \int_0^t p_{i0}(x)\,dx + K.$$

Take $t = 0$ to show $K = i^2$

$$\implies i^2 = EL_i^2(0) = 2(\lambda - \mu) \int_0^0 EL_i(t)\,dx + (\lambda + \mu)0 - \mu \int_0^0 p_{i0}(x)\,dx + K = K$$

$$\implies EL_i^2(t) = 2(\lambda - \mu) \int_0^t EL_i(t)\,dx + (\lambda + \mu)t - \mu \int_0^t p_{i0}(x)\,dx + i^2.$$

□

A graph of $E[L_i(t)^2]$ appears in Figure 2.15 above.

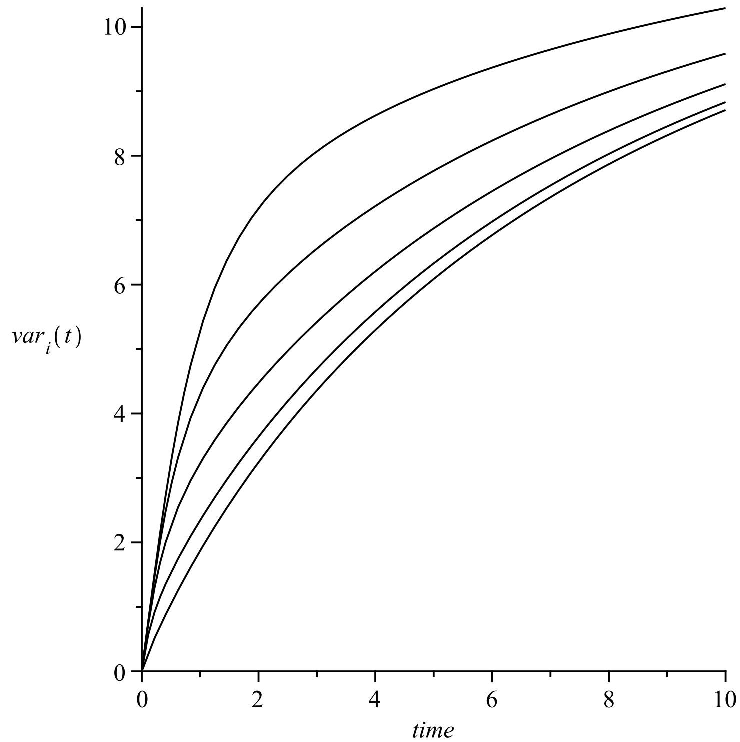

FIGURE 2.16. $Var_i[L(t)]$ for $\lambda = 3$,$\mu = 4$, $0 < t < 10$

From our results for $E[L_i(t)^2]$ and $EL_i(t)$, we can compute

$$var_i[L(t)] = E[L(i,t)^2] - (EL_i(t))^2.$$

The result appears in Figure 2.16 above. As expected, the variance is small for small t and increases as t gets larger. However, since the probabilities $p_{ij}(t) \to \pi_j$ as $t \to \infty$, we expect that the variance should approach a constant limit as $t \to \infty$. This limit should be $var(L) = E(L^2) - (E(L))^2$ where $E(L) = \sum_{j=0}^{\infty} j\pi_j$ and $E(L^2) = \sum_{j=0}^{\infty} j^2\pi_j$. The result is well known to be $var(L) = \dfrac{\rho}{(1-\rho)^2}$.

Next we consider results for $EL_i^3(t)]$.

THEOREM 2.7. *(new result) For an $M/M/1$ queueing system, with $\rho < 1$,*

$$EL_i^3(t) = 3(\lambda - \mu)\int_0^t EL_i^2(x)\,dx + 3(\lambda + \mu)\int_0^t EL_i(t)\,dx + (\lambda - \mu)t + \mu \int_0^t p_{i0}(x)\,dx + i^3. \tag{7}$$

PROOF.

$$EL_i^3(t) = \sum_{j=0}^{\infty} j^3 p_{ij}(t)$$

so,

$$\frac{d}{dt}EL_i^3(t) = \sum_{j=0}^{\infty} j^3 p'_{ij}(t).$$

Now we simplify the summands $\sum_{j=0}^{\infty} j^3 \mu p_{ij+1}(t)$ and $\sum_{j=0}^{\infty} j^3 \lambda p_{ij-1}(t)$, then

$$\sum_{j=0}^{\infty} j^3 \mu p_{ij+1}(t) = \mu \sum_{j=0}^{\infty}(j+1)^3 p_{ij+1}(t) - \mu \sum_{j=0}^{\infty} 3(j+1)^2 p_{ij+1}(t) + \mu \sum_{j=0}^{\infty} 3(j+1)p_{ij+1}(t) - \sum_{j=0}^{\infty} p_{ij+1}(t)$$
$$= \mu EL_i^3(t) - 3\mu EL_i^2(t) + 3\mu EL_i(t) - \mu(1 - p_{i0}(t))$$

and

$$\sum_{j=0}^{\infty} j^3 \lambda p_{ij-1}(t) = \lambda \sum_{j=0}^{\infty}(j-1)^3 p_{ij-1}(t) + 3\lambda \sum_{j=0}^{\infty}(j-1)^2 p_{ij-1}(t)$$

$$+ 3\lambda \sum_{j=0}^{\infty} (j-1) p_{ij-1}(t) + \lambda \sum_{j=0}^{\infty} p_{ij-1}(t)$$

$$= \lambda E L_i^3(t) + 3\lambda E L_i^2(t) + 3\lambda E L_i(t) + \lambda.$$

Thus,

$$\frac{d}{dt} E L_i^3(t) = \sum_{j=0}^{\infty} j^3 (\mu p_{ij+1}(t) + \lambda p_{ij-1}(t) - (\lambda+\mu) p_{ij}(t))$$

$$= \mu E L_i^3(t) - 3\mu E L_i^2(t) + 3\mu E L_i(t) - \mu(1 - p_{i0}(t)) + \lambda E L_i^3(t)$$

$$+ 3\lambda E L_i^2(t)$$

$$+ 3\lambda E L_i(t) + \lambda - (\lambda+\mu) E L_i^3(t)$$

$$= 3(\lambda-\mu) E L_i^2(t) + 3(\lambda+\mu) E L_i(t) + (\lambda-\mu) + \mu p_{i0}(t).$$

Integrate with respect to t to get

$$E L_i^3(t) = 3(\lambda-\mu) \int_0^t E L_i^2(x)\, dx + 3(\lambda+\mu) \int_0^t E L_i(t)\, dx + (\lambda-\mu)t + \mu \int_0^t p_{i0}(x)\, dx + K.$$

Take $t = 0$ to show $K = i^3$

$$\Longrightarrow i^3 = E L_i^3(t) = 3(\lambda-\mu) \int_0^t E L_i^2(x)\, dx + 3(\lambda+\mu) \int_0^t E L_i(t)\, dx$$

$$+ (\lambda-\mu)t + \mu \int_0^t p_{i0}(x)\, dx + K = K$$

$$\Longrightarrow E L_i^3(t) = 3(\lambda-\mu) \int_0^t E L_i^2(x)\, dx + 3(\lambda+\mu) \int_0^t E L_i(t)\, dx + (\lambda-\mu)t$$

$$+ \mu \int_0^t p_{i0}(x)\, dx + i^3.$$

□

We can now obtain a graph for the skewness of $L(i,t)$. (See Wikipedia.)

Recall that for a random variable X,

$$skewness(X) = \frac{E[(X - E[X])^3]}{(E[(X - E[X])^2])^{3/2}}.$$

Thus

$$skewness_i(L(t)) = \frac{E_i[(L(t) - E_i[L(t)])^3]}{(E_i[(L(t) - E_i[L(t)])^2])^{3/2}}.$$

First we find $\lim_{t\to 0^+} skewness_i(L(t))$ for $i > 0$ and $i = 0$.

PROPERTY 2.8. *(new result) For an $M/M/1$ queueing system, with $\lambda < \mu$,*

$$(a) \lim_{t\to 0^+} skewness_i(L(t)) = -\infty \text{ for } i > 0,$$

$$(b) \lim_{t\to 0^+} skewness_i(L(t)) = +\infty \text{ for } i = 0.$$

PROOF. (a)

$$E(X(\Delta t)) = (i-1)p_{ii-1}(\Delta t) + ip_{ii}(\Delta t) + (i+1)p_{ii+1}(\Delta t) + o(\Delta t)$$

$$= (i-1)\mu\Delta t + i(1 - \lambda\Delta - \mu\Delta t) + (i+1)(\lambda\Delta t) + o(\Delta t)$$

$$= i - (\mu - \lambda)\Delta t + o(\Delta t).$$

$$E[(X(\Delta t) - E(X(\Delta)))^2] = \sum_{j=i-1}^{i+1} (j - (1 - (\mu - \lambda)\Delta t))^2 p_{ij}(\Delta t) + o(\Delta t)$$

$$= (1-(\mu-\lambda)\Delta t)^2\mu\Delta t + ((\mu-\lambda)\Delta t)^2(1-\lambda\Delta t-\mu\Delta t)$$

$$+ (1+(\mu-\lambda)\Delta t)^2\lambda\Delta t + o(\Delta t)$$

$$= (\mu+\lambda)\Delta t + o(\Delta t).$$

$$E[(X(\Delta t)-E(X(\Delta t)))^3] = \sum_{j=i-1}^{i+1}(j-(1-(\mu-\lambda)\Delta t))^3 p_{ij}(\Delta t) + o(\Delta t)$$

$$= -(1-(\mu-\lambda)\Delta t)^3\mu\Delta t + ((\mu-\lambda)\Delta t)^2(i-\lambda\Delta t-\mu\Delta t)$$

$$+ (1+(\mu-\lambda)\Delta t)^3\lambda t + o(\Delta t)$$

$$= (-\mu+\lambda)\Delta t + o(\Delta t).$$

$$So, \quad \frac{E[X(\Delta t)-E(X(\Delta t))^3]}{(E[(X(\Delta t)-E(X(\Delta))^2])^{3/2}} = \frac{(-\mu+\lambda)\Delta t + o(\Delta t)}{(\mu+\lambda)^{3/2}\Delta t^{3/2} + o(\Delta t)}$$

$$= \frac{-\mu+\lambda+\frac{o(\Delta t)}{\Delta t}}{(\mu+\lambda)^{3/2}\Delta t^{1/2} + \frac{o(\Delta t)}{\Delta t}}$$

$$\longrightarrow -\infty \quad as \quad \Delta t \longrightarrow 0^+.$$

(b)

If $i = 0$,

$$E(X(\Delta t)) = \lambda\Delta t + o(\Delta t).$$

$$E[(X(\Delta t)-E(X(\Delta)))^2] = \sum_{j=0}^{i}(j-\lambda\Delta t)^2 p_{0j}(\Delta t) + o(\Delta t)$$

$$= (\lambda\Delta t)^2(1-\lambda\Delta t) + (1-\lambda\Delta t)^2\lambda\Delta t + o(\Delta t)$$

$$= \lambda\Delta t + o(\Delta t).$$

$$E[(X(\Delta t) - E(X(\Delta t)))^3] = \sum_{j=0}^{1} (j - \lambda\Delta t)^3 p_{ij}(\Delta t) + o(\Delta t)$$

$$= -(\lambda\Delta t)^3 \lambda\Delta t + (1 - \lambda\Delta t)^3 \lambda\Delta t + o(\Delta t)$$

$$= \lambda\Delta t + o(\Delta t).$$

$$So, \quad skewness_i(L(t)) = \frac{\lambda\Delta t + o(\Delta t)}{(\lambda\Delta t + o(\Delta t))^{3/2}}$$

$$\longrightarrow +\infty \quad as \quad \Delta t \longrightarrow 0^+.$$

□

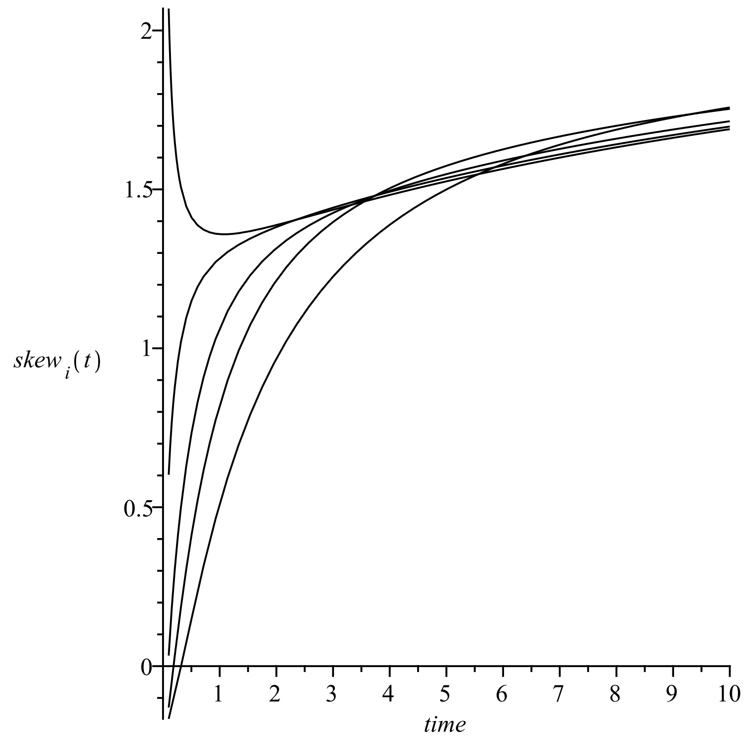

FIGURE 2.17. $skew_i[L(t)]$ for $\lambda = 3$,$\mu = 4$, $0.1 < t < 10$

Hence, we illustrate our graph on the interval (.1,10) rather than on the interval (0,10).

Skewness measures the lack of symmetry of the pdf. A pdf which is positively skewed has a long tail on the right. If the number of customers at time zero is $i = 0$, then the function must be positively skewed (and in fact is $+\infty$ according to our property). If t is large, it is possible that the number of customers is large, but the number of customers has a minimum of zero. In this case we expect the skewness to be positive, and this happens. If t is small and the initial number of customers is $i > 0$, then the algebra gives us a negative skewness for small values of t and this appears on the graph.

Van de Coevering obtains additional expressions for $EL_i(t)$, $E_i(L(t)^2)$, involving the functions $A_i(t)$ and $B_i(t)$ which are defined below. We add a function $C_i(t)$ so we can deal with $E_i(L(t)^3)$ as well. Define

$$A_i(t) = \frac{2}{\pi}\rho^{(j-i)/2} \int_0^\pi \frac{e^{-\mu t\gamma(y)}}{\gamma(y)^2} sin(iy)\sin(y)\,dy, \tag{8}$$

$$B_i(t) = \frac{2}{\pi}\rho^{(j-i)/2} \int_0^\pi \frac{e^{-\mu t\gamma(y)}}{\gamma(y)^3} sin(iy)\sin(y)\,dy, \tag{9}$$

$$C_i(t) = \frac{2}{\pi}\rho^{(j-i)/2} \int_0^\pi \frac{e^{-\mu t\gamma(y)}}{\gamma(y)^4} sin(iy)\sin(y)\,dy. \tag{10}$$

PROPERTY 2.9. *For an $M/M/1$ queue, with $\lambda < \mu$,*

$$EL_i(t) = A_i(t) - \rho A_{i+1}(t) + \rho/(1-\rho). \tag{11}$$

PROOF. Recall from (1) that

$$p_{ij}(t) = \frac{2}{\pi}\rho^{(j-i)/2} \int_0^\pi \frac{e^{-\mu t\gamma(y)}}{\gamma(y)} a_i(y)a_j(y)\,dy + \begin{cases} (1-\rho)\rho^j & \rho < 1 \\ 0 & \rho \geq 1, \end{cases}$$

for

$$\gamma(y) = 1 + \rho - 2\sqrt{\rho}\cos(y) \text{ and } a_k(y) = \sin(ky) - \sqrt{\rho}\sin((k+1)y).$$

Substituting the expression for $p_{i0}(t)$ from (1) into (3) gives:

$$\begin{aligned} EL_i(t) &= (\lambda-\mu)t + \mu\int_0^t p_{io}(x)\,dx + i \\ &= (\lambda-\mu)t + \mu\int_0^t [\frac{2}{\pi}\rho^{-i/2}\int_0^\pi \frac{\mathrm{e}^{-\mu x\gamma(y)}}{\gamma(y)}(sin(iy) \end{aligned}$$

$$- \rho^{1/2} sin((i+1)y))(-\rho^{1/2}\sin y)\,dy + 1 - \rho]\,dx + i$$

$$= (\lambda-\mu)t + i + \mu(1-\rho)t + \mu\frac{2}{\pi}\rho^{-i/2}\int_0^t\int_0^\pi \frac{\mathrm{e}^{-\mu x\gamma(y)}}{\gamma(y)}(\sin(iy)$$

$$- \rho^{1/2}\sin((i+1)y))(-\rho^{1/2}\sin y)\,dydx$$

$$= (\lambda-\mu)t + i + \mu(1-\rho)t - \mu\frac{2}{\pi}\rho^{(1-i)/2}\int_0^\pi\int_0^t \frac{\mathrm{e}^{-\mu x\gamma(y)}}{\gamma(y)}(\sin(iy)\sin y$$

$$- \rho^{1/2}\sin((i+1)y)\sin y)\,dxdy$$

$$= (\lambda-\mu)t + i + \mu(1-\rho)t - \frac{2}{\pi}\rho^{(1-i)/2}\int_0^\pi [-\frac{\mathrm{e}^{-\mu t\gamma(y)}}{\gamma^2(y)} + \frac{1}{\gamma^2(y)}]$$

$$[\sin(iy)\sin y - \rho^{1/2}\sin((i+1)y)\sin y]\,dy$$

$$= (\lambda-\mu)t + i + \mu(1-\rho)t - \frac{2}{\pi}\rho^{(1-i)/2}\int_0^\pi \frac{\mathrm{e}^{-\mu t\gamma(y)}}{\gamma^2(y)}\sin(iy)\sin y\,dy$$

$$- \frac{2}{\pi}\int_0^\pi \rho^{(2-i)/2}\frac{\mathrm{e}^{-\mu t\gamma(y)}}{\gamma^2(y)}\sin((i+1)y)\sin y\,dy - \frac{2}{\pi}\rho^{(1-i)/2}\int_0^\pi$$

$$\frac{1}{\gamma^2(y)}\sin(iy)\sin y\,dy + \frac{2}{\pi}\rho^{(2-i)/2}\int_0^\pi \frac{1}{\gamma^2(y)}\sin((i+1)y)\sin y\,dy$$

$$= A_i(t) - \rho A_{i+1}(t) - A_i(0) + \rho A_{i+1}(0) + i. \tag{12}$$

As $A_i(t) \to 0$ as $t \to \infty$ we have $EL_i(t) \to \rho/(1-\rho)$ for all i, so

$$EL_i(t) = A_i(t) - \rho A_{i+1}(t) - A_i(0) + \rho A_{i+1}(0) + i$$

$$\Rightarrow \rho/(1-\rho) = 0 - 0 - A_i 0 + \rho A_{i+1}(0) + i$$

$$\Rightarrow \rho A_{i+1}(0) - A_i(0) = \rho/(1-\rho) + i. \tag{13}$$

Substitute (12) into (11) to get

$$EL_i(t) = A_i(t) - \rho A_{i+1}(t) + \rho/(1-\rho) - i - (\lambda-\mu)t + i + (\lambda-\mu)t$$

$$= A_i(t) - \rho A_{i+1}(t) + \rho/(1-\rho).$$

□

LEMMA 2.10. *For an $M/M/1$ queue with $\rho < 1$,*

$$\lim t \to \infty, EL_i^2(t) = \frac{(\lambda\mu + \lambda^2)}{(\mu-\lambda)^2} = \frac{\rho^2+\rho}{(1-\rho)^2}.$$

PROOF. As $t \to \infty$, $EL_i^2(t) \to E(L^2)$.

$$E(L^2) = \sum_{i=0}^{\infty} i^2 \pi_i = \sum_{i=0}^{\infty} i^2 (1-\rho)\rho^i = (1-\rho)\rho \sum_{i=0}^{\infty} i^2 \rho^{i-1}$$

$$= \rho(1-\rho)[\sum_{i=0}^{\infty} i(i-1)\rho^{i-2}\rho + \sum_{i=0}^{\infty} i\rho^{i-1}]$$

$$= \sum_{i=0}^{\infty} (i^2-i)\rho^{i-2}\rho + \sum_{i=0}^{\infty} i\rho^{i-1}$$

$$= \rho(1-\rho)[\frac{2\rho}{(1-\rho)^3} + \frac{1}{(1-\rho)^2}]$$

$$= \frac{\rho^2+\rho}{(1-\rho)^2}$$

□

PROPERTY 2.11. *For an $M/M/1$ queue, with $\lambda < \mu$,*

$$EL_i^2(t) = 2(\rho - 1)[-B_i(t) + \rho B_{i+1}(t)] - EL_i(t) + \frac{2\rho}{(1-\rho)^2}. \tag{14}$$

PROOF.

$$\begin{aligned}
\int_0^t EL_i(x)\,dx &= \int_0^t A_i(x) - \rho A_{i+1}(x) - A_i(0) + \rho/(1-\rho)\,dx \\
&= \rho t/(1-\rho) + \int_0^t A_i(x) - \rho A_{i+1}(x)\,dx \\
&= \rho t/(1-\rho) + \int_0^t \frac{2}{\pi}\rho^{(1-i)/2}\int_0^\pi \frac{\mathrm{e}^{-\mu t\gamma(y)}}{\gamma^2(y)}\sin(iy)\sin y\,dydx \\
&\quad - \rho t/(1-\rho) + \int_0^t \frac{2}{\pi}\rho^{i/2}\int_0^\pi \frac{\mathrm{e}^{-\mu t\gamma(y)}}{\gamma^2(y)}\sin((i+1)y)\sin y\,dydx \\
&= \rho t/(1-\rho) - \frac{1}{\mu}(B_i(t) - B_i(0)) + \frac{\rho}{\mu}B_{i+1}(t) - \frac{1}{\mu}B_i(0) \\
&= \rho t/(1-\rho) + \frac{1}{\mu}[-Bi(t) + \rho B_{i+1}(t) + B_i(0) - \rho B_{i+1}(0)]. \qquad (15)
\end{aligned}$$

By (3),(10) and (14) get:

$$\begin{aligned}
EL_i^2(t) &= 2(\lambda-\mu)\int_0^t EL_i(x)\,dx + (\lambda+\mu)t - \mu\int_0^t p_{i0}(x)\,dx + i^2 \\
&= 2(\lambda-\mu)\frac{1}{\mu}[-B_i(t) + \rho B_{i+1}(t) + B_i(0) - \rho B_{i+1}(0)] \\
&\quad + 2(\lambda-\mu)\rho t/(1-\rho) + (\lambda+\rho)t + (\lambda-\mu)t + i - EL_i(t) + i^2 \\
&= 2(\rho-1)\frac{1}{\mu}[-B_i(t) + \rho B_{i+1}(t) + B_i(0) - \rho B_{i+1}(0)] \\
&\quad + 2(\rho-1)\frac{\rho}{1-\rho}t + 2\lambda t - EL_i(t) + i + i^2 \\
&\Rightarrow \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-(\lambda-\mu)} = \frac{\lambda/\mu}{(\mu-\lambda)/\mu} = \frac{\lambda}{\mu-\lambda}
\end{aligned}$$

$$= 2(\rho-1)\frac{1}{\mu}[-B_i(t)+\rho B_{i+1}(t)+B_i(0)-\rho B_{i+1}(0)] - 2(\rho-1)\frac{\lambda}{\lambda-\mu}t + 2\lambda t$$

$$- EL_i(t) + i + i^2$$

$$= 2(\rho-1)[-B_i(t)+\rho B_{i+1}(t)+B_i(0)-\rho B_{i+1}(0)] - EL_i(t) + i + i^2. \qquad (16)$$

Letting $t \to \infty$ and using that $EL_i^2(t) \to \frac{(\lambda\mu+\lambda^2)}{(\mu-\lambda)^2}$,

$$\frac{(\lambda\mu+\lambda^2)}{(\mu-\lambda)^2} = 2(\rho-1)[0+0+B_i(0)-\rho B_{i+1}(0)] - \frac{\rho}{1-\rho} + i + i^2$$

$$\Longleftrightarrow 2(\rho-1)[B_i(0)-\rho B_{i+1}(0)] = \frac{(\lambda\mu+\lambda^2)}{(\mu-\lambda)^2} + \frac{\rho}{1-\rho} - i - i^2 = \frac{2\lambda\mu}{(\mu-\lambda)^2} - i - i^2$$

Thus,

$$2(\rho-1)[B_i(0)-\rho B_{i+1}(t)(0)] = \frac{2\lambda\mu}{(\mu-\lambda)^2} - i - i^2. \qquad (17)$$

Substituting (16) into (15) gets:

$$EL_i^2(t) = 2(\rho-1)[-B_i(t)+\rho B_{i+1}(t)] + \frac{2\mu\lambda}{(\mu-\lambda)^2} - i - i^2 - EL_i(t) + i + i^2$$

$$= 2(\rho-1)[-B_i(t)+\rho B_{i+1}(t)] - EL_i(t) + \frac{2\rho}{(1-\rho)^2}.$$

□

LEMMA 2.12. *(new result) For an $M/M/1$ queue, with $\lambda < \mu$,*

$$lim_{t\to\infty} EL_i^3(t) = E(L^3) = \frac{\rho^3+4\rho^2+\rho}{(1-\rho)^3}.$$

PROOF.

$$
\begin{aligned}
E(L^3) &= \sum_{i=0}^{\infty} i^3 \pi_i \\
&= \sum_{i=0}^{\infty} i^3 (1-\rho)\rho^i \\
&= (1-\rho)[\rho^3 \sum_{i=0}^{\infty} i(i-1)(i-2)\rho^{i-3} + 3\rho^3 \sum_{i=0}^{\infty} i^2 \rho^{i-3} - 2\rho^3 \sum_{i=0}^{\infty} i\rho^{i-3}] \\
&= (1-\rho)[\rho^3 \sum_{i=0}^{\infty} i(i-1)(i-2)\rho^{i-3} + 3 \sum_{i=0}^{\infty} i^2 \rho^{i} - 2\rho \sum_{i=0}^{\infty} i\rho^{i}] \\
&= (1-\rho)\rho^3 \sum_{i=0}^{\infty} i(i-1)(i-2)\rho^{i-3} + 3(1-\rho) \sum_{i=0}^{\infty} i^2 \rho^{i} - 2\rho(1-\rho) \sum_{i=0}^{\infty} i\rho^{i} \\
&= (1-\rho)\rho^3 \frac{6}{(1-\rho)^4} + (1-\rho)\frac{rho^2+\rho}{(1-\rho)^3} - 2\rho(1-\rho)\frac{1}{(1-\rho)^2} \\
&= \frac{\rho^3 + 4\rho^2 + \rho}{(1-\rho)^3}
\end{aligned}
$$

□

PROPERTY 2.13. *(new result) For an* $M/M/1$ *queue, with* $\lambda < \mu$,

$$
\begin{aligned}
EL_i^3(t) = 6\mu(\rho-1)^2[-C_i(t) + \rho C_{i+1}(t)] + \frac{5\rho^3 - \rho^2 + 2\rho}{(1-\rho)^3} \\
- \frac{3}{\rho-1} EL_i^2(t) - \frac{4-\rho}{\rho-1} EL_i(t). \qquad (18)
\end{aligned}
$$

PROOF. Together with (3), (5) and (13) get,

$$
\begin{aligned}
EL_i^3(t) = 3(\lambda-\mu)\int_0^t EL_i^2(x)\,dx + 3(\lambda+\mu)\int_0^t EL_i(x)\,dx + (\lambda-\mu)t \\
+ \mu \int_0^t p_{i0}(x)\,dx + i^3
\end{aligned}
$$

$$= 3(\lambda-\mu)\int_0^t 2(\rho-1)[-B_i(x)+\rho B_{i+1}(x)] - EL_i(x) + 2\rho(1-\rho)^{-2}\,dx$$

$$+ 3(\mu+\lambda)\int_0^t EL_i(x)\,dx + (\lambda-\mu)t + EL_i(t) - (\lambda-\mu)t - i + i^3$$

$$= 6(\lambda-\mu)(\rho-1)\int_0^t [-B_i(x)+\rho B_{i+1}(x)]\,dx - 3(\lambda-\mu)\int_0^t EL_i(x)\,dx$$

$$+ 6(\lambda-\mu)\rho(1-\rho)^{-2}t + 3(\mu+\lambda)\int_0^t EL_i(x)\,dx + (\lambda-\mu)t + EL_i(t)$$

$$- (\lambda-\mu)t - i + i^3$$

$$= 6\mu(\rho-1)^2\int_0^t [-B_i(x)+\rho B_{i+1}(x)]\,dx - 6\mu\int_0^t EL(i,x)\,dx$$

$$+ 6\mu(\rho-1)\rho(1-\rho)^{-2}t + EL_i(t) - i + i^3$$

$$= 6\mu(\rho-1)^2\int_0^t [-B_i(x)+\rho B_{i+1}(x)]\,dx - 6\mu/\mu[-B_i(t)+\rho B_{i+1}(t)$$

$$+ B_i(0) - \rho B_{i+1}(0)] - 6\mu\rho t/(1-\rho) + 6\mu\rho t/(1-\rho) + EL_i(t) - i + i^3$$

$$= 6\mu(\rho-1)^2\int_0^t [-B_i(x)+\rho B_{i+1}(x)]\,dx$$

$$- \frac{3(EL_i^2(t) + EL_i(t) - 2\rho(1-\rho)^{-2})}{\rho-1} + EL_i(t) - i + i^3$$

$$= 6\mu(\rho-1)^2[-C_i(t) + C_i(0) + \rho C_{i+1}(t) - \rho C_{i+1}(0)] - \frac{3}{\rho-1}EL_i^2(t)$$

$$- \frac{4-\rho}{\rho-1}EL_i(t) - i + i^3,$$

when $t \to \infty$, the equation becomes

$$\frac{\rho^3+4\rho^2+\rho}{(1-\rho)^3} = 6\mu(\rho-1)^2[C_i(0) - \rho C_{i+1}(0)] - \frac{3}{\rho-1}\frac{\lambda\mu+\lambda^2}{(\mu-\lambda)^2}$$

$$- \frac{4-\rho}{\rho-1}\frac{\rho}{1-\rho} - i + i^3$$

$$\iff 6\mu(\rho-1)^2[C_i(0)-\rho C_{i+1}(0)] = \frac{\rho^3+4\rho^2+\rho}{(1-\rho)^3} - \frac{3\rho+3\rho^2}{(1-\rho)^2}$$

$$+ \frac{4\rho-\rho^2}{(1-\rho)^2} + i - i^3 = \frac{\rho^3+4\rho^2+\rho}{(1-\rho)^3} + \frac{-4\rho+\rho^2}{(1-\rho)^2} + i - i^3$$

$$= \frac{5\rho^3-\rho^2+2\rho}{(1-\rho)^3} + i - i^3.$$

Thus,

$$EL_i^3(t) = 6\mu(\rho-1)^2[-C_i(t)+\rho C_{i+1}(t)] + \frac{5\rho^3-\rho^2+2\rho}{(1-\rho)^3} + i - i^3 - \frac{3}{\rho-1}EL_i^2(t)$$

$$- \frac{4-\rho}{\rho-1}EL_i(t) - i + i^3$$

$$= 6\mu(\rho-1)^2[-C_i(t)+\rho C_{i+1}(t)] + \frac{5\rho^3-\rho^2+2\rho}{(1-\rho)^3}$$

$$- \frac{3}{\rho-1}EL_i^2(t) - \frac{4-\rho}{\rho-1}EL_i(t).$$

□

CHAPTER 3

Task versus Queue?

MODEL:

We have an $M/M/1$ queueing system. A customer arrives and sees i people in the system. This customer must receive service from the server and has to complete an additional task of fixed length D. The customer has two choices – perform the task first or join the queue first.

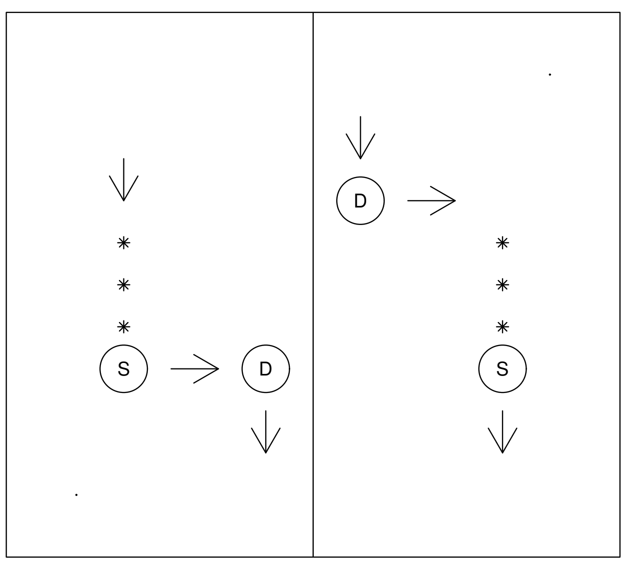

FIGURE 3.1. Which is better?

(a) Let T_{QD} be the total system time if the customer join Q first, then does the task

(b) Let T_{DQ} be the total system time if the customer join D first, then does the queue

Assume i is the number of customers initially observed.

Let $p_{ij}(t) = \text{Prob}(j$ customers at time $t|$ there are i customers at time $0)$

Let $E(T_{QD})$ be the expected system time if the queue is done first; $E(T_{DQ})$ be the expected system time if the task is done first.

This question was addressed by Hlynka and Molinaro (2010). However, no graphical analysis. A graphical analysis provides an easy way of describing the results.

THEOREM 3.1. *Given the setting above,*

(a) $$E(T_{QD}) = \frac{i+1}{\mu} + D$$

(b) $$E(T_{DQ}) = D + \frac{EL_i(D)+1}{\mu}$$

PROOF. Let X_i, $i = 1, 2, 3, \ldots$ be independent and identically distributed service time random variables.

(a) If the arriving customer joins the queue first, then the expected total time to complete the queue part is $E(\sum_{j=1}^{i+1} X_j)$ because of the memoryless property and because the arriving customer must also be served. Then the customer must complete the task with time length D. Thus

$$E(T_{QD}) = E(\sum_{j=1}^{i+1} X_i + D) = \sum_{j=1}^{i+1} E(X_i) + D = \sum_{j=1}^{i+1} \frac{1}{\mu} + D = \frac{i+1}{\mu} + D.$$

(b) If the arriving customer performs the task first, then upon completion of the task, the number of customers in the queueing system will be $L_i(D)$. Therefore, the time to complete the queue part is $\sum_{j=1}^{L_i(D)+1} X_j$. Hence,

$$EL_{DQ} = E(D + \sum_{j=1}^{L_i(D)+1} X_j) = D + E(L_i(D)+1)E(X_j) = D + \frac{E(L_i(D)+1)}{\mu}.$$

□

THEOREM 3.2. *(new result) For the setting above,*

$$E(T_{DQ}) < E(T_{QD}) \textit{ iff } EL_i(D) < i.$$

PROOF. This follows immediately from the previous theorem. □

PROCEDURE:

Suppose an arriving customer observes i customers in the $M_\lambda/M_\mu/1$ queueing system upon arrival.

(a) Draw the curve $y = E(L_i(t))$.

(b) Draw the line segment from $(0, i)$ until it intersects the curve (if it does) at (D^*, i).

(c) If the task has time length D which is less than D^*, then perform the task first (in order to minimize the expected total system time for the arriving customer.

PROOF. From the previous theorem, $E(T_{DQ}) < E(T_{QD})$ iff $EL_i(D) < i$. From Abate and Whitt (1987), the curve must have one of the three forms as illustrated by Diagram 2.11. Hence if there is an intersection point (D^*, i), then $EL_i(D) < i$ iff $D < D^*$. □

We illustrate this in Figure 3.2.

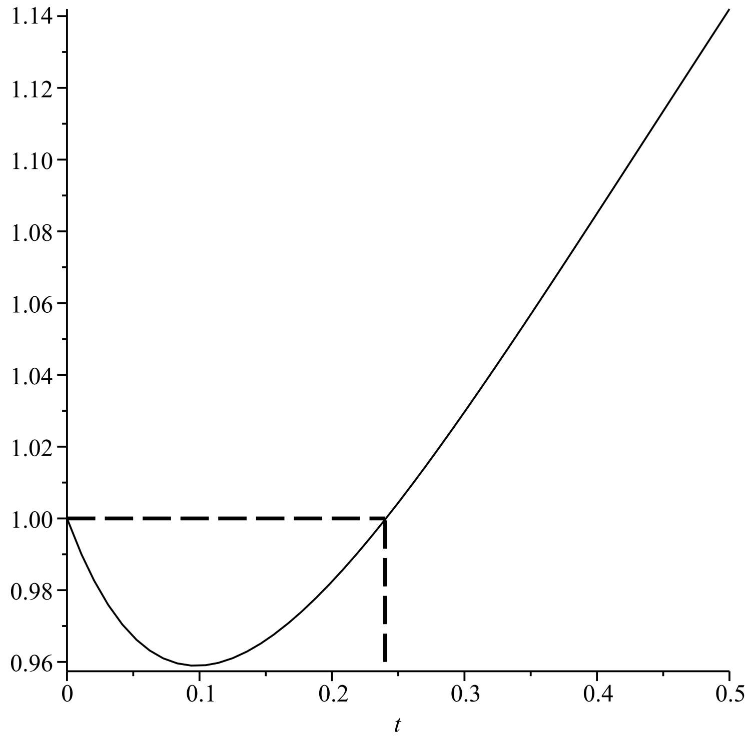

FIGURE 3.2. $EL_1(t)$ for $\lambda = 3,\mu = 4$, $0 < t < 0.5$

All the diagrams are plotted by maple.

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