Comments on Level Crossing Methods

by

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Abstract

This paper studies some topics from the book Level Crossing Methods in Stochastic Models, by P.H. Brill. The purpose of this major paper is to provide a very brief overview of the method, to add details to the presentation in the book, and present complete derivations of the level crossing method for some stochastic models. In particular, details are presented for simpler models such as $M/M/1$,$M/E_2/1$,$M/E_3/1$ and $M/M/2$, using level crossing methods.

Acknowledgements

I would like to express my gratitude to my co-supervisors Dr. Myron Hlynka and Dr. Percy H.Brill for giving me the opportunity to work on this project. Their sincerity, encouragement and guidance enabled me to develop an understanding of the subject. Dr. Hlynka provided assistance, pressure and encouragement – helping me when I needed it and leaving me on my own when I needed to figure things out by myself. Moreover, in completing this major paper, he spent lots of time helping me to correct it.

I want to thank my parents for their love, support and teaching me to think outside the box. I would not be in Canada, if it were not for them.

Last but not least, I offer my regards and blessings to all of those who supported me in any respect during the completion of the major paper.

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Introduction

1.1 Origin of Level Crossing Method

This section "presents a condensed version of the original development of the level crossing method for deriving probability distributions of state variables in stochastic models (LC)." Level crossing methods for obtaining probability distributions in stochastic models, were originated by Brill while working on his PhD thesis (1975). In his thesis, he developed LC with the more general system point method. "Thus LC is actually an essential component of the system point method. A more precise nomenclature for the overall technique is the system point level crossing method (SPLC). The analysis formulates Lindley recursions for successive customer waits and their probability distributions . The approach utilizes inequalities, conditional probabilities, and the law of total probability. It also involves multiple integration, transformation of variable, differentiation, and limit operations." (Brill, 2008)

To study the virtual wait in queues, the LC method starts by constructing a typical sample path of the virtual wait process . Then LC theorems are applied. "These theorems utilize sample-path structure to write an integral equation, or system of integral equations, for the steady-state pdf, by inspection! The LC approach can save an enormous amount of time when analyzing complex stochastic models. LC provides a common systematic procedure for studying a wide variety of stochastic models. It focuses attention on sample paths. Therefore it often leads to new insights into the model dynamics and its subtleties. In complex models, construction of a sample path may itself be a challenge. However, the benefit of this construction is that it often leads to a deeper understanding of the model."

1.1.1 Lindley Recursion

The following is from Brill(2008),

> Let W_n, S_n, T_{n+1} denote respectively the waiting time of customer n before service, the service time of customer n, and the time interval $\tau_{n+1} - \tau_n$ between the arrival instants (epochs) τ_n, τ_{n+1} of customers n and $n+1$ at the system, $n = 1, 2, ...$ The well known Lindley recursion for the waiting time is
>
> $$W_{n+1} = max\{W_n + S_n - T_{n+1}, 0\}, n = 1, 2, \ldots$$
>
> **Definition 1.1.1.**
>
> $$F_n(x) = P(W_n \leq x), x \geq 0,$$
> $$f_n(x) = \frac{d}{dx} F_n(x), x > 0,$$
> $$P_n(0) = F_n(0),$$
> $$B(y) = P(S_n \leq y), y \geq 0, n = 1, 2, \ldots,$$
> $$\bar{B}(y) = 1 - B(y), y \geq 0.$$

Where $F_n(\cdot)$ is the cdf of W_n;$f_n(\cdot)$ is the pdf on the positive part of W_n; $F_n(\infty) = P_n(0) + \int_{x=0}^{\infty} f_n(x)dx = 1, n = 1, 2, \ldots$ Assume that the input parameters of the queue are such that the steady state cdf $F(\cdot)$ and pdf $\{P_0, f(\cdot)\}$ of the wait exist, and $\lim_{n\to\infty} F_n(x) = F(x)$, $x \geq 0$, $\lim_{n\to\infty} P_n(0) = P_0$, $lim_{n\to\infty} f_n(x) = f(x)$, $x > 0$. We define $f(\cdot)$ to be right continuous. Thus $f(x^+) = f(x)$, $x > 0$. For consistency, we extend the domain of $f(\cdot)$ to include $x = 0$, and define $f(0^+) = f(0)$. Note that $f(0)$ adds zero probability to P_0.

1.2 Sample Path and System Point

"When applying the system point level crossing method (abbreviated SPLC) to analyze stochastic models, intuitive notions of sample path transitions often suffice.For some models, however, more precise notions of such transitions are useful. Pertinent sample-path transitions include downcrossings, upcrossings, and tangents of state-space levels."

From Brill (2000), "A sample path of the process $W(t)$ is a single realization of the process over time. Its value at time-point t is an outcome of the random variable W(t), say X(t). We denote an arbitrary sample path by the function $X(t), t \geq 0$, which is real-valued and right continuous on the reals. The function X has jump or removable discontinuities on a sequence of strictly increasing time points $\tau_n, n = 0, 1, ...$, where $\tau_0 = 0$ without loss of generality. Ordinarily, the time points τ_n may represent input or output epochs of the content in dams, arrival epochs of customers in queues, or demand or replenishment epochs of stock-on-hand in inventories, etc. Assume that a sample path decreases continuously on time segments between jump points, described by $dX(t)/dt = -r(X(t))$, $\tau_n \leq t < \tau_{n+1}, n = 0, 1, ...$ wherever the derivative exists, and where $r(x) \geq 0$ for all $x \in (-\infty, \infty)$. Note for example, that for the standard virtual wait process in queues, the state space is $[0, \infty), r(x) = 1(x > 0)$ and r(0) = 0. In an $< s, S >$ continuous review inventory system with no lead time or backlogging, and the stock on hand decays continuously at constant rate $k \geq 0$, $r(x) = k$ for all $x \in (s, S]$. Here $s \geq 0$ is the reorder point and S is the order up-to-level. If there is a lead time and backlogging is allowed, the state space is $(-\infty, S]$ and $r(x) = 0$ for $x < s$." The following diagram for the virtual wait of an M/G/1 queueing system illustrates jump upcrossings and continuous downcrossings.

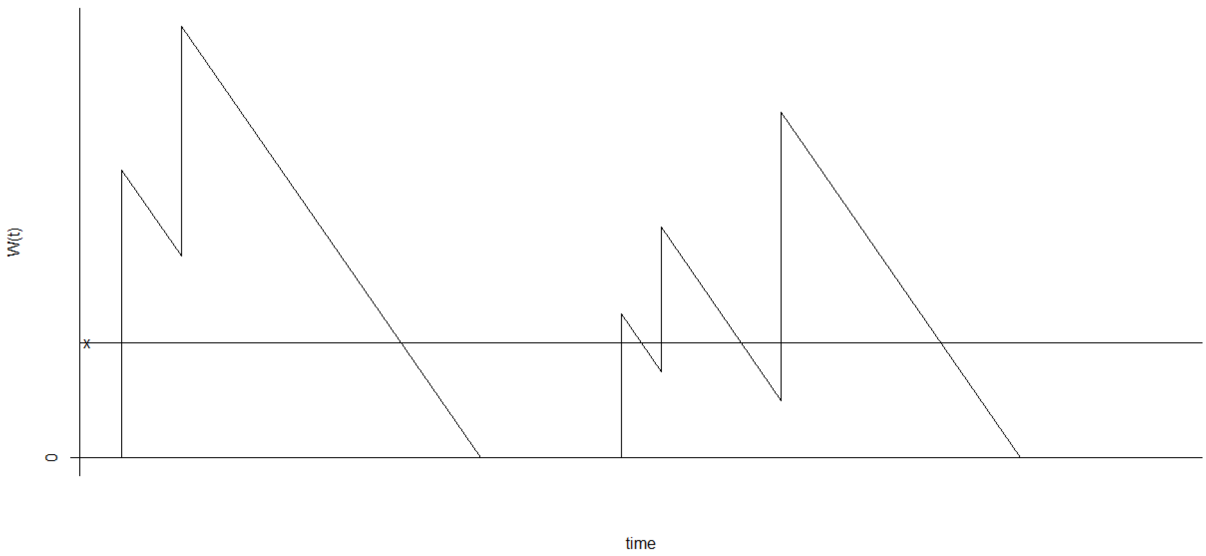

Figure 1.1: Sample path of $W(t), t \geq 0$ indicating of level x

1.3 Level Crossings by Sample Paths

As in Brill (2000), "it is sufficient to consider two types of level crossings from an intuitive viewpoint: continuous and jump level crossings. A continuous downcrossing of level x occurs at a time point $t_0 > 0$ if $\lim_{t \to t_{0^-}} X(t) = x$ and $X(t) > x$ and is monotone decreasing for all t in a small time interval ending at t_0. Intuitively, one may visualize the sample path as decreasing continuously to level x from above and just reaching level x at the instant t_0. A jump downcrossing of level x occurs at a time point $t_0 > 0$ if $\lim_{t \to t_{0^-}} X(t) > x$ and$X(t_0) \leq x$.

Intuitively, one may visualize the sample path as moving strictly above level x for all t in a small time interval ending at t_0, and then jumping vertically downward to a level below x, or to x itself, at the instant t_0."

There are "two basic level crossing theorems which greatly assist in setting up an integral equation for the steady state pdf g. The results will be stated separately for sample-path downcrossings and sample-path upcrossings." Combining "these results with a conservation law for level crossings, to construct the desired integral equation for g."

From Brill (2000) "Consider a continuous review $\langle s, S \rangle$ inventory system where $s \geq 0$ is the reorder point and S is the order-up-to level. Assume that demands for stock occur at a Poisson rate λ and demand sizes are iid (independent and identically distributed) exponential random variables with mean $\frac{1}{\mu}$. Assume that the stock decays at constant rate $k \geq 0$ when the stock is in the state-space interval $(s, S]$ and there is no lead time. The ordering policy is: If the stock either decays continuously to, or jumps downward below or to level s, then an order is placed and received immediately, replenishing the stock up to level S. It is required to derive the steady state pdf g of the stock on hand."

1.3.1 Downcrossings

From Brill (2000),

"Let $D_t^c(x)$ denote the total number of continuous downcrossings of level x and $D_t^j(x)$, the number of jump downcrossings of level x during $(0, t)$ due to the external Poisson rate λ_d. The following result holds."

Theorem 1.3.1. *Brill (1975, 2000), for r(x)=1*

With probability 1

$$\lim_{t \to \infty} D_t^c(x)/t = r(x)g(x) \tag{1.3.1}$$

$$\lim_{t \to \infty} D_t^j(x)/t = \lambda_d \int_{y=x}^{\infty} \bar{B}_d(y-x)g(y)dy \tag{1.3.2}$$

1.3.2 Upcrossings

From Brill (2000),

"Let $U_t^j(x)$ denote the total number of upcrossings of level x during $(0, t)$ due to the external Poisson rate λ_u. In the present model, these will be jump upcrossings."

Theorem 1.3.2. *(Brill, 1975, 2000)*

With probability 1

$$\lim_{t \to \infty} U_t^j(x)/t = \lambda_u \int_{y=-\infty}^{x} \bar{B}_u(x-y)g(y)dy \tag{1.3.3}$$

1.3.3 Simulation

In this section, we do some simulations for sample path of virtual wait, the following figure shows up. And we find the histogram of them.

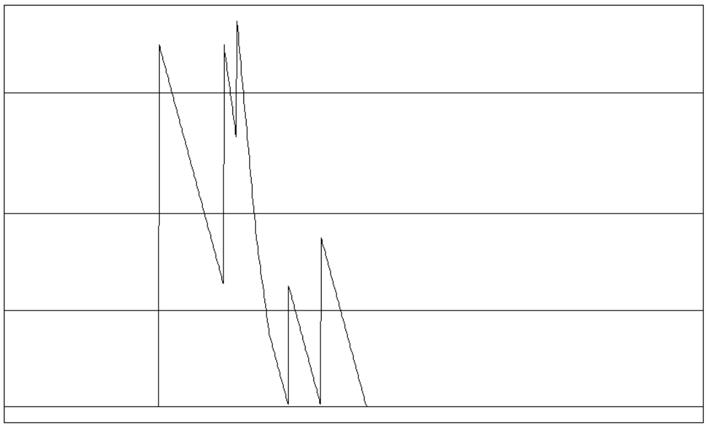

Figure 1.2: simulation

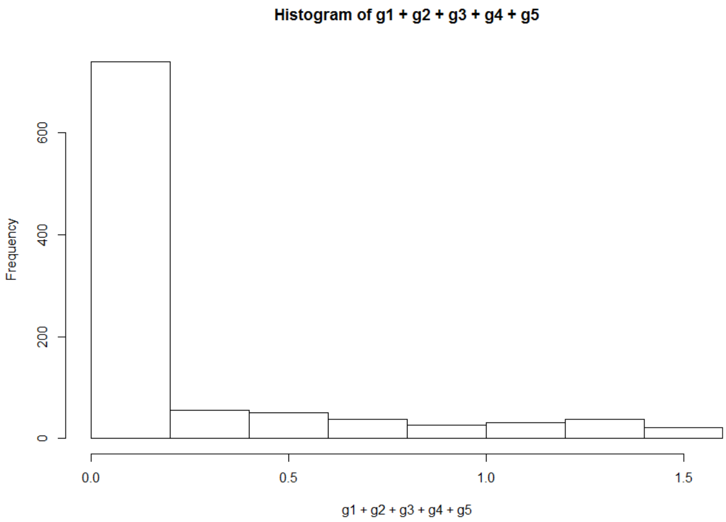

Figure 1.3: histogram

1.3.4 A Conservation Law and an Integral Equation

The following is taken directly from Brill (2000). For every state-space level x and every sample path, the following conservation law holds. In the long run,

Total downcrossing rate = Total upcrossing rate.

The foregoing conservation law applies to typical sample paths and every state-space level x. It enables the analyst to write down an integral equation for the pdf g in which every term has a precise mathematical interpretation as a long-run rate of sample-path crossings of levels. Thus direct substitution into the above conservation law gives

$$\lim_{t\to\infty} D_t^c(x)/t + \lim_{t\to\infty} D_t^j(x)/t = \lim_{t\to\infty} U_t^j(x)/t \tag{1.3.4}$$

Then, substituting from (1.3.1)(1.3.2)and(1.3.3) immediately enables us to write down the following integral equation for the pdf g. For all x

$$r(x)g(x) + \lambda_d \int_{y=x}^{\infty} \bar{B}_d(y-x)g(y)dy = \lambda_u \int_{y=-\infty}^{x} \bar{B}_u(x-y)g(y)dy \tag{1.3.5}$$

Chapter 2

Stock Inventory Example

2.1 Example

This section,we fill the full details of derivation of a solving stock inventory example form Brill (2000).

Example 2.1.1. "Consider a continuous review $\langle s, S \rangle$ inventory system where $s \geq 0$ is the reorder point and S is the order-up-to level. Assume that demands for stock occur at a Poisson rate λ and demand sizes are iid (independent and identically distributed) exponential random variables with mean $\frac{1}{\mu}$. Assume that the stock decays at constant rate $k \geq 0$ when the stock is in the state-space interval $(s, S]$ and there is no lead time. The ordering policy is: If the stock either decays continuously to, or jumps downward below or to level s, then an order is placed and received immediately, replenishing the stock up to level S. It is required to derive the steady state pdf g of the stock on hand."

Solution: We may specialize the results for the general model to this inventory model. "Now, the state space is essentially reduced to $(s, S]$, r(x)=k, $\lambda_d = \lambda$, and

$\lambda_u = 0$. Although $\lambda_u = 0$,the ordering policy ensures that upward jumps – all of them up to level S – occur whenever the stock falls to level s or below s. Upcrossings (for any level x) occur whenever a downcrossing of level s occurs. This can happen in two ways (continuous decay or jump demand). The rate at which it decays to level s is kg(s). The rate at which it jumps below level s due to demands is $\lambda \int_{y=s}^{S} e^{-\mu(y-s)} g(y)dy$. Here λ is the demand rate and $g(y)$ is the density of being at level y. In order to downcross level s, the demand size must be at least $y-s$. Integrating over all possible cases gives our expression for jump downcrossings.

Next consider a fixed level $x \in (s, S]$. As in the previous case the continuous downcrossing rate is k times the density of begin at level x. A jump downcrossing occurs if the inventory level is in $(x, S]$, and a demand occurs (with rate λ) and the demand is at least $y - x$. Integrating over all cases gives the jump downcrossing rate. See diagram. The total downcrossing rate of level x is given by

$$kg(x) + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy$$

.

From the immediately preceding discussion, the total upcrossing rate of every level $x \in (s, S]$,is precisely equal to the total downcrosssing rate of the reorder point, level s. Applying the conservation law for level crossings yields the desired integral equation for g, namely for all $x \in (s, S]$."

$$kg(x) + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy = kg(s) + \lambda \int_{y=s}^{S} e^{-\mu(y-s)} g(y)dy \qquad (2.1.1)$$

.

"Since all the probability for the stock-on-hand is concentrated on $(s, S]$, the normalizing condition is"

$$\int_{x=s}^{S} g(x)dx = 1$$

.

We next explain the following lines from Brill (2000).

"By solving the equations,we get g as"

$$g(x) = A(1 + (\frac{\lambda}{\mu\kappa})e^{(-(\frac{\lambda}{\kappa}+\mu)(S-x))})$$

and

$$1/A = (S - s) + \frac{\lambda/(\kappa\mu)}{\mu + \lambda/\kappa}(1 - e^{(-(\frac{\lambda}{\kappa}+\mu)(S-s))})$$

Proof. Note that the downward rate at level s (which results from either a continuous or a jump downcrossing) must equal the downward rate at level S (which can only result from a continuous downcrossing). This follows since each downcrossing at s immediately results in the inventory level jumping to S. Thus we know, using (2.1.1),

$$kg(s) + \lambda \int_{y=s}^{S} e^{-\mu(y-s)} g(y)dy = kg(S) \quad (2.1.2)$$

. Combining (2.1.1)and (2.1.2), we get

$$kg(x) + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy = kg(S) \quad (2.1.3)$$

so

$$\lambda \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy = kg(S) - kg(x) \quad (2.1.4)$$

Then, we take the derivative of (2.1.3) with respect to x, we have

$$kg'(x) - \lambda g(x) + \mu(\lambda \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy) = 0 \tag{2.1.5}$$

Substitute (2.1.4) into (2.1.5), we get

$$\begin{aligned} kg'(x) - \lambda g(x) + \mu[kg(S) - kg(x)] &= 0 \\ kg'(x) - (\lambda + \mu k)g(x) + \mu k g(S) &= 0 \end{aligned}$$

This is a first order nonhomogeneous differential equation. Using integrating by factors, we get the solution of $g(x)$ is given by

$$\begin{aligned} g(x) &= \frac{\int_x^S e^{-(\frac{\lambda}{\kappa}+\mu)x}(-\mu g(S))dx}{e^{-(\frac{\lambda}{\kappa}+\mu)x}} \\ &= -e^{(\frac{\lambda}{\kappa}+\mu)x}\mu g(S) \int e^{-(\frac{\lambda}{\kappa}+\mu)x}dx \\ &= -\mu g(S)e^{(\frac{\lambda}{\kappa}+\mu)x}[-\frac{e^{-(\frac{\lambda}{\kappa}+\mu)x}}{\frac{\lambda}{\kappa}+\mu} + C] \\ &= \frac{\mu g(S)}{\frac{\lambda}{\kappa}+\mu} - C\mu g(S)e^{(\frac{\lambda}{\kappa}+\mu)x} \end{aligned}$$

Then,we need to solve for C,let $x = S$

$$\begin{aligned} g(S) &= \frac{\mu g(S)}{\frac{\lambda}{\kappa}+\mu} - C\mu g(S)e^{(\frac{\lambda}{\kappa}+\mu)S} \\ C &= -\frac{1}{\mu}\frac{\lambda/\kappa}{\lambda/\kappa+\mu}e^{-(\frac{\lambda}{\kappa}+\mu)S} \end{aligned}$$

Thus, we get the following,

$$
\begin{aligned}
g(x) &= \frac{\mu g(S)}{\frac{\lambda}{\kappa}+\mu} + \frac{\lambda/\kappa}{\lambda/\kappa+\mu} g(S) e^{-(\frac{\lambda}{\kappa}+\mu)(S-x)} \\
&= \frac{\mu g(S)}{\frac{\lambda}{\kappa}+\mu}[1 + \frac{\lambda}{\kappa\mu} e^{-(\frac{\lambda}{\kappa}+\mu)(S-x)}]
\end{aligned}
$$

And we know the condition $\int_s^S g(x)dx = 1$, then we use this to solve for $g(S)$,

$$
\begin{aligned}
&\int_s^S \frac{\mu g(S)}{\frac{\lambda}{\kappa}+\mu}[1 + \frac{\lambda}{\kappa\mu} e^{-(\frac{\lambda}{\kappa}+\mu)(S-x)}]dx = 1 \\
&\frac{\mu g(S)}{\frac{\lambda}{\kappa}+\mu}(S-s) + \frac{\lambda/\kappa}{\lambda/\kappa+\mu} g(S) \int_s^S e^{-(\frac{\lambda}{\kappa}+\mu)(S-x)}dx = 1 \\
&\frac{\mu g(S)}{\frac{\lambda}{\kappa}+\mu}(S-s) + \frac{\lambda/\kappa}{(\lambda/\kappa+\mu)^2} g(S)[1 - e^{-(\frac{\lambda}{\kappa}+\mu)(S-s)}] = 1 \\
&\frac{\mu g(S)}{\frac{\lambda}{\kappa}+\mu}\{(S-s) + \frac{\lambda/\kappa\mu}{\lambda/\kappa+\mu}[1 - e^{-(\frac{\lambda}{\kappa}+\mu)(S-s)}]\} = 1
\end{aligned}
$$

Now, we consider $\frac{\mu g(S)}{\frac{\lambda}{\kappa}+\mu} = A$

$$
1/A = (S-s) + \frac{\lambda/\kappa\mu}{\lambda/\kappa+\mu}[1 - e^{-(\frac{\lambda}{\kappa}+\mu)(S-s)}] \tag{2.1.6}
$$

□

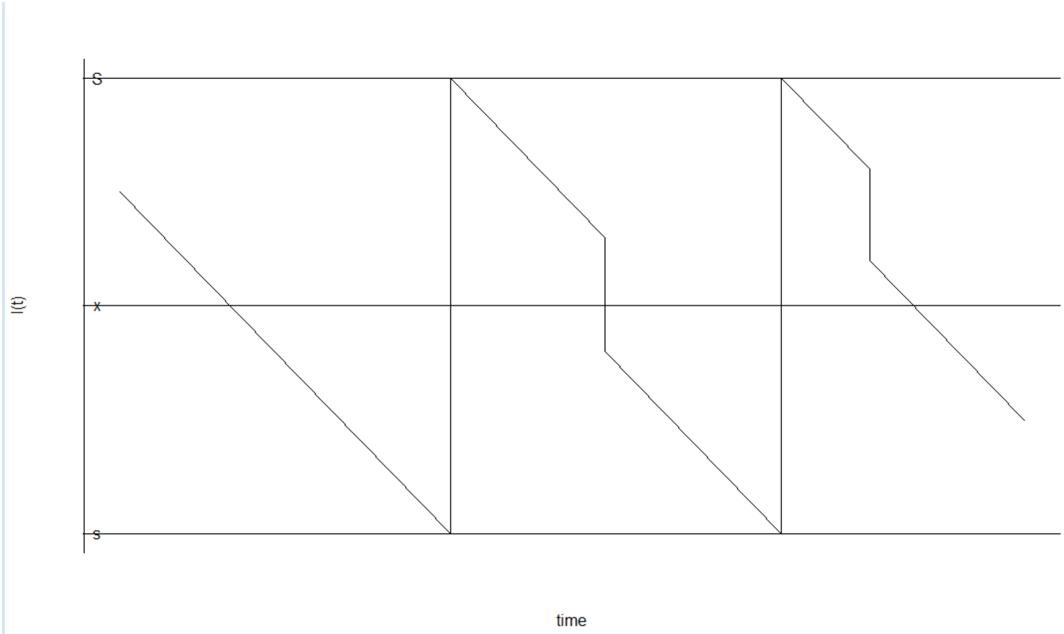

Figure 2.1: Stock inventory

Chapter 3

M/G/1 Queue

This chapter considers the virtual wait process in M/G/1 queues. There is a basic Level Crossing theorem for the steady-state pdf of wait in M/G/1 queues which is defined by Brill(2008). Then we use this theorem to illustrate LC analyses of $M/M/1$ and $M/E_k/1$ models in the steady state distribution of wait.

Theorem 3.0.2. *For an $M/G/1$ queue with arrival rate λ and service time S having cdf $B(\cdot)$, where $\lambda E(S) < 1$, the steady state pdf of the virtual wait $\{P_0; f(x), x > 0\}$, is given by*

$$f(x) = \lambda \bar{B}(x) P_0 + \lambda \int_0^x \bar{B}(x-y) f(y) dy, x > 0, \tag{3.0.1}$$

where $f(x)$ is the downcrossing rate of level x, $\lambda \bar{B}(x) P_0$ is upcrossing rate of level x, from level 0 and $\lambda \int_0^x \bar{B}(x-y) f(y) dy$ is upcrossing rate of level x, from levels in $(0, x)$.

$$f(0) = \lambda P_0, \tag{3.0.2}$$

$$P_0 + \int_0^\infty f(y) dy = 1. \tag{3.0.3}$$

3.1 $M/E_k/1$ Queue

Now using the above theorem, we consider a particular case, namely, an $M/E_k/1$ queue with arrival rate λ and service time S having pdf

$$b(x) = e^{-\mu x}\frac{(\mu x)^k \mu}{k!},$$

$x > 0$, $\mu > 0$, and $\lambda < \frac{\mu}{k}$.

The cdf of the service time is

$$B(x) = \int_{y=0}^{x} e^{-\mu y}\frac{(\mu y)^k \mu}{k!}dy$$

and the complementary cdf is

$$1 - B(x) = e^{-\mu x}(\sum_{i=0}^{k-1}\frac{(\mu x)^i}{i!}), x \geq 0.$$

Substituting into (3.0.1), the integral equation for the steady-state pdf of wait, is

$$f(x) = \lambda P_0 e^{-\mu x}(\sum_{i=0}^{k-1}\frac{(\mu x)^i}{i!}) + \lambda \int_{y=0}^{x} e^{-\mu(x-y)}(\sum_{i=0}^{k-1}\frac{(\mu(x-y))^i}{i!})f(y)dy, x > 0 \quad (3.1.1)$$

where $P_0 = 1 - \lambda E(S) = 1 - \frac{k\lambda}{\mu}$.

Note that P_0 is the probability that the system is in state 0, λ is the arrival rate, and $e^{-\mu x}(\sum_{i=0}^{k-1}\frac{(\mu x)^i}{i!})$ is the probability that a jump from level 0 exceeds x.

3.1.1 $M/E_2/1$

In this section we set k=2 in (3.1.1),which becomes a $M/E_2/1$ queue. The integral equation for $f(x)$ is

$$f(x) = \lambda P_0 e^{-\mu x}(1+\mu x) + \lambda \int_{y=0}^{x} e^{-\mu(x-y)}(1+\mu(x-y))f(y)dy \tag{3.1.2}$$

Differentiating (3.1.2) with respect to x twice results in the second order differential equation, we get

$$f'(x) = -\lambda\mu P_0 e^{-\mu x}\mu x + \lambda f(x) + \lambda \int_{y=0}^{x} -\mu^2(x-y)e^{-\mu(x-y)}f(y)dy$$

$$f''(x) = -\lambda\mu^2 P_0 e^{-\mu x} + \lambda\mu^3 x P_0 e^{-\mu x} + \lambda f'(x) + \lambda \int_{y=0}^{x} [-\mu^2 e^{-\mu(x-y)} + \mu^3(x-y)e^{-\mu(x-y)}]f(y)dy$$

Now multiply $f(x)$ by μ^2, we have

$$\mu^2 f(x) = \lambda\mu^2 P_0 e^{-\mu x}(1+\mu x) + \lambda \int_{y=0}^{x} \mu^2 e^{-\mu(x-y)}(1+\mu(x-y))f(y)dy$$

Then multiply $f'(x)$ by $-\mu$, we have

$$-\mu f'(x) = \lambda\mu^3 x P_0 e^{-\mu x} - \lambda\mu f(x) + \lambda \int_{y=0}^{x} \mu^3(x-y)e^{-\mu(x-y)}f(y)dy$$

Thus, combining those equations, we can get the following results,

$$f''(x) = 2[-\mu f'(x) + \lambda\mu f(x)] - \mu^2 f(x) + \lambda f'(x)$$

$$= (\lambda - 2\mu)f'(x) + (2\lambda - \mu)\mu f(x)$$

$$f''(x) + (2\mu - \lambda)f'(x) + (\mu^2 - 2\lambda\mu)f(x) = 0, \tag{3.1.3}$$

This is a second order homogeneous differential equation, let $f(x) = e^{rt}$, then

$f'(x) = re^{rt}$, $f''(x) = r^2e^{rt}$, substituting into (3.1.3), we get

$$r^2 + (2\mu - \lambda)r + (\mu^2 - 2\lambda\mu) = 0$$

$$r = \frac{(\lambda - 2\mu) \pm \sqrt{(2\mu - \lambda)^2 - 4\mu(\mu - 2\lambda)}}{2} = \frac{(\lambda - 2\mu) \pm \sqrt{\lambda^2 + 4\lambda\mu}}{2}$$

Then,

$$f(x) = a_1e^{r_1x} + a_2e^{r_2x} \tag{3.1.4}$$

We know the condition (3.0.2), so

$$f(0) = \lambda P_0 = \lambda(1 - \frac{2\lambda}{\mu})$$

$$f(0) = a_1 + a_2 = \lambda(1 - \frac{2\lambda}{\mu})$$

and by (3.0.3), we have

$$\int_{y=0}^{\infty} f(y)dy = 1 - P_0 = \frac{2\lambda}{\mu}$$

$$\int_{x=0}^{\infty}(a_1e^{r_1x} + a_2e^{r_2x})dx = \frac{2\lambda}{\mu}$$

$$(\frac{a_1}{r_1}e^{r_1x} + \frac{a_2}{r_2}e^{r_2x}) \mid_{x=0}^{\infty} = 1 - P_0$$

$$(0 - \frac{a_1}{r_1}) + (0 - \frac{a_2}{r_2}) = 1 - P_0$$

$$-\frac{a_1r_2 + a_2r_1}{r_1r_2} = 1 - P_0$$

$$a_2 = \lambda P_0 - a_1$$

$$a_1 = \frac{r_1r_2(P_0 - 1 - \frac{\lambda}{r_2}P_0)}{r_2 - r_1}$$

So our final expression (3.1.4) now has all of its components.

3.1.2 $M/E_3/1$

Now we set k=3 in (3.1.1),which becomes a $M/E_3/1$ queue. The integral equation for $f(x)$ is

$$f(x) = \lambda P_0 e^{-\mu x}(1 + \mu x + \frac{\mu^2 x^2}{2}) + \lambda \int_{y=0}^{x} e^{-\mu(x-y)}(1 + \mu(x-y) + \frac{1}{2}\mu^2(x-y)^2)f(y)dy$$

Differentiating $f(x)$ with respect to x three times results in the third order differential equation, we get the following equations and then solve these equations.

$$f'(x) = -\frac{1}{2}\lambda\mu^3 x^2 P_0 e^{-\mu x} + \lambda f(x) - \lambda \int_{y=0}^{x} \frac{1}{2}\mu^3(x-y)^2 e^{-\mu(x-y)} f(y)dy$$

$$\begin{aligned} f''(x) =& \frac{1}{2}\lambda\mu^4 x^2 P_0 e^{-\mu x} - \lambda\mu^3 x P_0 e^{-\mu x} + \lambda f'(x) \\ & - \lambda \int_{y=0}^{x} [-\frac{1}{2}\mu^4(x-y)^2 e^{-\mu(x-y)} + \mu^3(x-y)e^{-\mu(x-y)}]f(y)dy \\ f'''(x) =& -\frac{1}{2}\lambda\mu^5 x^2 P_0 e^{-\mu x} + 2\lambda\mu^4 x P_0 e^{-\mu x} - \lambda\mu^3 P_0 e^{-\mu x} + \lambda f''(x) \\ & - \lambda \int_{y=0}^{x} [\frac{1}{2}\mu^5(x-y)^2 e^{-\mu(x-y)} - 2\mu^4(x-y)e^{-\mu(x-y)} + \mu^3 e^{-\mu(x-y)}]f(y)dy \end{aligned}$$

Now we multiply $f(x)$ by μ^3 , multiply $f'(x)$ by μ^2 and multiply $f''(x)$ by $-\mu$, we get the following equations,

$$\mu^3 f(x) = \lambda\mu^3 P_0 e^{-\mu x}(1+\mu x+\frac{\mu^2 x^2}{2})+\lambda \int_{y=0}^{x} \mu^3 e^{-\mu(x-y)}(1+\mu(x-y)+\frac{1}{2}\mu^2(x-y)^2)f(y)dy$$

$$\mu^2 f'(x) = -\frac{1}{2}\lambda\mu^5 x^2 P_0 e^{-\mu x} + \lambda\mu^2 f(x) - \lambda \int_{y=0}^{x} \frac{1}{2}\mu^5(x-y)^2 e^{-\mu(x-y)} f(y)dy$$

$$
\begin{aligned}
-\mu f''(x) = & -\frac{1}{2}\lambda\mu^5 x^2 P_0 e^{-\mu x} + \lambda\mu^4 x P_0 e^{-\mu x} - \lambda\mu f'(x) \\
& + \lambda \int_{y=0}^{x} [-\frac{1}{2}\mu^5 (x-y)^2 e^{-\mu(x-y)} + \mu^4 (x-y) e^{-\mu(x-y)}] f(y) dy
\end{aligned}
$$

Combining these equations, we can get

$$f'''(x) = [-\mu f''(x) + \lambda\mu f'(x)] + [\mu^2 f'(x) - \lambda\mu^2 f(x)] + \mu^3 f(x)$$

$$f'''(x) + (\mu - \lambda) f''(x) + (-\mu^2 - \lambda\mu) f'(x) + (\lambda\mu^2 - \mu^3) f(x) = 0$$

This is a third order homogeneous differential equation, let $f(x) = e^{rt}$, then $f'(x) = re^{rt}$, $f''(x) = r^2 e^{rt}$, we have

$$r^3 + (\mu - \lambda) r^2 + (-\mu^2 - \lambda\mu) r + (\lambda\mu^2 - \mu^3) = 0$$

Since this is a cubic equation and it has λ and μ, it is awesome to solve, thus we take $\lambda = 1, \mu = 4$ to make this simple to solve, and then we get

$$r^3 + 3r^2 - 20r - 48 = 0$$

The following is performs the calculation for solving the cubic equation. This method is taken from H. S. Hall and S. R. Knight(2007)

Firstly, let $r = x - 1$, then

$$(x-1)^3 + 3(x-1)^2 - 20(x-1) - 48 = 0$$

$$x^3 - 23x - 26 = 0$$

Secondly, let $x = y + z$, then

$$x^3 = y^3 + z^3 + 3yz(y + z) = y^3 + z^3 + 3yzx$$

Finally, we have this equation

$$y^3 + z^3 + (3yz - 23)x - 26 = 0$$

From this, we get

$$3yz - 23 = 0$$

$$y^3 + z^3 = 26$$

$$y^3 z^3 = \frac{23^3}{27}$$

$$u^2 - 26u + \frac{23^3}{27} = 0,$$

and y^3, z^3 are the roots for u, at last we get

$$u = 13 \pm \sqrt{13^2 - \frac{23^3}{27}}$$

That is $y^3 = 13 + \sqrt{13^2 - \frac{23^3}{27}}$ and $z^3 = 13 - \sqrt{13^2 - \frac{23^3}{27}}$. Since $13^2 - \frac{23^3}{27}$ is an negative value, so the roots for y^3 and z^3 are complex values. For solving the complex roots, we have the following methods, recall

$$r^n = \alpha + i\beta = R[\cos(\theta) + i\sin(\theta) = Re^{i(\theta + 2k\pi)},$$

therefore,

$$r = R^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}}, k = 0, 1, 2, \ldots, n.$$

where

$$R = \sqrt{\alpha^2 + \beta^2}, \tan(\theta) = \frac{\beta}{\alpha}$$

Since the calculation is very awesome, so we haven't presented here to solve for y and z, then, go back to the original $r = y+z-1$. This solution actually contains 9 possible values since each cube root corresponds to 3 values. Of course, there are actually only 3 roots for $r^3 + 3r^2 - 20r - 48 = 0$. If we examine this equation graphically, we see that all three roots are real. Call them r_1, r_2, r_3. See graph.

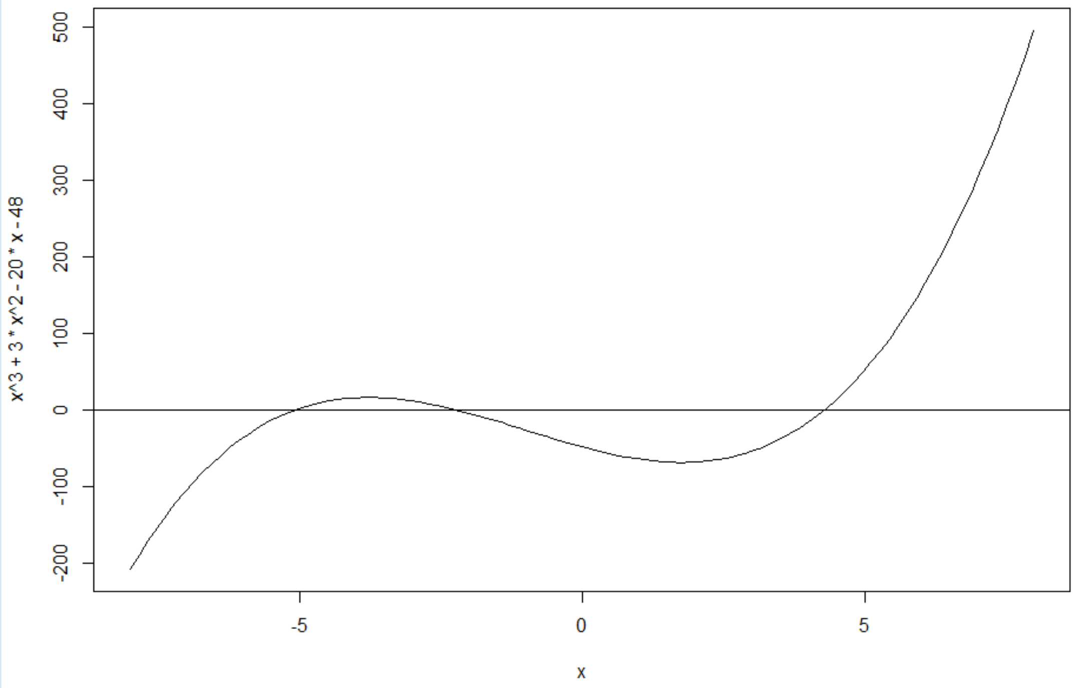

Figure 3.1: graph of $r^3 + 3r^2 - 20r - 48 = 0$

The final expression is

$$f(x) = ae^{r_1 x} + be^{r_2 x} + ce^{r_3 x}.$$

Now we find that if r_1 is positive then our condition don't satisfy. Therefore, $a = 0$. By (3.0.2), so

$$f(0) = \lambda P_0 = \lambda(1 - \frac{3\lambda}{\mu})$$

$$f(0) = b + c = \lambda(1 - \frac{3\lambda}{\mu})$$

and by (3.0.3), we have

$$\int_{y=0}^{\infty} f(y)dy = 1 - P_0 = \frac{3\lambda}{\mu}$$

$$\int_{x=0}^{\infty} (be^{r_2 x} + ce^{r_3 x})dx = \frac{3\lambda}{\mu}$$

$$(\frac{b}{r_2}e^{r_2 x} + \frac{c}{r_3}e^{r_3 x}) \mid_{x=0}^{\infty} = 1 - P_0$$

$$(0 - \frac{b}{r_2}) + (0 - \frac{c}{r_3}) = 1 - P_0$$

$$-\frac{br_3 + cr_2}{r_2 r_3} = 1 - P_0$$

$$c = \lambda P_0 - b$$

$$b = \frac{r_2 r_3 (P_0 - 1 - \frac{\lambda}{r_3} P_0)}{r_3 - r_2}$$

So our final expression now has all of its components.

3.2 M/M/1

We now derive some steady-state results for the standard $M/M/1$ queue with FCFS discipline. The cdf of service time is

$$B(x) = 1 - e^{-\mu x}, x \geq 0$$

So,

$$1 - B(x) = e^{-\mu x}$$

Substituting into (3.0.1), we obtain

$$f(x) = \lambda P_0 e^{-\mu x} + \lambda \int_{y=0}^{x} e^{-\mu(x-y)} f(y) dy$$

Differentiating both sides with respect to x, we get

$$f'(x) = -\lambda \mu P_0 e^{-\mu x} + \lambda f(x) + \lambda \int_{y=0}^{x} -\mu e^{-\mu(x-y)} f(y) dy$$

Combing these two equations, we get

$$f'(x) = \lambda f(x) - \mu f(x)$$

$$f'(x) + (\mu - \lambda) f(x) = 0$$

This is a first order homogeneous differential equation, let $f(x) = Ae^{-(\mu-\lambda)x}$,substitute to the equation,then

$$Ae^{-(\mu-\lambda)x} = \lambda P_0 e^{-\mu x} + \lambda \int_{y=0}^{x} e^{-\mu(x-y)} Ae^{-(\mu-\lambda)y} dy$$

$$Ae^{-(\mu-\lambda)x} = \lambda P_0 e^{-\mu x} + A\lambda e^{-\mu x} \int_{y=0}^{x} e^{\lambda y} dy$$

$$Ae^{-(\mu-\lambda)x} = \lambda P_0 e^{-\mu x} + Ae^{-\mu x}(e^{\lambda x} - 1)$$

$$Ae^{-\mu x} = \lambda P_0 e^{-\mu x}$$

$$A = \lambda P_0$$

$$P_0 = 1 - \frac{\lambda}{\mu}$$

Finally, we get $f(x)$ is given by

$$f(x) = \lambda(1 - \frac{\lambda}{\mu})e^{-(\mu-\lambda)x}$$

Chapter 4

M/M/c Queue

This chapter derives some results for the standard M/M/c queue as a special case of the generalized model. And we provide steady-state analyses of M/M/c variants using LC. From Brill(2008), for the positive-wait states,the model equation is

$$f(x) = \lambda P_{c-1} e^{-c\mu x} + \lambda \int_{y=0}^{x} e^{-c\mu(x-y)} f(y) dy, x > 0 \tag{4.0.1}$$

The normalizing condition is

$$F(\infty) = \sum_{n=0}^{c-1} P_n + \int_{y=0}^{\infty} f(x) dx = 1 \tag{4.0.2}$$

For solving (4.0.1), we differentiate both sides with respect to x, we obtain

$$f'(x) = -c\mu\lambda P_{c-1} e^{-c\mu x} + \lambda f(x) + \lambda \int_{y=0}^{x} -c\mu e^{-c\mu(x-y)} f(y) dy \tag{4.0.3}$$

Combining (4.0.1)and (4.0.3), we get

$$f'(x) = -c\mu f(x) + \lambda f(x) \tag{4.0.4}$$

$$f'(x) + (c\mu - \lambda)f(x) = 0 \tag{4.0.5}$$

This is a first order homogeneous differential equation, let

$$f(x) = Ae^{-(c\mu-\lambda)x}$$

Now, we substitute $x = 0$ into (4.0.1), we get

$$f(0) = \lambda P_{c-1} \tag{4.0.6}$$

So, our $A = \lambda P_{c-1}$, therefore, our solution for $f(x)$ os given by

$$f(x) = \lambda P_{c-1}e^{-(c\mu-\lambda)x} \tag{4.0.7}$$

The following diagram for the virtual wait of an $M/M/c$ queueing system is taken from p. 211 of Brill, 2008. It illustrates c lines and one page.

In the diagram, the system begins on Line 0 with 0 customers at time 0. Then a customer arrives and the system point jumps to line 1. It stays on line 1 until another customer arrives and the system point jumps to line 2. (Note that the system point could have jumped back to line 0 if the customer had completed service.) The system point keeps moving until it eventually is on line $c - 1$ with $c - 1$ customers in service and $c - 1$ servers working. If the next event is an arrival then there are c customers and the system point jumps onto page $c - 1$ at a height which is an exponentially distributed values with rate $c\mu$. Before a service completion takes place, suppose there is another arrival, meaning that there are $c + 1$ customers in the system. Then the system point jumps on page $c - 1$ by an exponentially distributed amount with rate $c\mu$. The system point slides down with slope -1. By the time the system point hits zero, we have had two service completions. One of the completions was from the original set of c customers. At that point, the waiting customer entered service leaving all c servers busy and no waiting customer. We are not sure which of the c customers completes first, but when that event happens, the system point jumps back to Line $c-1$. The path stays with $c-1$ customers until the next event which is an arrival. So the system point jumps to page $c-1$. The level is exponentially distributed with rate $c\mu$. in the diagram, there is one more arrival and we have the same situation as previously.

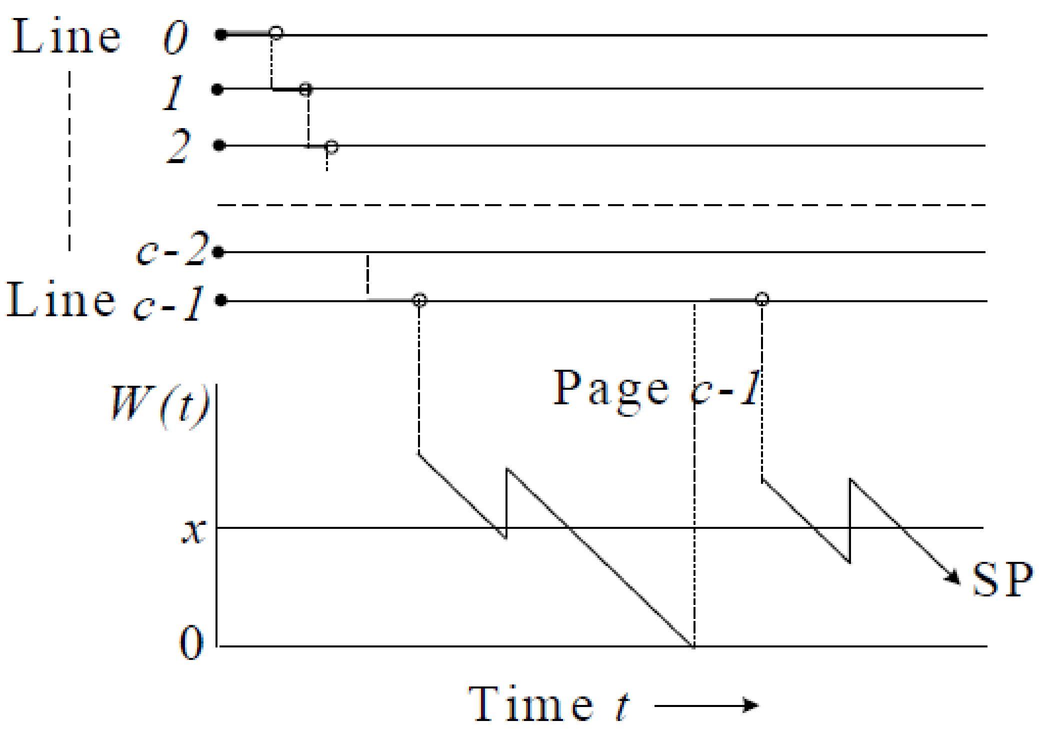

Figure 4.1: M/M/C

4.1 M/M/2

Setting c=2 in (4.0.1) gives an $M/M/2$ queue. The integral equation for $f(x)$ is

$$f(x) = \lambda P_1 e^{-2\mu x} + \lambda \int_{y=0}^{x} e^{-2\mu(x-y)} f(y)dy \tag{4.1.1}$$

Using rate in= rate out,we get

$$\lambda P_0 = \mu P_1 \tag{4.1.2}$$

$$(\lambda + \mu)P_1 = \lambda P_0 + f(0) \tag{4.1.3}$$

Here $f(x)$ is the downcrossing rate of level x on Page 1, and P_1 is probability of being on Line 1, and P_0 is the probability of being on Line 0. The upcrossing rate of level x on Page 1 can occur in two ways. One way has the system point is on Line 1 (with probability P_0) and there is an arrival (with rate λ) and the jump size is exponentially distributed with rate 2μ and exceeds level x. The second way has the system point on Page 1 at some level y between 0 and x and an arrival occurs. The jump size must be at least $x - y$ with probability $e^{-2\mu(x-y)}$ (corresponding to rate 2μ). We integrate over all such possible cases.

By (4.0.1), we obtain

$$P_0 + P_1 + \int_0^{\infty} f(x)dx = 1$$

According to (4.1.1), we can easily see the relationship between $f'(x)$ and $f(x)$, written as

$$f'(x) = -2\mu f(x) + \lambda f(x) = (\lambda - 2\mu)f(x)$$

This is a first order homogeneous differential equation, let

$$f(x) = ke^{-(2\mu-\lambda)x}$$

And by (4.1.2) and (4.1.3), we get

$$f(0) = \lambda P_1$$

So,

$$f(x) = \lambda P_1 e^{-(2\mu-\lambda)x}$$

By (4.0.2), we have

$$P_0 + P_1 + \int_0^\infty f(x)dx = 1,$$

$$P_0 + \frac{\lambda}{\mu}P_0 + \int_0^\infty \lambda\frac{\lambda}{\mu}P_0 e^{-(2\mu-\lambda)x}dx = 1$$

Solving these equations, we get

$$P_0 = \frac{2\mu-\lambda}{2\mu+\lambda}$$

So,

$$f(x) = \frac{\lambda^2(2\mu-\lambda)}{\mu(2\mu+\lambda)}e^{-(2\mu-\lambda)x}$$

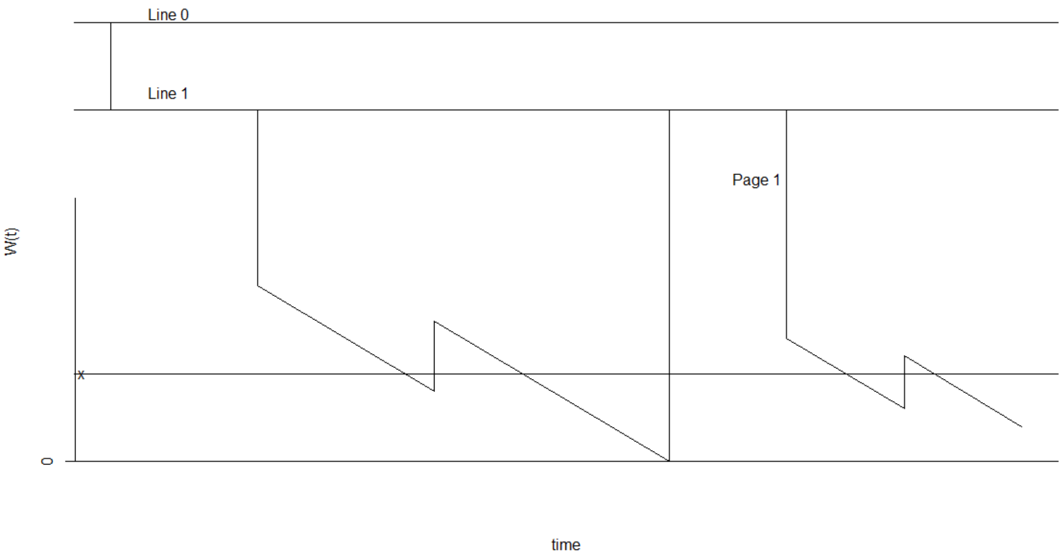

Figure 4.2: M/M/2

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VITA AUCTORIS

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