CALCULATING TRANSIENT PROBABILITIES OF A $M/M/1$ QUEUE: R PROGRAM AND TEST BANK

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Abstract

We calculate transient probabilities of an $M/M/1$ queue. These calculations are mainly for the cases in which n (the number of people initially observed in a queue) and i (the number of people observed after returning to a queue) are non zero. We present the formulas and the programming code in R used to evaluate these probabilities. The numerical results may serve as a test bank for others to check their own programs and results.

1. Introduction

Problem: We wish to find transient probabilities of an $M/M/1$ queue for different values of n, i, t, λ and μ. In particular, we evaluate the formula when values of n and i are non-zero.

The following notation will be used throughout the report

- t is the length of time measured from the initial time 0.
- n is the number of people in the system at time 0.
- i is the number of people in the system at time t.
- ρ is the ratio $\frac{\lambda}{\mu}$.

We assume that we have an $M/M/1$ queueing system. Interarrival times and service times are exponentially distributed.

Numerical transient probability calculations exist appear in the literature but we could only find such results when $n = 0$ and $i = 0$. Transient probabilities are used in a wide range of applications.

Parthasarathy and Lenin [5] discuss the application of transient probabilities to multi server queues. Both Tarabia [3] and Conolly and Langaris [2] create formulas that shorten computational time for these probabilities. Both of these articles provide sample calculations showing the CPU time for their their formulae. However, the authors only provide calculations where both n and i are zero and they show very limited values for these calculations.

Comolly and Langaris [2] did not specify the value of λ and μ but only ρ. Looking at $\rho = .5$, $\lambda = 1$, $\mu = 2$ and $\rho = .5$, $\lambda = 2$, $\mu = 4$, the speed of the system is doubled in the second case, and the transient probability for the first matches that of the second only when t is halved in the seocnd case. By comparing calculations in Conolly and Langaris [2], the table where $\rho = 0.5$ will match the column where $\lambda = 0.5, \mu = 1$ but not the column where $\lambda = 1, \mu = 2$. It is important to specify λ and μ in the calculations.

We were unable to match the calculations done in Tarabia [3], likely because λ and μ were not specified. The calculations only specified the values of ρ that were used.

We reproduced most of the values of Conolly and Langaris [2], together with cases where n and i are non-zero. We also give the code used to evaluate these probabilities using R programming language.

2. Formula

Assume n is the number of customers initially observed and i is the number of customers observed after time t. We evaluate $p_i(t|n$ at t=0), the probability of seeing i customers after time t in a queue that initially had n customers, for different values of i, n, λ, μ.

The formula by Sharma [1] can be used to evaluate $p_i(t|n \text{ at time } 0)$

$$\begin{aligned} p_i(t|n \text{ at time } 0) &= \sum_{k=0}^{\infty} p_{i+k,k}(i,t) \\ &= (1-\rho)\rho^i + e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty} \left(\frac{(\lambda t)^k}{k!} \sum_{m=0}^{i+k+n} (k-m) \frac{(\mu t)^{m-1}}{m!} \right) \\ &+ e^{-(\lambda+\mu)t} \sum_{k=0}^{\infty} (\lambda t)^{i+k-n} (\mu t)^k \left(\frac{1}{k!(i+k-n)!} - \frac{1}{(i+k)!(k-n)!} \right). \end{aligned}$$

However there is an inconsistency between this formula and that used in the special case $i = 0$ in Sharma (p. 17). We will use an equivalent formula that removes any confusion. Developed by Conolly and Langaris [2] this is a generalized version of Sharma and Shobha (where our initial state n can be any number, not just 0).

This formula is used to do all calculations in this report:

$$\begin{aligned} p_i(t) = (1-\rho)\rho^i + e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty} \left(\frac{(\lambda t)^k}{k!} \sum_{m=0}^{k+i+n+1} (k-m) \frac{(\mu t)^{m-1}}{m!} \right) \\ + e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty} \frac{(\lambda t)^{k+1} (\mu t)^{k+max(i,n)}}{k!} \left(\frac{(\lambda t)^{-min(i,n)-1}}{(k+|n-i|)!} - \frac{(\mu t)^{min(i,n)+1}}{(k+n+i+2)!} \right). \end{aligned} \tag{2.1}$$

To evaluate these large summations we use the programming language R. We cap off these summations at values between 50 and 80 and claim this does not affect our final answer to any significance. Since R cannot handle factorials over 170 we rearrange formula (2.1) so that R does cancellation during the calculation and thus can evaluate formulas while avoiding calculations with large factorials. Factorials will be paired up with a variable whose exponent is equal to the factorial. For example we the calculate expressions of the form: $\frac{x^{m+2}}{(m+2)!}$.

Focusing on what is inside the final summation of formula (2.1) we have:

$$\frac{(\lambda t)^{k+1} (\mu t)^{k+max(i,n)}}{k!} \left(\frac{(\lambda t)^{-min(i,n)-1}}{(k+|n-i|)!} - \frac{(\mu t)^{min(i,n)+1}}{(k+n+i+2)!} \right).$$

We rearrange this to give

$$\frac{(\mu t)^k}{k!}\frac{(\lambda t)^k(\mu t)^{max(i,n)}(\lambda t)^{-min(i,n)}}{(k+|n-i|)!}-\frac{(\lambda t)^k}{k!}\frac{(\lambda t)(\mu t)(\mu t)^k(\mu t)^{max(i,n)}(\mu t)^{min(i,n)}}{(k+n+i+2)!}.$$

To remove the max and min functions we can separate into cases $i<n$ and $i\geq n$ and rearrange so our fractions are in the proper form. Which gives us the following:

CASE i<n

$$\frac{(\mu t)^k}{k!}\frac{(\lambda t)^k(\mu t)^n(\lambda t)^{-i}}{(k+n-i)!}-\frac{(\lambda t)^k}{k!}\frac{(\lambda t)(\mu t)(\mu t)^k(\mu t)^n(\mu t)^i}{(k+n+i+2)!}$$
$$=\frac{(\mu t)^k}{k!}\frac{(\lambda t)^{k+n-i}}{(k+n-i)!}\left(\frac{\mu}{\lambda}\right)^n-\frac{(\lambda t)^k}{k!}\frac{(\mu t)^{k+n+i+2}}{(k+n+i+2)!}\left(\frac{\lambda}{\mu}\right).$$

CASE i≥n

$$\frac{(\mu t)^k}{k!}\frac{(\lambda t)^k(\mu t)^i(\lambda t)^{-n}}{(k+i-n)!}-\frac{(\lambda t)^k}{k!}\frac{(\lambda t)(\mu t)(\mu t)^k(\mu t)^i(\mu t)^n}{(k+n+i+2)!}$$
$$=\frac{(\mu t)^k}{k!}\frac{(\lambda t)^{k+i-n}}{(k+i-n)!}\left(\frac{\mu}{\lambda}\right)^i-\frac{(\lambda t)^k}{k!}\frac{(\mu t)^{k+i+n+2}}{(k+i+n+2)!}\left(\frac{\lambda}{\mu}\right).$$

We can put all of this together to give us the entire formula that we will be using for our calculations:

CASE i<n

$$p_i(t|\text{ n at time 0})=(1-\rho)\rho^i+e^{-(\lambda+\mu)t}\rho^i\sum_{k=0}^{\infty}\left(\frac{(\lambda t)^k}{k!}\sum_{m=0}^{k+i+n+1}\frac{k-m}{\mu t}\frac{(\mu t)^m}{m!}\right)$$
$$+e^{-(\lambda+\mu)t}\rho^i\sum_{k=0}^{\infty}\left(\frac{(\mu t)^k}{k!}\frac{(\lambda t)^{k+n-i}}{(k+n-i)!}\left(\frac{\mu}{\lambda}\right)^n-\frac{(\lambda t)^k}{k!}\frac{(\mu t)^{k+n+i+2}}{(k+n+i+2)!}\left(\frac{\lambda}{\mu}\right)\right).$$

CASE i≥n

$$p_i(t|\text{ n at time 0})=(1-\rho)\rho^i+e^{-(\lambda+\mu)t}\rho^i\sum_{k=0}^{\infty}\left(\frac{(\lambda t)^k}{k!}\sum_{m=0}^{k+i+n+1}\frac{k-m}{\mu t}\frac{(\mu t)^m}{m!}\right)$$
$$+e^{-(\lambda+\mu)t}\rho^i\sum_{k=0}^{\infty}\left(\frac{(\mu t)^k}{k!}\frac{(\lambda t)^{k+i-n}}{(k+i-n)!}\left(\frac{\mu}{\lambda}\right)^i-\frac{(\lambda t)^k}{k!}\frac{(\mu t)^{k+i+n+2}}{(k+i+n+2)!}\left(\frac{\lambda}{\mu}\right)\right).$$

This allows us to create a function in R that does division before exponentiation and factorials. By doing division first, R can handle the values computationally since these values do not grow as quickly. This new function can calculate factorials (that large exponentials are being divided by) of almost any size, without exhausting R's storage.

Using this new (computationally acceptable by R) formula we evaluate these probabilities for different values of n, i, μ, λ, and t.

3. R Code

The following code was used to calculate the final formulas (in either case):

```
divid2 <- function(v3, y){          #calculates(v3^y)/y!
v <- 1
    if(y == 0){
    return(v)
    }
    else{
        for(p in 1 : y){
        v1 <- v3/p
        v <- v * v1
        }
    return(v)
    }
}

dsum3 <- function(l, u, t, i, n, w){
v <- 0; s <- 0
    for(k in 0 : w){
    v5 <- divid2(l * t, k)
    v <- 0
        for(m in 0 : (k + i + n + 1)){
        v4 <- (k - m) * (1/(u * t)) * divid2(u * t, m)
        v <- v + v4
        }
    s <- s + v * v5
    }
return(s)
}
```

$fsum3 <- function(l, u, t, i, n, w)\{$
$v <- 0$
$if(i < n)\{$
$for(k \text{ in } 0 : w)\{$
$v1 <- divid2(u * t, k)$
$v2 <- divid2(l * t, k + n - i) * ((u/l)^n)$
$v3 <- divid2(l * t, k)$
$v4 <- divid2(u * t, k + i + n + 2) * l/u$
$v <- v + v1 * v2 - v3 * v4$
$\}$
$\}$
$else\{$
$for(k \text{ in } 0 : w)\{$
$v1 <- divid2(u * t, k)$
$v2 <- divid2(l * t, k + i - n) * ((u/l)^i)$
$v3 <- divid2(l * t, k)$
$v4 <- divid2(u * t, k + i + n + 2) * l/u$
$v <- v + v1 * v2 - v3 * v4$
$\}$
$\}$
$return(v)$
$\}$

$inner3 <- function(l, u, t, i, n, w)\{$
$v1 <- (1 - l/u) * ((l/u)^i)$
$v2 <- exp(-(l + u) * t) * ((l/u)^i)$
$v3 <- dsum3(l, u, t, i, n, w)$
$v4 <- fsum3(l, u, t, i, n, w)$
$v <- v1 + v2 * v3 + v2 * v4$
$return(v)$
$\}$

```
final3 <- function(l, u, t, w1, n, w){
v <- 0;
    for(i in 0 : w1){
    v1 <- inner3(l, u, t, i, n, w) * (i + 1)/u
    v <- v + v1
    }
s <- v + t
return(s)
}

acase <- function(u, t, n){
v <- (1/u) * (n + 1) + t
return(v)
}
```

4. Calculations

The following calculations were evaluated using R programming language with the code shown above. We calculated the probability of observing i customers at time t, given that there were n customers at time 0, the service rate is μ, the interarrival rate is λ.

The tables are set up in the following way:

(1) Each page represents 3 tables for a specific value of n-the initial number of customers
(2) Each table represents a different value of i-the number of customers at time t
(3) Each column represents the arrival and service rates of the queue (λ and μ).
(4) The rows represent different values of t.
(5) The cells are the probabilities of seeing i customers after t time units, when there were initially n customers in line.

Table for $P(i = 0)$, $n = 0$ (Probability of i customers seen after time t)

	$\lambda = 0.2$ $\mu = 2$ $\rho = 0.1$	$\lambda = 1$ $\mu = 2$ $\rho = 0.5$	$\lambda = 1.8$ $\mu = 2$ $\rho = 0.9$	$\lambda = 0.1$ $\mu = 1$ $\rho = 0.1$	$\lambda = 0.5$ $\mu = 1$ $\rho = 0.5$	$\lambda = 0.9$ $\mu = 1$ $\rho = 0.9$	$\lambda = 1$ $\mu = 1$ $\rho = 1$
$t = 0.5$	0.938596941	0.726255083	0.559449122	0.961401837	0.821086477	0.700908462	0.673670023
$t = 1$	0.916237344	0.633795374	0.427231054	0.938596941	0.726255083	0.559449122	0.523777612
$t = 1.5$	0.907322878	0.59014586	0.362710238	0.924791301	0.670133972	0.479185433	0.439827067
$t = 2$	0.903485196	0.565156683	0.323148286	0.916237344	0.633795374	0.427231054	0.385752761
$t = 2.5$	0.901728191	0.549166124	0.295806131	0.910821157	0.608579005	0.390433427	0.34751308
$t = 3$	0.900884278	0.538195211	0.27548955	0.907322878	0.59014586	0.362710238	0.318708892
$t = 3.5$	0.900463627	0.530303897	0.259643233	0.905022422	0.576132987	0.340888538	0.296026979
$t = 4$	0.900247811	0.524430933	0.246846663	0.903485196	0.565156683	0.323148286	0.277574275
$t = 10$	0.900000279	0.50328898	0.178951504	0.900073975	0.516452025	0.227266133	0.177286534

Table for $P(i = 1)$, $n = 0$ (Probability of i customers seen after time t)

	$\lambda = 0.2$ $\mu = 2$ $\rho = 0.1$	$\lambda = 1$ $\mu = 2$ $\rho = 0.5$	$\lambda = 1.8$ $\mu = 2$ $\rho = 0.9$	$\lambda = 0.1$ $\mu = 1$ $\rho = 0.1$	$\lambda = 0.5$ $\mu = 1$ $\rho = 0.5$	$\lambda = 0.9$ $\mu = 1$ $\rho = 0.9$	$\lambda = 1$ $\mu = 1$ $\rho = 1$
$t = 0.5$	0.058880259	0.221247612	0.29851779	0.037721004	0.159287753	0.242096914	0.257849192
$t = 1$	0.078174593	0.257281827	0.298537901	0.058880259	0.221247612	0.29851779	0.308508323
$t = 1.5$	0.085115729	0.263209345	0.277664734	0.07103102	0.246614162	0.305378988	0.308609259
$t = 2$	0.087838811	0.263028192	0.258606584	0.078174593	0.257281827	0.298537901	0.296377341
$t = 2.5$	0.088990065	0.261536793	0.242965345	0.082472301	0.261662831	0.288259647	0.281924173
$t = 3$	0.089507556	0.259895071	0.230162803	0.085115729	0.263209345	0.277664734	0.268025072
$t = 3.5$	0.089751804	0.258401589	0.219535786	0.086775878	0.263431316	0.267700419	0.255372912
$t = 4$	0.089871608	0.2571179	0.210572457	0.087838811	0.263028192	0.258606584	0.244038652
$t = 10$	0.089999879	0.251173948	0.158171076	0.089963573	0.255131296	0.196246779	0.168535912

Table for $P(i = 2)$, $n = 0$ (Probability of i customers seen after time t)

	$\lambda = 0.2$ $\mu = 2$ $\rho = 0.1$	$\lambda = 1$ $\mu = 2$ $\rho = 0.5$	$\lambda = 1.8$ $\mu = 2$ $\rho = 0.9$	$\lambda = 0.1$ $\mu = 1$ $\rho = 0.1$	$\lambda = 0.5$ $\mu = 1$ $\rho = 0.5$	$\lambda = 0.9$ $\mu = 1$ $\rho = 0.9$	$\lambda = 1$ $\mu = 1$ $\rho = 1$
$t = 0.5$	0.002446962	0.04494333	0.106826652	0.000863237	0.018099052	0.049174455	0.058094085
$t = 1$	0.005290747	0.082400731	0.163905197	0.002446962	0.04494333	0.106826652	0.122030256
$t = 1.5$	0.007033458	0.101185018	0.180415451	0.004000129	0.066712403	0.143346386	0.159565865
$t = 2$	0.00797251	0.110795218	0.183222124	0.005290747	0.082400731	0.163905197	0.17875084
$t = 2.5$	0.00846131	0.116069103	0.181307184	0.006288951	0.093417186	0.174907705	0.187562648
$t = 3$	0.00871477	0.119156462	0.177752632	0.007033458	0.101185018	0.180415451	0.19070933
$t = 3.5$	0.008847195	0.121060029	0.173716355	0.007578101	0.106744232	0.182748989	0.190753668
$t = 4$	0.008917183	0.12228245	0.169660265	0.00797251	0.110795218	0.183222124	0.189090573
$t = 10$	0.008999904	0.124956495	0.13709447	0.0089749	0.123644418	0.162114324	0.152329974

Table for $P(i=0)$, $n=1$
(Probability of i customers seen after time t)

	$\lambda=0.2$ $\mu=2$ $\rho=0.1$	$\lambda=1$ $\mu=2$ $\rho=0.5$	$\lambda=1.8$ $\mu=2$ $\rho=0.9$	$\lambda=0.1$ $\mu=1$ $\rho=0.1$	$\lambda=0.5$ $\mu=1$ $\rho=0.5$	$\lambda=0.9$ $\mu=1$ $\rho=0.9$	$\lambda=1$ $\mu=1$ $\rho=1$
$t=0.5$	0.588802588	0.442495225	0.331686433	0.377210042	0.318575507	0.268996571	0.257849192
$t=1$	0.781745935	0.514563655	0.331708779	0.588802588	0.442495225	0.331686433	0.308508323
$t=1.5$	0.851157291	0.52641869	0.308516371	0.710310203	0.493228324	0.339309987	0.308609259
$t=2$	0.87838811	0.526056383	0.287340648	0.781745935	0.514563655	0.331708779	0.296377341
$t=2.5$	0.889900648	0.523073586	0.269961494	0.824723011	0.523325663	0.320288496	0.281924173
$t=3$	0.895075563	0.519790143	0.255736448	0.851157291	0.52641869	0.308516371	0.268025072
$t=3.5$	0.897518043	0.516803178	0.243928651	0.867758782	0.526862631	0.29744491	0.255372912
$t=4$	0.898716076	0.514235799	0.233969396	0.87838811	0.526056383	0.287340648	0.244038652
$t=10$	0.899998785	0.502347897	0.17574564	0.899635725	0.510262593	0.218051977	0.168535912

Table for $P(i=1)$, $n=1$
(Probability of i customers seen after time t)

	$\lambda=0.2$ $\mu=2$ $\rho=0.1$	$\lambda=1$ $\mu=2$ $\rho=0.5$	$\lambda=1.8$ $\mu=2$ $\rho=0.9$	$\lambda=0.1$ $\mu=1$ $\rho=0.1$	$\lambda=0.5$ $\mu=1$ $\rho=0.5$	$\lambda=0.9$ $\mu=1$ $\rho=0.9$	$\lambda=1$ $\mu=1$ $\rho=1$
$t=0.5$	0.374263971	0.373646518	0.346458968	0.592824161	0.538709074	0.486550174	0.473914915
$t=1$	0.187398877	0.284033182	0.277639161	0.374263971	0.373646518	0.346458968	0.337299545
$t=1.5$	0.126500171	0.266097205	0.25465548	0.254482384	0.310330454	0.299149208	0.290783674
$t=2$	0.104822185	0.260690736	0.239387775	0.187398877	0.284033182	0.277639161	0.268126259
$t=2.5$	0.096440646	0.258230745	0.227297063	0.148987657	0.272087714	0.264486825	0.253151555
$t=3$	0.092956415	0.256717992	0.217256027	0.126500171	0.266097205	0.25465548	0.24139315
$t=3.5$	0.091417538	0.255620777	0.208732753	0.113044651	0.262758819	0.24649806	0.231407735
$t=4$	0.090703565	0.254760034	0.201388672	0.104822185	0.260690736	0.239387775	0.222626196
$t=10$	0.090000537	0.250854073	0.155533053	0.090187252	0.253478268	0.189341183	0.161080596

Table for $P(i=2)$, $n=1$
(Probability of i customers seen after time t)

	$\lambda=0.2$ $\mu=2$ $\rho=0.1$	$\lambda=1$ $\mu=2$ $\rho=0.5$	$\lambda=1.8$ $\mu=2$ $\rho=0.9$	$\lambda=0.1$ $\mu=1$ $\rho=0.1$	$\lambda=0.5$ $\mu=1$ $\rho=0.5$	$\lambda=0.9$ $\mu=1$ $\rho=0.9$	$\lambda=1$ $\mu=1$ $\rho=1$
$t=0.5$	0.03515122	0.144747354	0.211153432	0.029226169	0.125960145	0.195222763	0.208917346
$t=1$	0.028111139	0.134754064	0.197047592	0.03515122	0.144747354	0.211153432	0.222134655
$t=1.5$	0.019746074	0.127272882	0.188498252	0.032745	0.140922	0.203998362	0.213042619
$t=2$	0.01464238	0.124660841	0.182837783	0.028111139	0.134754064	0.197047592	0.204690834
$t=2.5$	0.011916186	0.123968034	0.177947334	0.023541646	0.130159784	0.192135447	0.198391282
$t=3$	0.01051194	0.123922081	0.173409075	0.019746074	0.127272882	0.188498252	0.193289355
$t=3.5$	0.009791878	0.12406828	0.169169291	0.016818048	0.125589709	0.185508027	0.188798635
$t=4$	0.009419857	0.124250123	0.165225591	0.01464238	0.124660841	0.182837783	0.184642841
$t=10$	0.009000417	0.124994972	0.135159107	0.009122367	0.124555468	0.158191782	0.147145827

Table for $P(i=0)$, $n=2$ (Probability of i customers seen after time t)							
	$\lambda=0.2$ $\mu=2$ $\rho=0.1$	$\lambda=1$ $\mu=2$ $\rho=0.5$	$\lambda=1.8$ $\mu=2$ $\rho=0.9$	$\lambda=0.1$ $\mu=1$ $\rho=0.1$	$\lambda=0.5$ $\mu=1$ $\rho=0.5$	$\lambda=0.9$ $\mu=1$ $\rho=0.9$	$\lambda=1$ $\mu=1$ $\rho=1$
$t=0.5$	0.244696176	0.179773319	0.131884755	0.086323654	0.072396207	0.060709204	0.058094085
$t=1$	0.529074673	0.329602925	0.202352096	0.244696176	0.179773319	0.131884755	0.122030256
$t=1.5$	0.703345829	0.404740071	0.222735125	0.400012867	0.266849611	0.176970847	0.159565865
$t=2$	0.797250993	0.443180872	0.226200153	0.529074673	0.329602925	0.202352096	0.17875084
$t=2.5$	0.846131033	0.464276414	0.223836029	0.628895107	0.373668744	0.215935438	0.187562648
$t=3$	0.871476999	0.476625847	0.219447694	0.703345829	0.404740071	0.222735125	0.19070933
$t=3.5$	0.884719544	0.484240115	0.214464635	0.757810112	0.426976928	0.225616035	0.190753668
$t=4$	0.891718296	0.489129801	0.209457117	0.797250993	0.443180872	0.226200153	0.189090573
$t=10$	0.899990432	0.49982598	0.169252433	0.897490021	0.494577671	0.200141141	0.152329974

Table for $P(i=1)$, $n=2$ (Probability of i customers seen after time t)							
	$\lambda=0.2$ $\mu=2$ $\rho=0.1$	$\lambda=1$ $\mu=2$ $\rho=0.5$	$\lambda=1.8$ $\mu=2$ $\rho=0.9$	$\lambda=0.1$ $\mu=1$ $\rho=0.1$	$\lambda=0.5$ $\mu=1$ $\rho=0.5$	$\lambda=0.9$ $\mu=1$ $\rho=0.9$	$\lambda=1$ $\mu=1$ $\rho=1$
$t=0.5$	0.351512202	0.289494707	0.234614924	0.292261693	0.25192029	0.216914181	0.208917346
$t=1$	0.281111394	0.269508128	0.218941769	0.351512202	0.289494707	0.234614924	0.222134655
$t=1.5$	0.197460743	0.254545764	0.209442502	0.327450003	0.281843999	0.226664846	0.213042619
$t=2$	0.146423795	0.249321682	0.203153093	0.281111394	0.269508128	0.218941769	0.204690834
$t=2.5$	0.119161859	0.247936068	0.19771926	0.23541646	0.260319567	0.21348383	0.198391282
$t=3$	0.105119399	0.247844162	0.19267675	0.197460743	0.254545764	0.209442502	0.193289355
$t=3.5$	0.097918775	0.248136559	0.187965879	0.168180478	0.251179418	0.20612003	0.188798635
$t=4$	0.094198571	0.248500246	0.18358399	0.146423795	0.249321682	0.203153093	0.184642841
$t=10$	0.090004167	0.249989945	0.150176786	0.091223667	0.249110935	0.175768646	0.147145827

Table for $P(i=2)$, $n=2$ (Probability of i customers seen after time t)							
	$\lambda=0.2$ $\mu=2$ $\rho=0.1$	$\lambda=1$ $\mu=2$ $\rho=0.5$	$\lambda=1.8$ $\mu=2$ $\rho=0.9$	$\lambda=0.1$ $\mu=1$ $\rho=0.1$	$\lambda=0.5$ $\mu=1$ $\rho=0.5$	$\lambda=0.9$ $\mu=1$ $\rho=0.9$	$\lambda=1$ $\mu=1$ $\rho=1$
$t=0.5$	0.367032786	0.349993643	0.318866756	0.591465585	0.533316185	0.478861991	0.465859473
$t=1$	0.160203177	0.21730688	0.220902223	0.367032786	0.349993643	0.318866756	0.309838084
$t=1.5$	0.079833926	0.170655534	0.190283587	0.237915169	0.264111147	0.253701001	0.247541257
$t=2$	0.044275172	0.149736404	0.176701165	0.160203177	0.21730688	0.220902223	0.21624627
$t=2.5$	0.027210391	0.139263574	0.168998753	0.111473902	0.188974228	0.202076039	0.198081131
$t=3$	0.018619194	0.133635801	0.163707422	0.079833926	0.170655534	0.190283587	0.186409289
$t=3.5$	0.014164835	0.13044043	0.159570943	0.058702522	0.158305972	0.182363571	0.178254135
$t=4$	0.011808815	0.128542238	0.156079245	0.044275172	0.149736404	0.176701165	0.172125848
$t=10$	0.00900328	0.125087135	0.131205644	0.009856361	0.126625468	0.150223584	0.137224754

Table for $P(i = 0)$, $n = 5$ (Probability of i customers seen after time t)

	$\lambda = 0.2$ $\mu = 2$ $\rho = 0.1$	$\lambda = 1$ $\mu = 2$ $\rho = 0.5$	$\lambda = 1.8$ $\mu = 2$ $\rho = 0.9$	$\lambda = 0.1$ $\mu = 1$ $\rho = 0.1$	$\lambda = 0.5$ $\mu = 1$ $\rho = 0.5$	$\lambda = 0.9$ $\mu = 1$ $\rho = 0.9$	$\lambda = 1$ $\mu = 1$ $\rho = 1$
$t = 0.5$	0.003358083	0.002379606	0.001685622	0.000164301	0.000164301	0.000113284	0.000108139
$t = 1$	0.045506772	0.025309344	0.014007828	0.003358083	0.003358083	0.001685622	0.001546321
$t = 1.5$	0.15390446	0.07312394	0.034068823	0.016490883	0.016490883	0.006337155	0.005620466
$t = 2$	0.303816606	0.131651383	0.054387882	0.045506772	0.045506772	0.014007828	0.012073471
$t = 2.5$	0.456624499	0.18920661	0.071765322	0.092095767	0.092095767	0.023630621	0.019878697
$t = 3$	0.587608753	0.240452042	0.08569899	0.15390446	0.15390446	0.034068823	0.028069991
$t = 3.5$	0.688469107	0.283987521	0.096605762	0.226246461	0.226246461	0.044488797	0.036002376
$t = 4$	0.760871068	0.320171486	0.105079455	0.303816606	0.303816606	0.054387882	0.043326832
$t = 10$	0.899573619	0.474570463	0.132983401	0.84322969	0.84322969	0.116774039	0.083361826

Table for $P(i = 2)$, $n = 5$ (Probability of i customers seen after time t)

	$\lambda = 0.2$ $\mu = 2$ $\rho = 0.1$	$\lambda = 1$ $\mu = 2$ $\rho = 0.5$	$\lambda = 1.8$ $\mu = 2$ $\rho = 0.9$	$\lambda = 0.1$ $\mu = 1$ $\rho = 0.1$	$\lambda = 0.5$ $\mu = 1$ $\rho = 0.5$	$\lambda = 0.9$ $\mu = 1$ $\rho = 0.9$	$\lambda = 1$ $\mu = 1$ $\rho = 1$
$t = 0.5$	0.056879516	0.042077457	0.03107115	0.0120951	0.010152383	0.008520674	0.008155344
$t = 1$	0.163124744	0.107117133	0.068716902	0.056879516	0.042077457	0.03107115	0.028794971
$t = 1.5$	0.206946961	0.139904726	0.086423646	0.11424457	0.077980312	0.052786835	0.047823831
$t = 2$	0.192900225	0.150172055	0.094314032	0.163124744	0.107117133	0.068716902	0.061304013
$t = 2.5$	0.154670322	0.150415332	0.098370491	0.194209037	0.127321058	0.079402353	0.070110136
$t = 3$	0.114421233	0.147116457	0.100884238	0.206946961	0.139904726	0.086423646	0.075784145
$t = 3.5$	0.081170215	0.143051871	0.102683173	0.204947143	0.146918136	0.091098265	0.079503718
$t = 4$	0.056654885	0.139271059	0.104074697	0.192900225	0.150172055	0.094314032	0.082023573
$t = 10$	0.009145102	0.125458854	0.108264886	0.028453612	0.133533298	0.106072602	0.089091859

Table for $P(i = 5)$, $n = 5$ (Probability of i customers seen after time t)

	$\lambda = 0.2$ $\mu = 2$ $\rho = 0.1$	$\lambda = 1$ $\mu = 2$ $\rho = 0.5$	$\lambda = 1.8$ $\mu = 2$ $\rho = 0.9$	$\lambda = 0.1$ $\mu = 1$ $\rho = 0.1$	$\lambda = 0.5$ $\mu = 1$ $\rho = 0.5$	$\lambda = 0.9$ $\mu = 1$ $\rho = 0.9$	$\lambda = 1$ $\mu = 1$ $\rho = 1$
$t = 0.5$	0.366999674	0.349440335	0.317673446	0.591463955	0.533283382	0.478776548	0.465759608
$t = 1$	0.159758536	0.211712187	0.211944219	0.366999674	0.349440335	0.317673446	0.308508326
$t = 1.5$	0.078337309	0.154988712	0.170016004	0.237753398	0.261797673	0.249454454	0.243000484
$t = 2$	0.04132453	0.121994891	0.145962787	0.159758536	0.211712187	0.211944219	0.207003227
$t = 2.5$	0.022770044	0.09966387	0.129958451	0.110576635	0.178789556	0.187547077	0.183547521
$t = 3$	0.01288858	0.083352589	0.118495082	0.078337309	0.154988712	0.170016004	0.166680254
$t = 3.5$	0.007425732	0.070908533	0.109938119	0.056504665	0.136678125	0.156620696	0.15379676
$t = 4$	0.004332104	0.061157375	0.103384665	0.04132453	0.121994891	0.145962787	0.143557601
$t = 10$	1.84E-05	0.021845092	0.078344329	0.001514014	0.047106845	0.094234505	0.094144169

5. Conclusion and Acknowledgement

The calculations were done for the purpose of having a bank of probabilities for non-zero values of n and i. Notice the first table gives the case when both n and i are zero. These values can be compared to calculations in Conolly and Langaris [2] and agree for the appropriate values of λ and μ (note that in [2] these values are not specified and only the value of ρ is given). This report was created to supply these transient probability calculations to those who need them for research, to provide an R program to allow others to perform their own calcuulations, and to provide a test bank to check calculations.

There are some interesting things to notice in the calculations. One observation relates to the results when the speed of the queue is doubled. If we double the speed of the system (double μ, double λ) the probabilities will agree with the original ones when t is halved. The calculations should be identical (with the exception of any rounding errors). See the case where $\lambda = 0.1, \mu = 1, t = 2$ and $\lambda = 0.2, \mu = 2, t = 1$ and many other examples.

Let $p_{n,i}(t)$ = be the probability of moving from state n to i in time t. When $\lambda = \mu$,we observe that $p_{n,i}(t)$ is the largest among $p_{n,1}(t), p_{n,2}(t), \ldots$; i.e. the probabilities are maximized in cases where the intial queue size matches the queue size at time t. (e.g. $n = 1, i = 1$ or $n = 5, i = 5$).

Notice that the probabilities follow logical patterns:

(1) As λ increases (or μ decreases), ceteris paribus, we are more likely to see more customers when we return to the queue.
(2) As μ increases (or λ decreases), ceteris paribus, we are more likely to see fewer customers when we return to the queue.
(3) Since $\lambda < \mu$ in all the cases we studied, in the majority of our calculations, the queues have a fewer customers at time t.
(4) Since the probabilities are transient probabilities, the results presented generally do not match steady state.

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6. References

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