Time to Ruin for Loss Reserves

by

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Abstract

This paper gives a procedure to determine the Laplace transform of the probability density function of the time that it takes for an insurance reserve to be depleted. This would happen when the total of claims received in a time period are greater than the reserve. This paper uses the probabilistic interpretation of a Laplace Transform to aid in the solution. A numerical inverse Laplace transform procedure is able to obtain the pdf.

Acknowledgements

I would like to express my thanks to the committee for their time.

Contents

List of Figures

CHAPTER 1

Introduction and Literature Search

We are interested in finding the probability density function (pdf) of the time to ruin for a risk model. This is a topic of considerable interest. In Stanford et al. (2005; [**18**]), the authors use “Erlangization” to find finite time ruin probabilities with phase-type claim amounts. ”The method is based on finding the probability of ruin prior to a phase-type random horizon, independent of the risk process. When the horizon follows an Erlang-l distribution, the method provides a sequence of approximations that converges to the true finite-time ruin probability as l increases.”

In Gerber and Shiu (1998; [**6**]), the authors study ”the joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. The time of ruin is analyzed in terms of its Laplace transforms, which can naturally be interpreted as discounting. Hence the classical risk theory model is generalized by discounting with respect to the time of ruin.”

In Tsai and Parker (2004; [**19**]), the authors ”study ruin probabilities based on the classical discrete time surplus process, in which the premium received in each period is assumed to be a constant. [They] apply Buhlmann credibility theory to calculate the so-called Buhlmann credibility premium as the renewal pure premium received in each period. With the dynamic premium scheme, [they] calculate the ruin probabilities, by Monte-Carlo simulation. Then, [they] compare each of these quantities with corresponding one calculated based on constant premium scheme, interpret the difference, and investigate some problems like, can the dynamic credibility premium scheme significantly affect the probability of ruin?"

In Frostig (2005; [**5**]), the author studies "two risk models with constant dividend barriers. In the two models claims arrive according to a Poisson process. In the first model the claim size has a phase type distribution. In the second model the claim size is exponentially distributed, but the arrival rate, the mean claim size, and the premium rate are governed by a random environment. The expected time to ruin and the amount of dividends paid until ruin occurs are obtained for both models."

In Lefevre and Loisel (2008; [**12**]), the authors are "concerned with the problem of ruin in the classical compound binomial and compound Poisson risk models. The primary purpose is to extend to those models an exact formula derived in [**13**] for the probability of (non-)ruin within finite time."

In Dickson and Waters (2002; [**3**]), the authors "study the distribution of time to ruin in the classical risk model. [They] consider some methods of calculating the distribution, in particular by using algorithms to calculate finite time ruin probabilities."

In Dickson (2008; [**2**]), the author uses "probabilistic arguments obtain an integral expression for the joint density of the time of ruin and the deficit at ruin. For the classical risk model, he obtains the bivariate Laplace transform of this joint density and inverts it in the cases of individual claims distributed as Erlang(2) and as a mixture of two exponential distributions. As a consequence, he obtains explicit solutions for the density of the time of ruin."

In Frangos et al. (2005; [**4**]), the authors "study the ruin probability at a given time for liabilities of diffusion type, driven by fractional Brownian motion with Hurst exponent in the range (0.5, 1). Using fractional Ito calculus they derive a partial differential equation, the solution of which provides the ruin probability. An analytical solution is found for this equation and the results obtained by this approach are compared with the results obtained by Monte-Carlo simulation."

There are numerous works that use the probabilistic interpretation of Laplace transforms.

Dantzig (1949; [**1**]) introduced the concept of a catastrophe for probability models. Runnenburg (1965 [**17**], 1975 [**16**]) revived interest in the method and gave numerous applications. Rade (1972; [**14**]) gave several applications of the method in queueing models. Kleinrock (1975; [**11**]) discusses the work of Runnenburg and the probabilistic interpretation of Laplace transforms in a section of his classic queueing book. Roy (1997; [**15**]) used the probabilistic interpretation of Laplace transforms to study busy periods of an M/G/1 queue. Horn (1999; [**9**]) used this interpretation to study order statistics of Erlang random variables. His results were extended in Hlynka et al (2010; [**7**]). Jahan (2008; [**10**]) used the interpretation to study a queueing control model.

CHAPTER 2

Laplace Transform and its Probabilistic Interpretation

The first part of this chapter is taken almost verbatim from Roy (1997; [**15**]).

DEFINITION 2.1. *The Laplace transform of a function $f(x)$ is denoted by $f^*(s)$ and is given by*

$$f^*(s) = \int_0^\infty e^{-sx} f(x)dx.$$

where $s > 0$.

DEFINITION 2.2. *The Laplace-Stieltjes transform of a function $f(x)$ is denoted by $f^*_{LS}(s)$ and is given by*

$$f^*_{LS}(s) = \int_0^\infty e^{-sx} dF(x).$$

PROPERTY 2.3. *If m_i denotes the i^{th} moment of X where the p.d.f. of X is $f(x)$, then*

$$m_i = (-1)^i \frac{d^i}{ds^i} f^*(s)|_{s=0}.$$

PROOF.

$$\begin{aligned}(-1)^i \frac{d^i}{ds^i} f^*(s)|_{s=0} &= (-1)^i \frac{d^i}{ds^i} \int_0^\infty e^{-sx} f(x)dx|_{s=0} \\ &= (-1)^i \int_0^\infty (\frac{d^i}{ds^i}|_{s=0}) f(x)dx \\ &= (-1)^i \int_0^\infty ((-x)^i e^{-sx}|_{s=0}) f(x)dx\end{aligned}$$

$$= \int_0^\infty x^i f(x)dx$$

$$= m_i.$$

□

In our interpretation, we will be interested in the probability that one random variable, say Y, exceeds another random variable, X, which we denote $P(Y > X)$. For continuous variables, this probability is defined by Hogg et al. ([**8**]) to be

$$P(Y > X) = \int_0^\infty \int_x^\infty f(x,y)dydx$$

where $f(x,y)$ is the joint p.d.f. of X and Y. Note that we restrict our random variables to those which are independent with non-negative supports.

$P(Y > X) = \int_0^\infty P(Y > x)f(x)dx$ (independence)

where $f(x)$ is the p.d.f. of X. We read this expression as the probability that Y exceeds a specific value of x taken as a weighted average with respect to $f(x)$ over all possible values for X.

PROPERTY 2.4. *The exponential distribution is "memoryless" - that is, if Y is an exponential random variable, then $P(Y > t + s|Y > s) = P(Y > t)$.*

PROOF.

$$
\begin{aligned}
P(Y > t + s | Y > s) &= \frac{P(Y > t + s)}{P(Y > s)} \\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\
&= \mathrm{e}^{-\lambda t} \\
&= P(Y > t).
\end{aligned}
$$

□

PROPERTY 2.5. *If* $X \sim exp(\lambda_1)$ *and* $Y \sim exp(\lambda_2)$ *where* X *and* Y *are independent, then* $\min(X, Y) \sim exp(\lambda_1 + \lambda_2)$.

PROOF. $X \sim exp(\lambda_1)$ and $Y \sim exp(\lambda_2)$ and $Z = \min(X, Y)$. The cumulative distribution function of Z is

$$
\begin{aligned}
F_z(z) &= P(Z \leq z) \\
&= P(\min(X, Y) \leq z) \\
&= 1 - P(\min(X, Y) > z) \\
&= 1 - P(X > z, Y > z) \\
&= 1 - P(X > z)P(Y > z)(by.independence) \\
&= 1 - e^{-\lambda_1 z} e^{-\lambda_2 z} \\
&= 1 - e^{-(\lambda_1 + \lambda_2) z}
\end{aligned}
$$

which we recognize as the cumulative distribution function of an exponential random variable with rate $\lambda_1 + \lambda_2$. □

PROPERTY 2.6. *If* $X \sim exp(\lambda_1)$ *and* $Y \sim exp(\lambda_2)$ *where* X *and* Y *are independent, then* $P(Y > X) = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$.

PROOF.

$$\begin{aligned} P(Y > X) &= \int_0^\infty \int_x^\infty \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} dy dx \\ &= \int_0^\infty \lambda_1 e^{(\lambda_1+\lambda_2)x} dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2}. \end{aligned}$$

□

THEOREM 2.7. *Let* X *and* Y *be independent random variables. Further, suppose that* $Y \sim exp(s)$ *and the p.d.f. of* X *is* $f(x)$. *Then*

$$f^*(s) = P(Y > X).$$

PROOF.

$$\begin{aligned} P(Y > X) &= \int_0^\infty \int_x^\infty f(x) s e^{-sy} dy dx \\ &= \int_0^\infty f(x)(-1)e^{-sy}|_{y=x}^{y=\infty} dx \\ &= \int_0^\infty f(x) e^{-sx} \end{aligned}$$

$$= f^*(s).$$

□

This gives a probabilistic interpretation for the Laplace transform. The Laplace transform of the pdf f(x) of a random variable X, with positive support, is the probability that X preceeds a "catastrophe" random variable Y which is exponentially distributed with rate s.

We next present a result using the probabilistic interpretation of the Laplace transform, and which uses the recursive type argument that will appear later in our ruin model analysis.

PROPERTY 2.8. *Let A, B and C be three states where transitions from state A to B happens at exponential rate μ_B, A to C happen at exponential rate μ_C, B to A happens at exponential rate λ_A and B to C happens at exponential rate λ_C then the Laplace transform of the probability density function of the time to move to state C if the system starts in state A can be expressed as* $f_A^*(s) = \dfrac{\mu_C}{\mu_C + \mu_B + s} + \dfrac{\mu_B}{\mu_C + \mu_B + s} f_B^*(s)$ *where* $f_B^*(s) = \dfrac{\lambda_C}{\lambda_C + \lambda_A + s} + \dfrac{\lambda_A}{\lambda_C + \lambda_A + s} f_A^*(s)$.
The solution for $f_A^*(s)$ *is*

$$f_A^*(s) = \frac{\mu_B\,\lambda_C + \lambda_A\,\mu_C + \lambda_C\,\mu_C + s\mu_C}{\lambda_C\,\mu_C + \mu_B\,\lambda_C + \lambda_C\,s + \lambda_A\,\mu_C + \lambda_A\,s + s\mu_C + s\mu_B + s^2}.$$

PROOF. Let T_A be the time to enter state C from A. Let T_B be thetime to enter C from B. Let $f_A^*(s)$ be the Laplace transform of the pdf of the time to enter C from B. Let T_B be the time to enter C from B. Let $f_B^*(s)$ be the Laplace transform of the pdf of the time to enter C from B. Then the Laplace transform of the pdf of T_A is the probability that $T_A < Y$ where Y is an exponential catastrophe random variable at rate s. When in state A one of two things can happen, either we can go straight to state C with probability $\frac{\mu_C}{\mu_C + \mu_B + s}$ or go to state B with probability $\frac{\mu_B}{\mu_C + \mu_B + s}$. However in the latter case, we must multiply by the probability that $T_B < Y$, namely by $f_B^*(s)$. If in state B we can go straight to C with probability $\frac{\lambda_C}{\lambda_C + \lambda_A + s}$ or go to state A with probability $\frac{\lambda_A}{\lambda_C + \lambda_A + s}$. In the latter case, we must multiply by the probability of going from A to C, namely by $f_A^*(s)$. Hence we get the two equations above. These equations are easy to solve to get an expression for $f_A^*(s)$. □

The previous property derives a special case of the phase type distribution.

CHAPTER 3

Loss Models and Laplace Transforms

In this chapter, we set up a Laplace transform model to describe the probability density function of the time to ruin measured from an initial reserve at level k. The goal here is primarily to create a methodology which can be used, if the parameters of the model are known.

Let T be the time to ruin, measured from the current time, which is taken as 0. We find the Laplace Transform of the pdf of T by using the probabilistic method of the previous chapter. Let Y be exponentially distributed with rate s. i.e. $f(y) = se^{-sy}$ for $s \geq 0$. We call Y the catastrophe random variable. We assume that the time until the next claim is exponentially distributed with rate λ. We assume that the claim size is a discrete random variable W with probability mass function $g(w)$, $w = 1, 2, \ldots$. We assume that W, Y, X are all independent. Define a time unit as the time to increase the reserve by 1 monetary unit, if there are no claims. For example, a monetary unit could be one billion dollars and a time unit could be 100 years.

Theorem 3.1. *Let $f(t,k)$ be the pdf of the time to ruin from an initial reserve of level k. Let the Laplace transform of the time to ruin (from level k) be $f^*(s,k)$. Let m be the level at which any additional revenues are not placed in the reserves. Then*

$$f^*(s,k) = e^{-(\lambda+s)1} f^*(s,1+k) + (1-e^{-(\lambda+s)}) \frac{\lambda}{\lambda+s} \left(\sum_{j=1}^{k} g(j) f^*(s,k-j) + \sum_{j=k+1}^{\infty} g(j)\right) \quad (1)$$

for $k = 1, \ldots, m-1$.

$$f^*(s,m) = e^{-(\lambda+s)1} f^*(s,m) + (1-e^{-(\lambda+s)}) \frac{\lambda}{\lambda+s} \left(\sum_{j=1}^{m} g(j) f^*(s,m-j) + \sum_{j=m+1}^{\infty} g(j)\right). \quad (2)$$

$$f^*(s,0) = e^{-(\lambda+s)1} f^*(s,1) + (1-e^{-(\lambda+s)}) \frac{\lambda}{\lambda+s}. \quad (3)$$

Proof. We consider the case $k = 1, \ldots, m-1$ first. Assume the reserve level is k at the start of a time period. By the results of chapter 4, the Laplace transform of the time to ruin is given by $P(T < Y)$ (i.e. the probability that the ruin occurs before catastrophe). In the first time unit, there are three possibilities -

(a) nothing happens and the reserves increase by one)

(b) a claim arrives (before a catastrophe)

(c) the catastrophe occurs (before a claim).

If (c) happens then $Y > T$, so we need not be concerned about that case.

Case (a) happens if there are no events (claims or catastrophes) in one time unit. Since both claims and catastrophes are exponential, and since the minimum of two independent exponentials is exponential with the sum of the rates, the probability of this case occurring is the right hand tail of an exponential distribution, namely $e^{-(\lambda+s)1}$. If this case happens, then the reserves increase by one so the probability of ruin from the new level $k+1$ is $f^*(s, k+1)$.

Case (b) happens if a claim or a catastrophe occurs in one time unit and the claim occurs before the catastrophe. The probability of a claim or catastrophe occurring in one time unit is 1 minus the probability of neither a claim nor a catastrophe, namely $1 - e^{-(\lambda+s)1}$. Given that an event (claim or catastrophe) occurred, the probability that the claim occurs before the catastrophe is $\dfrac{\lambda}{\lambda + s}$. If case (b) occurs, then the reserves drop by the amount of the claim(s) so the new level is $k - j$. We must sum over all j to cover all possibilities. For $j = 1$ to k, we reach the new level $k - j \geq 0$. For $j = k + 1$ to ∞, the new level is negative so ruin has already occurred.

If a claim occurs without ruin occurring, then we are assuming that the system resets itself to the new level at the point in time with the claim occurred.

The arguments for the two boundary cases $k = 0$ and $k = m$ are similar. □

From this theorem, we get the following corollary.

COROLLARY 3.2. *From the expressions given in Theorem 3.1, there is a linear system of equations in $m + 1$ unknowns $\{f^*(s, j)\}$ which can be solved.*

PROOF. The equations in Theorem 3.1 are linear in $\{f^*(s, j)\}$ and the coefficients involve λ and s. There are $m + 1$ independent linear equations in $m + 1$ unknowns. The solutions for $f^*(s, j)$, $j = 0, 1, \ldots, m$, will also involve λ and s. □

SPECIAL CASE 1: $g(1) = 1; g(j) = 0$ for $j \neq 1$, $m = 3$.

In this case, we consider three monetary levels $0, 1, 2, 3$. Level 0 means that the reserves are empty and if a claim is made during the time period, then ruin will occur. This is the most dangerous level. By contrast, level 3 is the least dangerous level and since the major claim amount is 1, it would take 4 major claims in the first time period to cause ruin. Level 3 is much safer than level. 1. This special case has features similar to the classic "gambler's ruin" problem. We apply Theorem 3.1 to get a linear system of equations for $f^*(s,0), f^*(s,1), f^*(s,2), f^*(s,3)$ and solve these using MAPLE.

$$f^*(s,0) = e^{-(\lambda+s)}f^*(s,1) + (1 - e^{-(\lambda+s)})\frac{\lambda}{\lambda+s} \tag{4}$$

$$f^*(s,1) = e^{-(\lambda+s)}f^*(s,2) + (1 - e^{-(\lambda+s)})\frac{\lambda}{\lambda+s}f^*(s,0) \tag{5}$$

$$f^*(s,2) = e^{-(\lambda+s)}f^*(s,3) + (1 - e^{-(\lambda+s)})\frac{\lambda}{\lambda+s}f^*(s,1) \tag{6}$$

$$f^*(s,3) = e^{-(\lambda+s)}f^*(s,3) + (1 - e^{-(\lambda+s)})\frac{\lambda}{\lambda+s}f^*(s,2) \tag{7}$$

Using MAPLE, the solution to (4), (5), (6), (7), is

$$f^*(s,0) = \frac{\left(-1 + \mathrm{e}^{-\lambda-s}\right)\lambda\left(s + \left(\mathrm{e}^{-\lambda-s}\right)^2\lambda - 2\,\mathrm{e}^{-\lambda-s}\lambda + \lambda\right)}{denom0} \tag{8}$$

where

$$denom0 = -s^2 - 2\,\left(e^{-\lambda-s}\right)^2 \lambda\,s + 3\,e^{-\lambda-s} s\lambda - 2\,\lambda\,s - 3\,\left(e^{-\lambda-s}\right)^2 \lambda^2 - \lambda^2 + 3\,e^{-\lambda-s}\lambda^2 + \left(e^{-\lambda-s}\right)^3 \lambda^2,$$

$$f^*(s,1) = \frac{\lambda^2\left(-1+e^{-\lambda-s}\right)\left(-2\,e^{-\lambda-s}\lambda + \left(e^{-\lambda-s}\right)^2 \lambda + s + \lambda - e^{-\lambda-s}s\right)}{denom1} \tag{9}$$

where $denom1 = (\lambda + s)\,A1$ and

$$A1 = -s^2 - 2\,\left(e^{-\lambda-s}\right)^2 \lambda\,s + 3\,e^{-\lambda-s} s\lambda - 2\,\lambda\,s - 3\,\left(e^{-\lambda-s}\right)^2 \lambda^2 - \lambda^2 + 3\,e^{-\lambda-s}\lambda^2 + \left(e^{-\lambda-s}\right)^3 \lambda^2,$$

$$f^*(s,2) = \frac{\lambda^3\left(-1+e^{-\lambda-s}\right)\left(1 - 2\,e^{-\lambda-s} + \left(e^{-\lambda-s}\right)^2\right)}{denom2} \tag{10}$$

where $denom2 = (\lambda + s)\,A2$ and

$$A2 = -s^2 - 2\,\left(e^{-\lambda-s}\right)^2 \lambda\,s + 3\,e^{-\lambda-s} s\lambda - 2\,\lambda\,s - 3\,\left(e^{-\lambda-s}\right)^2 \lambda^2 - \lambda^2 + 3\,e^{-\lambda-s}\lambda^2 + \left(e^{-\lambda-s}\right)^3 \lambda^2,$$

$$f^*(s,3) = \frac{\lambda^4\left(-1+\mathrm{e}^{-\lambda-s}\right)\left(1-2\,\mathrm{e}^{-\lambda-s}+\left(\mathrm{e}^{-\lambda-s}\right)^2\right)}{denom3} \tag{11}$$

where $denom3 = (\lambda+s)^2\,A3$ and

$$A3 = -s^2 - 2\,\left(\mathrm{e}^{-\lambda-s}\right)^2\lambda\,s + 3\,\mathrm{e}^{-\lambda-s}s\lambda - 2\,\lambda\,s$$
$$- 3\,\left(\mathrm{e}^{-\lambda-s}\right)^2\lambda^2 - \lambda^2 + 3\,\mathrm{e}^{-\lambda-s}\lambda^2 + \left(\mathrm{e}^{-\lambda-s}\right)^3\lambda^2.$$

Let us focus on $f^*(s,0)$ as representative of what happens to $f^*(s,0), f^*(s,1), f^*(s,2), f^*(s,3)$. This Laplace transform of $f^*(s,0)$ is a function of λ and s. It represents the probability that the time to ruin is less than the time of a catastrophe, if the reserves are at level 0. As the catastrophe rate s increases, the probability decreases so for fixed λ, $f^*(s,0)$ should decrease. As the claim rate λ increases, then the probability of ruin before the catastrophe increases so for fixed s, $f^*(s,0)$ should be increasing in λ. We see this in the three dimensional graph of $f^*(s,0)$ vs λ and s, as seen in Figure 3.1.

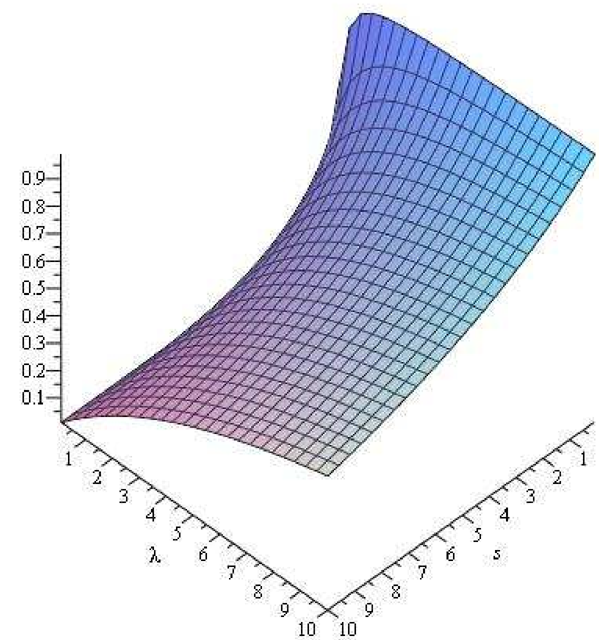

FIGURE 3.1. Laplace transform $f^*(s, 0)$ vs λ and s

We further consider $f^*(s,0)$ for a typical value of λ, namely $\lambda = 3$. As mentioned, $f^*(s,0)$ is the probability that ruin occurs before the catastrophe and must be a decreasing function of s. Since $f^*(s,0)$ is a probability, its values must be between 0 and 1. As the catastrophe rate s increases, the probability that ruin occurs before the catastrophe goes to zero and as the catastrophe rate s goes to zero, the probability that ruin occurs before the catastrophe goes to one. This can be seen in Figure 3.2.

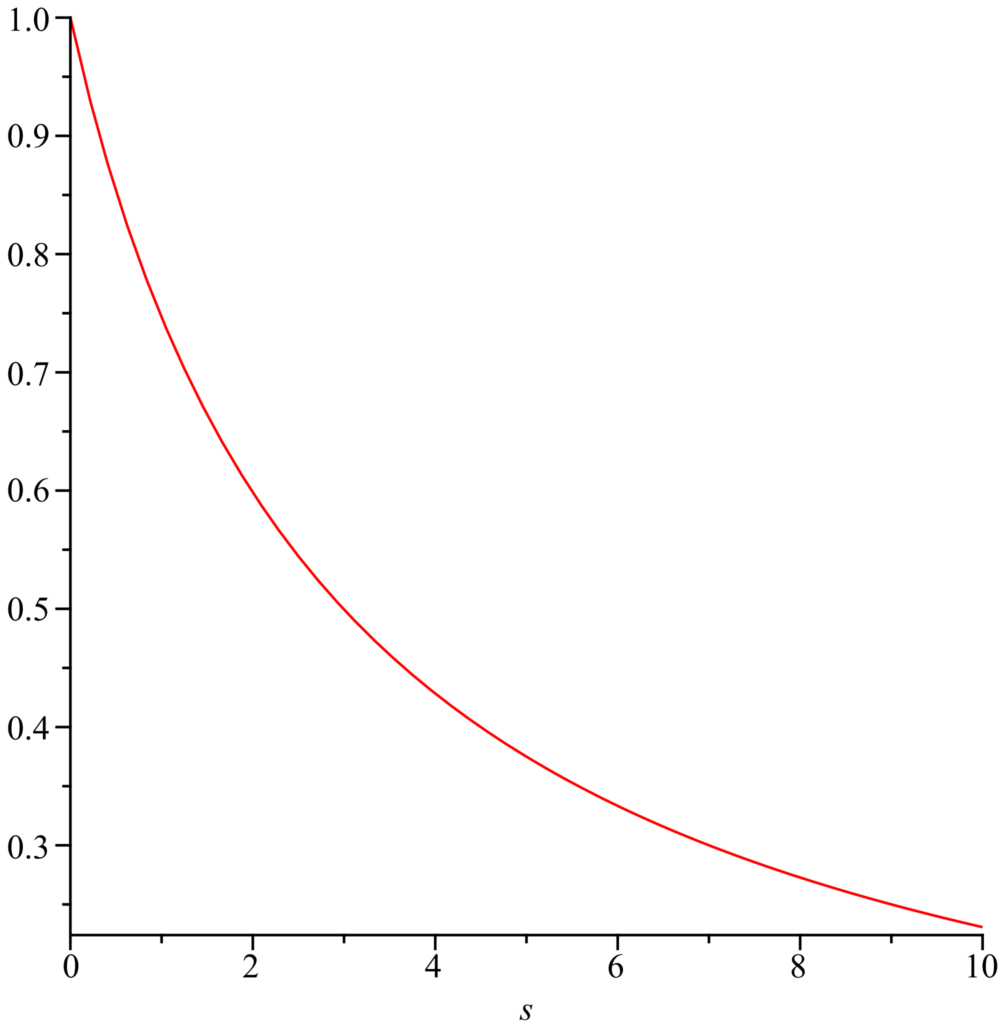

FIGURE 3.2. Laplace transform $f^*(s, 0)$ for $\lambda = 3$

Finally in Figure 3.3, we compare $f^*(s,0)$, $f^*(s,1)$, $f^*(s,2)$, $f^*(s,3)$, which give the Laplace transforms of the times to ruin for different starting monetary levels of reserves. We would expect that the larger the reserves, the lower the probability of ruin occurring before a catastrophe. So we expect that

$$f^*(s,3) > f^*(s,2) > f^*(s,1) > f^*(s,0)$$

Figure 3.3 shows these features.

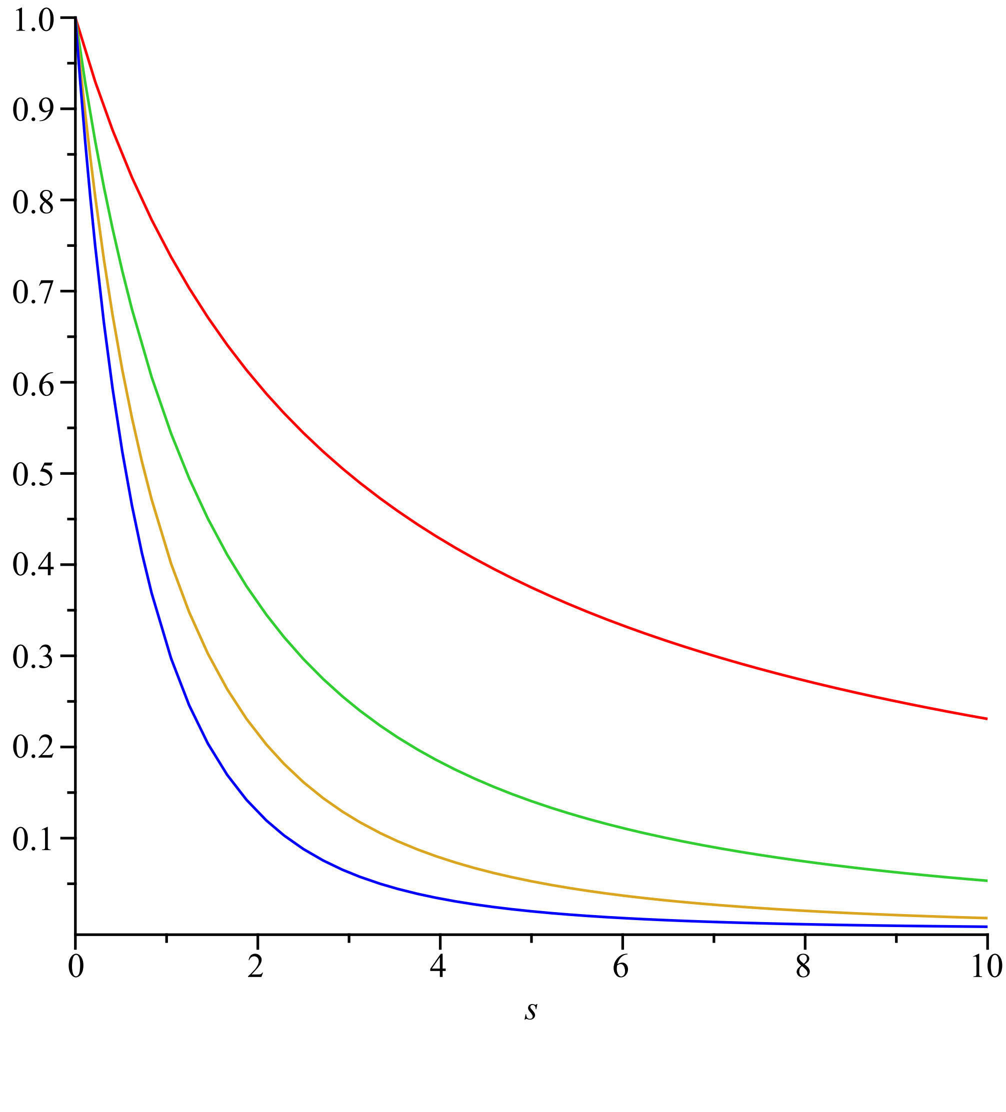

FIGURE 3.3. Laplace transforms $f^*(s,0), f^*(s,1), f^*(s,2), f^*(s,3)$ for $\lambda = 3$

One useful feature of the Laplace transform is that we can easily obtain the moments of the random variables of interest. Let T_0 be the time until ruin when the system begins in state 0. Then $E(T_0) = -\frac{f^*(s,0)}{dt}|_{t=0}$. So our next task is to compute $\frac{f^*(s,0)}{dt}$.

We compute $\frac{d}{ds}f^*(s,0) =$

$$-3\,\frac{e^{-3-s}\left(s+3\,(e^{-3-s})^2-6\,e^{-3-s}+3\right)}{-s^2-6\,(e^{-3-s})^2\,s+9\,e^{-3-s}s-6\,s-27\,(e^{-3-s})^2-9+27\,e^{-3-s}+9\,(e^{-3-s})^3}+$$

$$\frac{(-3+3\,e^{-3-s})\left(1-6\,(e^{-3-s})^2+6\,e^{-3-s}\right)}{-s^2-6\,(e^{-3-s})^2\,s+9\,e^{-3-s}s-6\,s-27\,(e^{-3-s})^2-9+27\,e^{-3-s}+9\,(e^{-3-s})^3}-$$

$$\frac{K\left(-2\,s+12\,(e^{-3-s})^2\,s+48\,(e^{-3-s})^2-9\,e^{-3-s}s-18\,e^{-3-s}-6-27\,(e^{-3-s})^3\right)}{\left(-s^2-6\,(e^{-3-s})^2\,s+9\,e^{-3-s}s-6\,s-27\,(e^{-3-s})^2-9+27\,e^{-3-s}+9\,(e^{-3-s})^3\right)^2} \tag{12}$$

where $K = (-3+3\,e^{-3-s})\left(s+3\,(e^{-3-s})^2-6\,e^{-3-s}+3\right)$

The graph appears in Figure 3.4. If we exam this figure at $s = 0$, then we have $-E(T_0)$. We observe from the graph that $E(T_0)$ is approximately 0.35. To get more accuracy, we substitute $s = 0$ in our expression (12) for $\frac{df^*(s,0)}{ds}$.

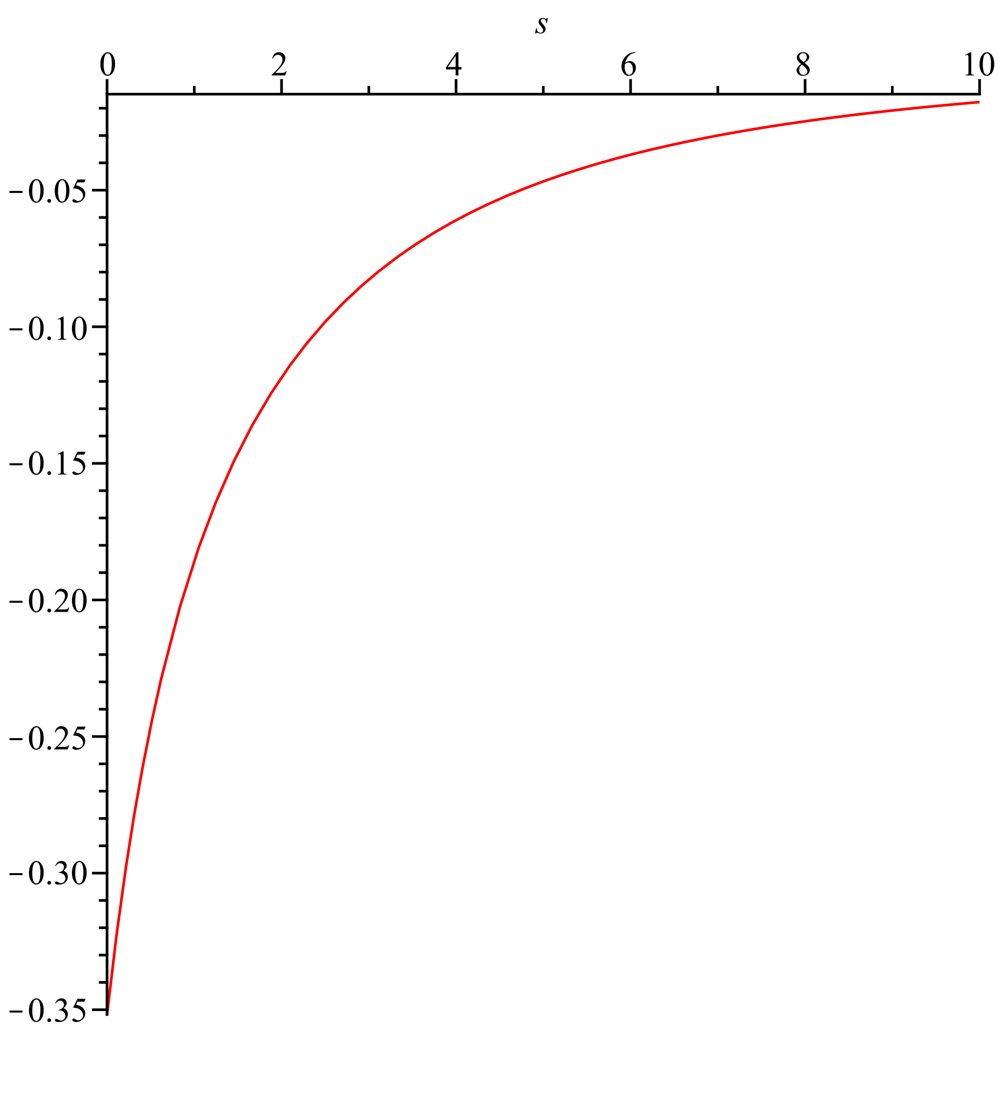

FIGURE 3.4. Graph of the derivative of $f^*(s, 0)$

Thus $\frac{df^*(s,0)}{ds}|_{s=0} =$

$$-3\,\frac{e^{-3}\left(3+3\,(e^{-3})^2-6\,e^{-3}\right)}{-9-27\,(e^{-3})^2+27\,e^{-3}+9\,(e^{-3})^3}+\frac{(-3+3\,e^{-3})\left(1-6\,(e^{-3})^2+6\,e^{-3}\right)}{-9-27\,(e^{-3})^2+27\,e^{-3}+9\,(e^{-3})^3}-$$

$$\frac{(-3+3\,e^{-3})\left(3+3\,(e^{-3})^2-6\,e^{-3}\right)\left(-6+48\,(e^{-3})^2-18\,e^{-3}-27\,(e^{-3})^3\right)}{\left(-9-27\,(e^{-3})^2+27\,e^{-3}+9\,(e^{-3})^3\right)^2} \tag{13}$$

and simplifying gives -0.3517616164. Thus the expected time to ruin conditional on starting in state 0 is 0.3517616164.

Similarly we can find expected values of the time to ruin for the other starting values. Define T_i to be the time to ruin starting in state i. Then we find

$E(T_0)$	$E(T_1)$	$E(T_2)$	$E(T_3)$
.352	.703	1.054	1.39

We realize that the expected time to move from state 3 to state 2 should be almost the same as the expected time to move from state 2 to state 1 which should be almost the same as the expected time to move from state 1 to 0, which should be almost the same as the expected time to move from state 0 to ruin. Thus it is no surprise to see the linearity of the expected times vs starting level.

3.1. Inverting the Laplace Transform

We now have the expected time to ruin but the Laplace transform contains more information than the expected value. If we invert the Laplace transform, then we would have the complete pdf of the time to ruin. Unfortunately, the complex expressions for the Laplace transforms make it impossible for MAPLE to handle this analytically. We thus turn to numerical techniques for the inversion of Laplace transforms. One technique that can be used is the Stehfest algorithm (Vogt, 2006; [**20**]). We applied the MAPLE code presented by Vogt for various values of s in $f^*(s,0)$ and obtained the following values of $f(t)$.

t	0.1	0.2	0.3	0.5	1.0	2.0
$f(t)$	2.22	1.64	1.22	0.67	0.09	0.02

This gives us a good idea of the shape of the pdf. Using the points above and a few additional points, we plot the pdf in Figure 3.5. We notice a curious bump near $t = 1.4$.

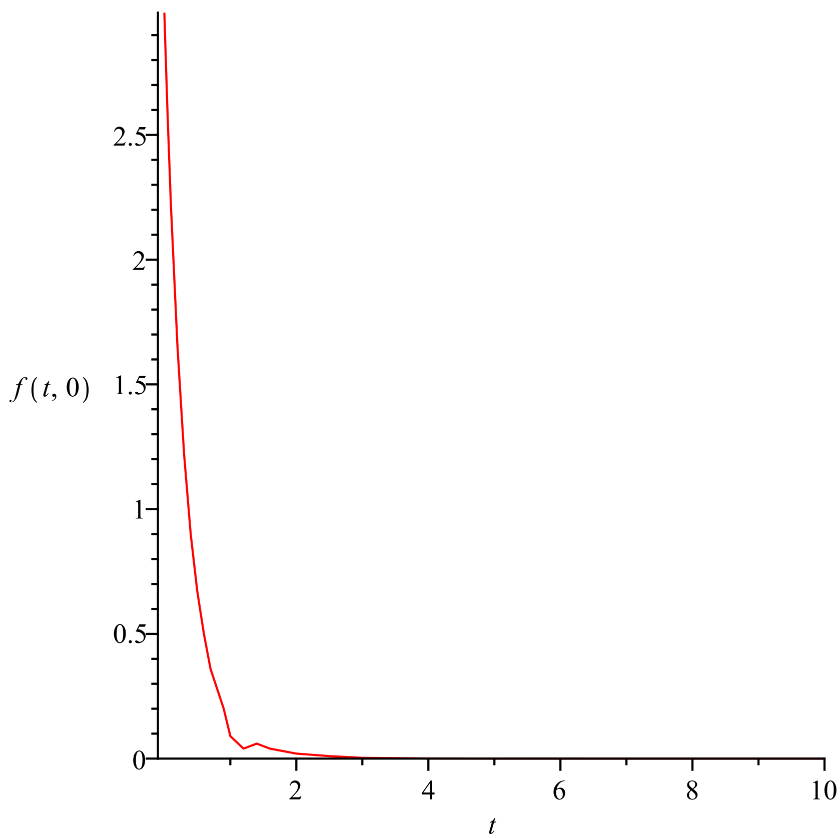

FIGURE 3.5. Graph of pdf $f(t,0)$ from the inverse Laplace Transform derivative of $f^*(s,0)$

SPECIAL CASE 2:

$g(j) = \frac{1}{5};$ for j = 1,2,...,5 and 0 otherwise, $m = 3$.

We note that in Special Case 1, that all drops due to a claim have size 1, which limits the applicability of the model. In contrast, Special Case 2 has more complexity, and requires more care. Our equations given in Theorem 3.1 become the following:

$$f^*(s,0) = e^{-(\lambda+s)}f^*(s,1) + (1 - e^{-(\lambda+s)})\frac{\lambda}{\lambda+s} \tag{14}$$

$$f^*(s,1) = e^{-(\lambda+s)}f^*(s,2) + (1 - e^{-(\lambda+s)})\frac{\lambda}{\lambda+s}(\frac{1}{5}f^*(s,0) + \sum_{j=2}^{5}\frac{1}{5} + \sum_{j=6}^{\infty}0) \tag{15}$$

$$f^*(s,2) = e^{-(\lambda+s)}f^*(s,3) + (1 - e^{-(\lambda+s)})\frac{\lambda}{\lambda+s}(\frac{1}{5}f^*(s,1) + \frac{1}{5}f^*(s,0) + \sum_{j=3}^{5}\frac{1}{5} + \sum_{j=6}^{\infty}0) \tag{16}$$

$$f^*(s,3) = e^{-(\lambda+s)}f^*(s,3) + (1 - e^{-(\lambda+s)})\frac{\lambda}{\lambda+s}(\frac{1}{5}f^*(s,2) + \frac{1}{5}f^*(s,1) + \frac{1}{5}f^*(s,0) + \sum_{j=4}^{5}\frac{1}{5} + \sum_{j=6}^{\infty}0) \tag{17}$$

The solution, from MAPLE, is

$$f^*(s,0) = \lambda\frac{num1}{denom4} \tag{18}$$

where

$$denom4 = \lambda^2 \left(e^{-\lambda-s}\right)^3 - \lambda^2 \left(e^{-\lambda-s}\right)^2 + 15\,\lambda^2 e^{-\lambda-s} + 15\,\lambda\, e^{-\lambda-s} s - 50\,\lambda\, s - 25\, s^2 - 25\,\lambda^2$$

and

$$num1 = 5\,\left(e^{-\lambda-s}\right)^2 s$$
$$+\,5\, e^{-\lambda-s} s - 25\, s + 5\,\left(e^{-\lambda-s}\right)^3 s + \lambda\,\left(e^{-\lambda-s}\right)^3 - \lambda\,\left(e^{-\lambda-s}\right)^2 - 25\,\lambda + 15\,\lambda\, e^{-\lambda-s}$$

$$f^*(s,1) = \lambda \frac{num2}{denom5} \tag{19}$$

where $denom5 = (\lambda + s)\, A4$,

$$A4 = \left(\lambda^2 \left(e^{-\lambda-s}\right)^3 - \lambda^2 \left(e^{-\lambda-s}\right)^2 + 15\,\lambda^2 e^{-\lambda-s} + 15\,\lambda\, e^{-\lambda-s} s - 50\,\lambda\, s - 25\, s^2 - 25\,\lambda^2\right)$$

and

$$num2 = \lambda^2 \left(e^{-\lambda-s}\right)^3 - \lambda^2 \left(e^{-\lambda-s}\right)^2 + 6\,\left(e^{-\lambda-s}\right)^2 \lambda\, s + 5\,\left(e^{-\lambda-s}\right)^2 s^2 +$$
$$15\,\lambda^2 e^{-\lambda-s} + 19\,\lambda\, e^{-\lambda-s} s + 5\, e^{-\lambda-s} s^2 - 25\,\lambda^2 - 45\,\lambda\, s - 20\, s^2$$

$$f^*(s,2) = \lambda \frac{num3}{denom6} \tag{20}$$

where $denom6 = (\lambda + s)\, A5$

$$A5 = \left(\lambda^2 \left(e^{-\lambda-s}\right)^3 - \lambda^2 \left(e^{-\lambda-s}\right)^2 + 15\,\lambda^2 e^{-\lambda-s} + 15\,\lambda\, e^{-\lambda-s} s - 50\,\lambda\, s - 25\, s^2 - 25\,\lambda^2\right)$$

and

$$num3 = \lambda^2 \left(e^{-\lambda-s}\right)^3 + s\lambda\, \left(e^{-\lambda-s}\right)^3 - \lambda^2 \left(e^{-\lambda-s}\right)^2 + 15\,\lambda^2 e^{-\lambda-s} +$$

$$18\,\lambda\, e^{-\lambda-s} s + 5\, e^{-\lambda-s} s^2 - 25\,\lambda^2 - 39\,\lambda\, s - 15\, s^2$$

$$f^*(s,3) = 1/5\,\lambda \frac{num4}{denom7} \tag{21}$$

where $denom7 = (\lambda + s)^2\, A6$,

$$A6 = (\lambda + s)^2 *$$

$$\left(\lambda^2 \left(e^{-\lambda-s}\right)^3 - \lambda^2 \left(e^{-\lambda-s}\right)^2 + 15\,\lambda^2 e^{-\lambda-s} + 15\,\lambda\, e^{-\lambda-s} s - 50\,\lambda\, s - 25\, s^2 - 25\,\lambda^2\right),$$

$$num4 = 5\,\lambda^3 \left(e^{-\lambda-s}\right)^3 + 9\,\lambda^2 \left(e^{-\lambda-s}\right)^3 s + 5\, \left(e^{-\lambda-s}\right)^3 s^2 \lambda - 5\,\lambda^3 \left(e^{-\lambda-s}\right)^2 +$$

$$8\,\lambda^2 s \left(e^{-\lambda-s}\right)^2 + 10\, \left(e^{-\lambda-s}\right)^2 s^2 \lambda + 75\,\lambda^3 e^{-\lambda-s} + 117\,\lambda^2 e^{-\lambda-s} s + 45\,\lambda\, e^{-\lambda-s} s^2$$

$$- 125\,\lambda^3 - 284\,\lambda^2 s - 210\,\lambda\, s^2 - 50\, s^3.$$

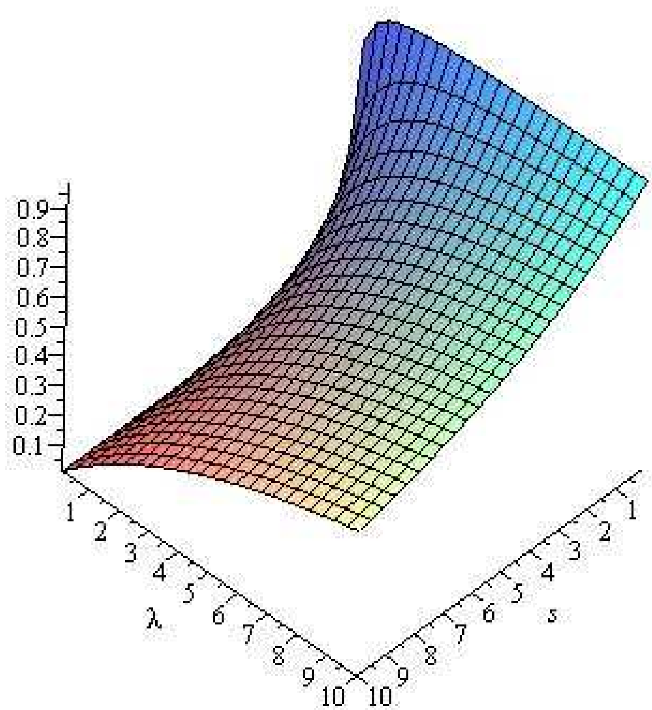

FIGURE 3.6. Laplace transform $f^*(s, 0)$ vs λ and s

As in Special Case 1, we obtain a two dimensional graph of $f^*(s, 0)$ vs λ and s. This appears in Figure 3.6.

We next present three graphs of $f^*(s,0), f^*(s,1), f^*(s,2), f^*(s,3)$ vs s when $\lambda = 0.5, 3.0, 8.0$. These appear in Diagram 3.7, Diagram 3.8, Diagram 3.9.

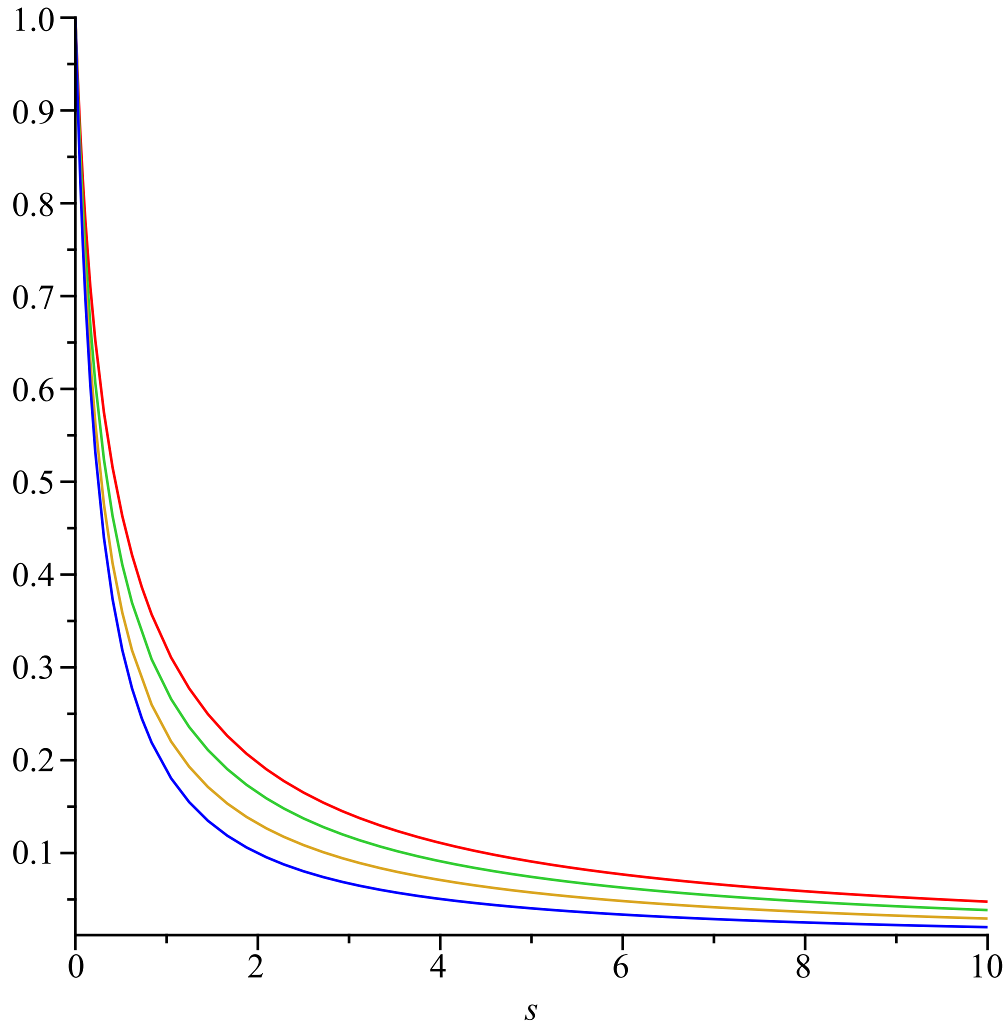

FIGURE 3.7. Laplace transform $f^*(s, 0), f^*(s, 1), f^*(s, 2), f^*(s, 3)$ for $\lambda = 0.5$

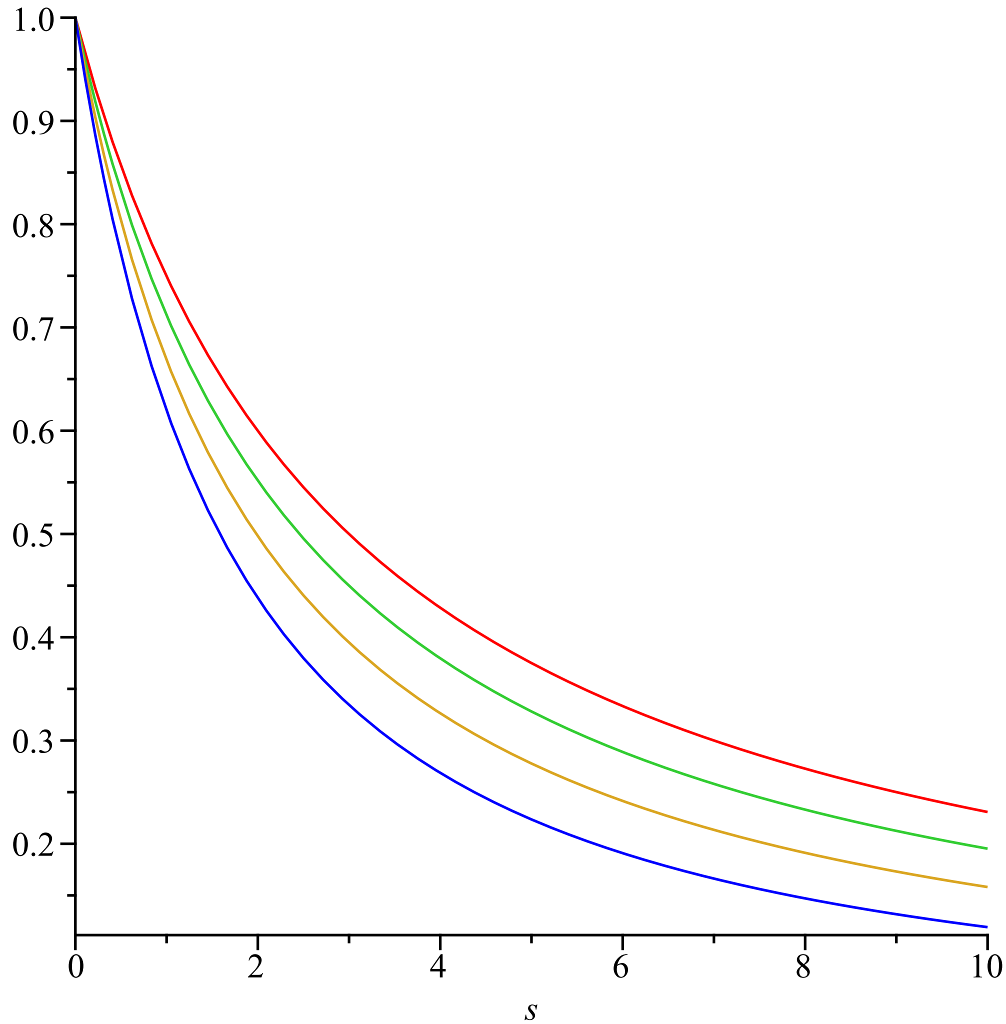

FIGURE 3.8. Laplace transform $f^*(s,0), f^*(s,1), f^*(s,2), f^*(s,3)$ for $\lambda = 3$

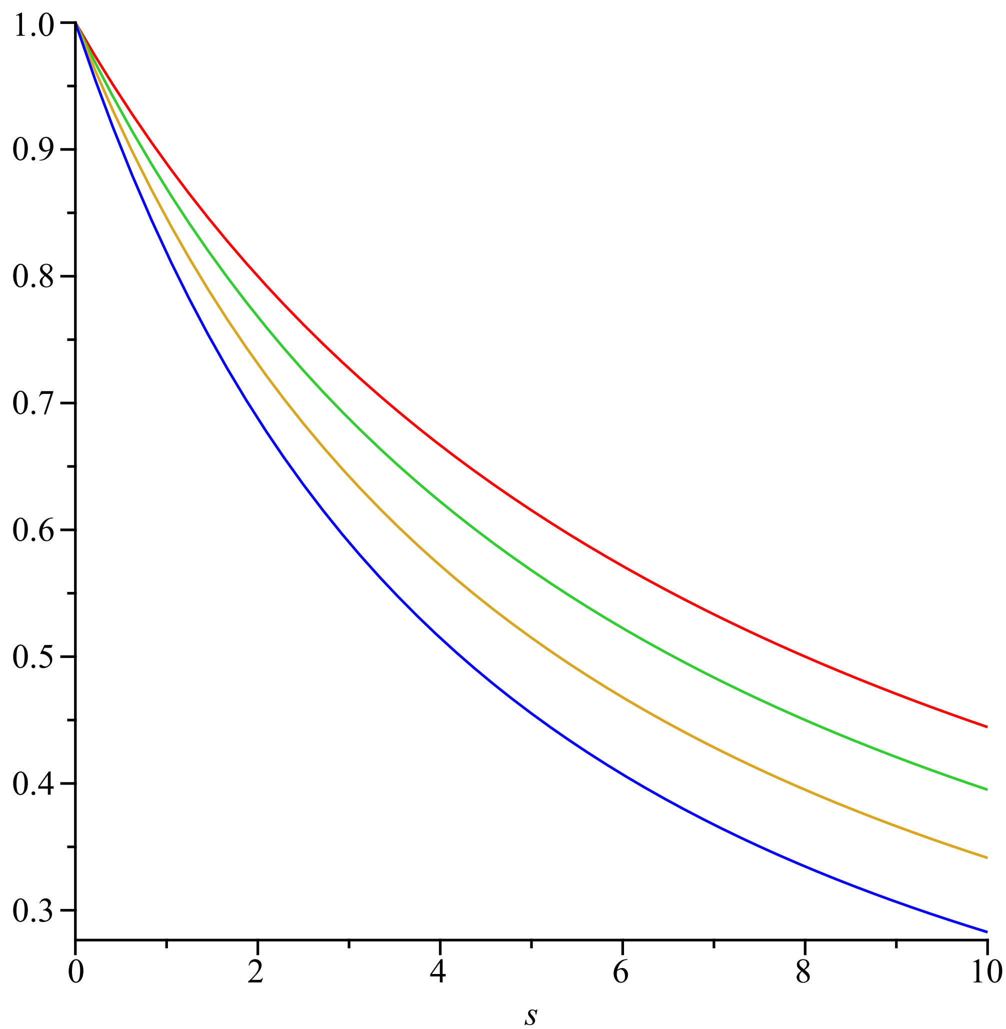

FIGURE 3.9. Laplace transform $f^*(s,0), f^*(s,1), f^*(s,2), f^*(s,3)$ for $\lambda = 8$

The above three graphs show the sensitivity of the Laplace transform to changes in the value of λ.

3.2. Conclusions

In this paper, we have used the probabilistic interpretation of Laplace transforms to find the Laplace transforms of the time to ruin from different initial starting reserve states. After solving for the Laplace transforms, we solved for the expected time to ruin in one case and indicated that the other cases can be solved in a similar manner. We then numerically inverted the Laplace transform to obtain the pdf of the time until ruin. To the best of our knowledge, this technique of using the probabilistic interpretation of the Laplace transform has not been used before in the analysis of ruin problems.

The actual cases were selected to illustrate the method, not because they necessarily reflect an actual structure that has occurred. Insurance companies have many different sizes and the types of risks hat are covered vary considerably. In order to use the technique for a particular company, an analysis of the structure of the claim sizes and interclaim time would be needed. alim sizes and

The important new contribution of this paper is the technique that presented, and applied to loss reserves. Property 2.8 is new, as is Theorem 3.1.

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