Transient results for *M*/*M*/1/*c* queues via path counting

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Abstract: We find combinatorially the probability of having n customers in an $M/M/1/c$ queueing system at an arbitrary time t when the arrival rate λ and the service rate μ are equal, including the case $c = \infty$. Our method uses path-counting methods and finds a bijection between the paths of the type needed for the queueing model and paths of another type which are easy to count. The bijection involves some interesting geometric methods.

Keywords: counting; *M*/*M*/1; *M*/*M*/1/*c*; paths; queueing; transient.

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1 Introduction

In queueing theory, the most commonly studied system is the *M*/*M*/1 system which has exponentially distributed interarrival times, exponentially distributed service times and a single server. The limiting probabilities, which are easy to obtain, of finding i customers in the system (for $i = 0,1,2,...$) form a geometric distribution and require the arrival rate λ to be less than the service rate μ. However in the transient case (finite time), the probabilities are most commonly derived using differential equations and are expressed in terms of Bessel functions (Gross et al., 2008). In the transient case, the restriction that $\lambda < \mu$ is not necessary. Transient results in queueing are receiving considerable attention in recent queueing literature.

In this article, we consider a combinatorial method for finding the transient probabilities at time t when $\lambda = \mu$. Successful combinatorial analyses for transient probabilities of an $M/M/1$ queue date back to Champernowne (1956) while further achievements in combinatorial queueing are documented in Takacs (1967) and Jain, Mohanty and Bohm (2007).

The type of geometric combinatorial argument presented here is new and quite appealing. It handles in a unified way, $M/M/1$ and $M/M/1/c$ cases starting with an initial number of customers that can be zero or non-zero. The form of many of the results is also new.

In Section 2, we present a brief literature review. In Section 3, we indicate the relationship between transient queueing probabilities and path counts. In Section 4, we find an interesting geometric transformation which aids in path counting for the $M/M/1$ case. In Section 5, we find a geometric transformation which aids in path counting in the $M/M/1/c$ case. Some additional interesting properties arise in this case. Conclusions and comments follow at the end.

2 Literature review

There is considerable current research on transient queueing models. In particular, there is interest in different forms of solutions (to a variety of queueing problems) and different ways of obtaining such solutions.

Sharma (1997) obtains a considerable number of transient queueing results often using Laplace transforms. Krinik and Mortensen (2007) find transient solutions, via recursion, for birth–death type models that allow catastrophes. Krishna Kumar and Pavai Madheswari (2005, 2007) consider transient analysis of an $M/M/1$ queue with catastrophes and server failures.

Bohm, Krinik and Mohanty (1997) use combinatorial counting methods to find expressions for the transient probabilities for $M/M/1$ and $M/M/c$ models. Griffiths, Leonenko and Williams (2006) find expressions for transient probabilities of $M/E_k/1$ queueing models. Their solution uses differential equations and generating functions.

Jain and Meitei (2005) remark on the variety of different solution forms that have been obtained for transient probabilities in the $M/M/1$ case. Tarabia and El-Baz (2006) obtain a relatively simple expression for the transient probabilities in the $M/M/1/c$ case by assuming that the solution has a particular form. Combinatorial expressions show up in their derivations.

Van Houdt and Blondia (2005) consider transient queue length and waiting time distributions for a discrete time queueing system. Margolius (2007) considers transient solutions to quasi-birth and death processes with time-varying periodic rates.

Joy and Jones (2005) use expressions for transient probabilities of $M^b/M/1$ batch arrival queues in order to compute the probability of serving at least k customers by time t. This probability is then proposed as one of several performance measures for hospital service.

There are many other papers for transient probabilities in queues expressed in a variety of forms and derived in a variety of ways.

3 Queueing model

In this section, we have given an expression for the number of customers in an *M*/*M*/1 queue at time t in terms of certain path counts.

Consider a single server *M*/*M*/1 queueing system with arrivals at rate λ and service at rate μ. Assume that at time zero, there are zero customers in the system. We are interested in the probability of finding i customers in the system at time t. If we want to simulate such a system after each arrival or service completion, we could generate two exponential random values, one interarrival time and one completion time, and choose the minimum to determine what the next event (an arrival or a departure) would be. However, if there are zero customers in the system and the next event (minimum) is a 'departure' then we interpret this to mean that the system stays with zero customers, but the clock moves forward by the minimum of the two random values. The embedded Markov chain is that associated with the process with respect to the rate $\lambda+\mu$.

Let $p_{0,i}(t)$ be the probability that at time t, there are i customers in the system, given that there were 0 customers at time 0. During the time length t, there will be some number of events k (arrivals or 'departures' with the count having a Poisson distribution). The k events will result in i customers present in the system at time t if there is a path of k steps from $(0,0)$ to (k,i) where the path has steps of type (*) below.

(*) For $m>0$, a step is a movement from an arbitrary feasible point (j,m) to $(j+1,m+1)$ or from (j,m) to $(j+1,m-1)$. If $m=0$, then a step is a movement from $(j,0)$ to $(j+1,0)$ or from $(j,0)$ to $(j+1,1)$.

Theorem 1. For an *M*/*M*/1 queue, assume $\lambda=\mu$. Then,

$$p_{0,i}(t)=\sum_{k=i}^{\infty}\frac{(2\mu t)^k\,\mathrm{e}^{-2\mu t}}{k!}\frac{N_{0,i}(k)}{2^k}=\sum_{k=i}^{\infty}\frac{(\mu t)^k\,\mathrm{e}^{-2\mu t}N_{0,i}(k)}{k!} \tag{1}$$

where $N_{0,i}(k)$ is the number of paths from (0, 0) to (k, i) in k steps.

If $\lambda\neq\mu$,

$$p_{0,i}(t)=\sum_{k=i}^{\infty}\sum_{f=0}^{k-i}\frac{((\lambda+\mu)t)^k\,\mathrm{e}^{-(\lambda+\mu)t}}{k!}N_{0,i,f}(k)\left(\frac{\lambda}{\lambda+\mu}\right)^{u}\left(\frac{\mu}{\lambda+\mu}\right)^{d+f} \tag{2}$$

where $N_{0,i,f}(k)$ is the number of paths from $(0,0)$ to (k,i) in k steps with exactly f flat steps, where u, d can be solved in terms of i,k,f from the restrictions $u+d+f=k$, $u-d=i$.

Proof. The Poisson probability of k events in time t is $(((\lambda+\mu)t)^k e^{-(\lambda+\mu)t})/k!$ for $k=0,1,\ldots$. The probability of moving up on any step is $\lambda/(\lambda+\mu)$. If $\lambda=\mu$ then the probability of moving up at each step is 1/2. If the movement on a step is not up then it must be down (or flat if the start of a step is $(*,0)$). To obtain Equation (1), we condition on k and sum over all cases.

If $\lambda \neq \mu$ then the probabilities of two paths which end at i will be equal if they have the same number of flat steps. So, we partition our summation into classes with the same number of flat steps f. We condition on k and then on f and sum over all cases to get (2). □

The main contribution of this article is the study of $N_{0,i}(k)$ (for both the *M*/*M*/1 and *M*/*M*/1/*c* cases), examining the geometric interpretation and finding an expression for it.

It is interesting to compare our result with the standard expression using Bessel functions given below (Gross et al., 2008).

$$p_{0,i}(t) = \mathrm{e}^{-(\lambda+\mu)t}\left[\left(\frac{\lambda}{\mu}\right)^{\frac{i}{2}} I_i(at) + \left(\frac{\lambda}{\mu}\right)^{\frac{i-1}{2}} I_{i+1}(at) + \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{i} \sum_{j=i+2}^{\infty}\left(\frac{\lambda}{\mu}\right)^{\frac{-j}{2}} I_j(at)\right]$$

where $a = 2\mu(\lambda/\mu)^{1/2}$ and $I_k(y) = \sum_{m=0}^{\infty}((y/2)^{k+2m})/((k+m)!m!)$. When $\lambda = \mu$, this expression reduces to

$$p_{0,i}(t) = \mathrm{e}^{-2\mu t}(I_i(2\mu t) + I_{i+1}(2\mu t)).$$

4 Path counting for *M*/*M*/1 queues

In Section 4.1, we find some results for path counts $\hat{N}_{0,i}(k)$ which are related to the path counts $N_{0,i}(k)$ that we seek. In Section 4.2, we give a transformation from one type of path to another. In Section 4.3, we give the inverse transformation.

4.1 Preliminaries

We begin with standard notation on paths and extend it to paths of the type we need. We also suggest the existence of a bijection that will help to obtain our path counts.

Definition 1. An upward step U from (a,b) is a line segment connecting the lattice points (a,b) and $(a+1,b+1)$. A downward step D is a line segment connecting (a,b) to $(a+1,b-1)$. A flat step F is a line segment connecting $(a,0)$ to $(a+1,0)$.

In the following definition, a path allows negative y coordinates.

Definition 2. An U-D path of length k is a polygonal path composed of a sequence of upward and downward steps only, beginning at the origin and terminating at the lattice point (k,i) where *k* and *i* are integers and $k+i \equiv 0 \bmod 2$.

Definition 3. Let P be a polygonal path of length k composed of upward, downward and flat paths. If P is restricted to the first quadrant and flat paths only occur on the horizontal axis, then P is called a U-D-F path.

Lemma 1. If k represents the length of a U-D path, with 0 as the initial height, and i as the terminal height then the number of such paths, denoted $\hat{N}_{0,i}(k)$, is given by

$$\hat{N}_{0,i}(k) = \binom{k}{\frac{k+i}{2}}. \tag{3}$$

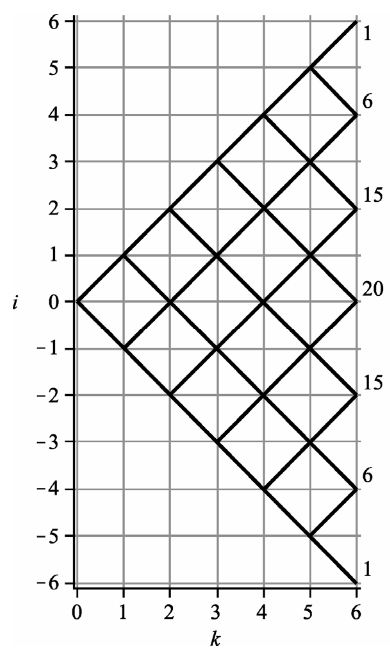
Proof. Let u and d represent the numbers of upward and downward steps, respectively, contained in an U-D path of length k terminating at height i. Then, we have $i = u - d$ and $k = u + d$. Solving for u gives $u = (k+i)/2$.

Since there are only two choices for each segment, upward or downward, the number of such paths is

$$\hat{N}_{0,i}(k) = \binom{k}{u} = \binom{k}{\frac{k+i}{2}}. \qquad \square$$

Let S_1 be the set of all lattice points at which a U-D path may terminate, that is, $S_1 = \{(k,i) \mid k > 0, -k \le i \le k, k+i \equiv 0 \bmod 2\}$. If each such lattice point is connected to its neighbours only by an upward or a downward step and is assigned a label α where α is the number of U-D paths terminating at that point, then we have a representation of Pascal's triangle (Figure 1).

Figure 1 S_1 with connecting steps



Thus, for example, after step 6, we have

$\widehat{N}_{0,-6}(6)$	$\widehat{N}_{0,-4}(6)$	$\widehat{N}_{0,-2}(6)$	$\widehat{N}_{0,0}(6)$	$\widehat{N}_{0,2}(6)$	$\widehat{N}_{0,4}(6)$	$\widehat{N}_{0,6}(6)$
1	6	15	20	15	6	1

Now suppose that S_1 is folded along the line $i=-1/2$ (or $Y=-1/2$ in the usual X-Y coordinate system), so that the each point (k,i) below the k-axis is mapped to the point $(k,-i-1)$. The result is a set of points, $S_2=\{(k,i)\,|\,k>0,\ 0\le i\le k,\ k,i\in\mathbb{Z}\}$ which is the set of all lattice points at which an U-D-F path may terminate. Figure 2 shows S_2 connected using the permissible steps. If each Pascal label α from S_1 is carried over to S_2, then α corresponds to the number of U-D-F paths terminating at that point.

Thus for example, if $N_{0,i}(6)$ represents the number of U-D-F paths from 0 to i after 6 steps, the counts can be shown to be

$N_{0,0}(6)$	$N_{0,1}(6)$	$N_{0,2}(6)$	$N_{0,3}(6)$	$N_{0,4}(6)$	$N_{0,5}(6)$	$N_{0,6}(6)$
20	15	15	6	6	1	1

Compare these values with $\widehat{N}_{0,i}(6)$ together with the folding over $i=-1/2$. The two results are the same!

This suggests that there may exist a bijective function that maps each U-D path to a unique U-D-F path in such a way that we can easily count the U-D-F paths. In fact, a U-D path can be transformed to the corresponding U-D-F path as shown below. Upward and downward steps are denoted as U and D. A sample path has been chosen to demonstrate the path transformation algorithm. It is quite likely that the bijection between the two types of path can have other uses than the use given here.

Figure 2 S_2 with connecting steps

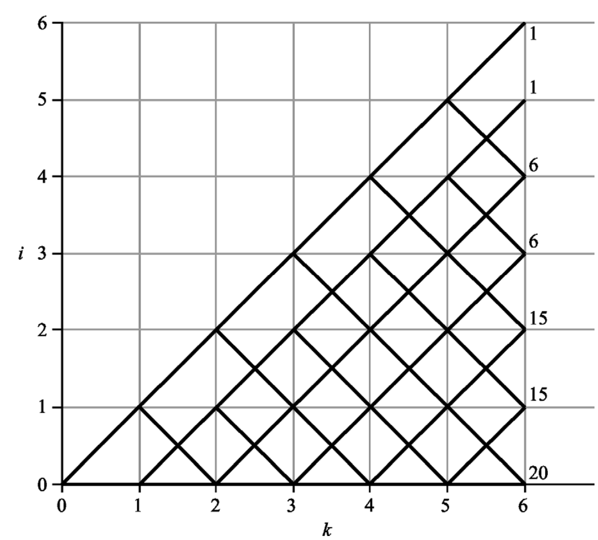
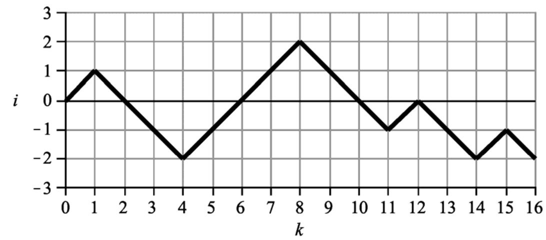
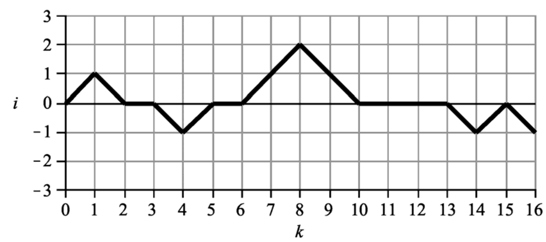


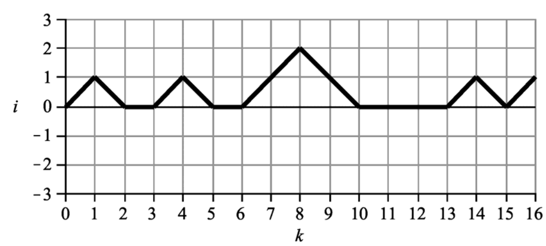
Figure 3 Converting an U-D path into U-D-F



(a) U-D path



(b) Intermediate path



(c) U-D-F Path

4.2 U-D paths to U-D-F paths

In this section, we indicate the steps of one direction of our bijective map between two types of paths.

1 Separate the U-D path into sections such that the divisions occur where the path touches or crosses the horizontal axis and identify the positive and negative sections (Figure 3a).

$^{+}$U D | $^{-}$D D U U | $^{+}$U U D D | $^{-}$D U | $^{-}$D D U D.

2 Change each letter which lies within a negative section and adjacent to a division into a flat path (Figure 3b).

$^{+}$U D | $^{-}$F D U F | $^{+}$U U D D | $^{-}$F F | $^{-}$F D U D.

3 Reverse the remaining letters in the negative sections (Figure 3c).

$^{+}$U D | $^{-}$F U D F | $^{+}$U U D D | $^{-}$F F | $^{-}$F U D U.

4.3 U-D-F paths to U-D paths

In this section, we demonstrate the other direction of the bijection and indicate the steps to move from U-D-F paths to U-D paths.

1 Identify the flat paths and group them in pairs from left to right (Figure 3c).

U D [F U D F] U U D D [F F] [F U D U.

2 Reverse the letters contained between each pair of flat paths (Figure 3b).

U D [F D U F] U U D D [F F] [F D U D.

3 Change the first flat path of each pair to a downward path and the second one to an upward path (Figure 3a).

U D [D D U U] U U D D [D U] [D D U D.

Since, we have demonstrated rules to convert paths back and forth, we have a bijection between U-D-F paths and U-D paths.

If the U-D path terminates at the point (k,i) where $k \geq 0$ then the corresponding U-D-F path also terminates at (k,i). However, an U-D path which ends at the point (k,i) with $i < 0$ corresponds to a U-D-F path that ends at the point $(k,-i-1)$.

If instead we begin with an U-D-F path that ends at (k,i) such that $k+i \equiv 0 \bmod 2$ then the corresponding U-D path also terminates at the point (k,i). Similarly, an U-D-F path which terminates at (k,i) such that $k+i \equiv 1 \bmod 2$ is transformed into an U-D path that ends at the point $(k,-i-1)$.

4.4 Counting U-D-F paths

The bijection that has been found in the previous sections allows us to count our U-D-F paths by transforming them to U-D paths which are easy to count.

Theorem 2. Let k represent the length of the U-D-F path and i represent the height at which the path terminates. Then, the number of such U-D-F paths, denoted $N_{0,i}(k)$, is given by

$$N_{0,i}(k) = \binom{k}{\left[\frac{k+i+1}{2}\right]}, \tag{4}$$

where $[*]$ is the greatest integer function.

Proof. Since, there exists a bijection between the U-D-F paths and the U-D paths, there are the same number of U-D-F paths as there are corresponding U-D paths. Using Lemma 1, we have

$$N_{0,i}(k) = \begin{cases} \widehat{N}_{0,i}(k) & \text{if } k+i \equiv 0 \mod 2 \\ \widehat{N}_{0,-i-1}(k) & \text{if } k+i \equiv 1 \mod 2 \end{cases} = \begin{cases} \binom{k}{\frac{k+i}{2}} & \text{if } k+i \equiv 0 \mod 2 \\ \binom{k}{\frac{k-i-1}{2}} & \text{if } k+i \equiv 1 \mod 2. \end{cases}$$

The property $\binom{n}{r} = \binom{n}{n-r}$ is employed to simplify the expression.

For $k+i \equiv 1 \mod 2$,

$$N_{0,i}(k) = \binom{k}{\frac{k-i-1}{2}} = \binom{k}{k - \frac{k-i-1}{2}} = \binom{k}{\frac{k+i+1}{2}}.$$

Since

$$\left\lceil \frac{k+i+1}{2} \right\rceil = \begin{cases} \frac{k+i}{2} & \text{if } k+i \equiv 0 \mod 2 \\ \frac{k+i+1}{2} & \text{if } k+i \equiv 1 \mod 2, \end{cases}$$

then in general, we have

$$N_{0,i}(k) = \binom{k}{\left\lceil \frac{k+i+1}{2} \right\rceil}.$$

□

4.5 *Finding transient probabilities for M/M/1 queues*

Example 1. Suppose customers arrive at an *M/M/*1 queueing system at a rate of $\lambda = 20$ customers per hour. The service time is $\mu = 20$ customers per hour. At time 0, there are 0 customers. After 6 min (0.1 hour), we the probability of having i customers in the system for $i = 0, 1, 2, ..., 5$.

Solution. Since, $\lambda = \mu$ we can use Equation (1) from Theorem 1 together with Equation (4) of Theorem 2. Using a computer algebra system and truncating at $k = 20$, we find the probabilities (to five decimal places) to be:

$p_{0,0}(0.1) = 0.38575$ $p_{0,1}(0.1) = 0.29638$

$p_{0,2}(0.1) = 0.17875$ $p_{0,3}(0.1) = 0.08706$

$p_{0,4}(0.1) = 0.03518$ $p_{0,5}(0.1) = 0.01207$

4.6 *Recursive expression for* $N_{0,i,f}(k)$

We can also handle computations for the transient queueing probabilities in the $\lambda \neq \mu$ case using Equation (2). This is done by building a recursive computational expression for $N_{0,i,f}(k)$ in Theorem 1 as follows.

Theorem 3. Let $N_{0,i,f}(k)$ be the number of paths from 0 to i in k steps with exactly f flat steps. Then, for $i \neq 0$,

$$N_{0,i,f}(k) = N_{0,i+1,f}(k-1) + N_{0,i-1,f}(k-1)$$

and

$$N_{0,0,f}(k) = N_{0,1,f}(k-1) + N_{0,0,f-1}(k-1).$$

However, this method would be less convenient than using the expressions presented in Sharma (1997), for example.

4.7 *Counting U-D-F paths starting from (0, m)*

In an *M*/*M*/1 queueing system, we might wish to know the probability of having i customers in the system if we start with $m(m \neq 0)$ customers at time 0 where the arrival rate λ equals the service rate μ. We will see that our method treats the $m = 0$ and the $m \neq 0$ case geometrically in the same way. Let $N_{m,i}(k)$ be the number of paths from $(0, m)$ to (k, i). The analogue to Theorem 1 is the following.

Theorem 4. With the conditions above, assume $\lambda = \mu$. Let $p_{m,i}(t)$ be the probability of having i customers in the system at time t if there are m customers at time 0. Then

$$p_{m,i}(t) = \sum_{k=0}^{\infty} \frac{(2\mu t)^k \mathrm{e}^{-2\mu t}}{k!} \frac{N_{m,i}(k)}{2^k}. \tag{5}$$

The count of the number of paths $N_{m,i}(k)$ uses the same folding at $i = -1/2$ as was used previously. For example, suppose we consider paths of length $k = 6$ starting from $(0, m)$ for $m = 2$. The counts of the U-D paths are

$\widehat{N}_{2,-4}(6)$	$\widehat{N}_{2,-2}(6)$	$\widehat{N}_{2,0}(6)$	$\widehat{N}_{2,2}(6)$	$\widehat{N}_{2,4}(6)$	$\widehat{N}_{2,6}(6)$	$\widehat{N}_{2,8}(6)$
1	6	15	20	15	6	1

Folding over $i = -1/2$, the U-D-F counts become

$N_{0,0}(6)$	$N_{0,1}(6)$	$N_{0,2}(6)$	$N_{0,3}(6)$	$N_{0,4}(6)$	$N_{0,6}(6)$	$N_{0,8}(6)$
15	6	20	1	15	6	1

Thus, we get our binomial counts, but in a strange order! The analogue of Theorem 2 is the following.

Theorem 5. Consider all U-D-F paths from $(0,m)$ to (k,i). Then, the number of such U-D-F paths, denoted $N_{m,i}(k)$, is given by

$$N_{m,i}(k)=\begin{cases}\binom{k}{\frac{k+i-m}{2}} & \text{if } k+i-m\equiv 0 \bmod 2\\ \binom{k}{\frac{k+i+m+1}{2}} & \text{if } k+i-m\equiv 1 \bmod 2.\end{cases} \tag{6}$$

5 Path counting for *M*/*M*/1/*c* queues

We next assume that we are dealing with an *M*/*M*/1/*c* queue with $c<\infty$. We want to obtain an expression for the probability of having i customers in the system at finite time t. In this case, there is a capacity of c customers in the system, including the customer in service. As in the *M*/*M*/1 case, we assume that each step consists of an arrival or a departure. If there are 0 customers in the system, and the next event is a departure, then we assume that the clock moves forward, but the path is Flat. Similarly, if there are c customers in the system and the next event is an arrival, then we assume that the clock moves forward but the path is Flat (at level c). Therefore, we may have two different types of flat path in an $M/M/1/c$ system.

Let $p_{0,i}(t)$ be the probability that at time t, there are i customers in the system, given that there were 0 customers at time 0. During the time length t, there will be some number of events k (arrivals or 'departures' with the count having a Poisson distribution). The k events will result in i customers present in the system at time t if there is a path of k steps from $(0,0)$ to (k,i) where the path has steps of type (*) below.

(*) For an $M/M/1/c$ queue with $0<m<c$, a step is a movement from an arbitrary feasible point (j,m) to $(j+1,m+1)$ or from (j,m) to $(j+1,m-1)$. If $m=0$, then a step is a movement from $(j,0)$ to $(j+1,0)$ or from $(j,0)$ to $(j+1,1)$. If $m=c$, then a step is a movement from (j,c) to $(j+1,c)$ or from (j,c) to $(j+1,c-1)$.

Our analogue of Theorem 1 is

Theorem 6. Consider an $M/M/1/c$ queue, assume $\lambda=\mu$. Then

$$p_{0,i}(t)=\sum_{k=i}^{\infty}\frac{(2\mu t)^k\,\mathrm{e}^{-2\mu t}}{k!}\frac{M_{0,i}(k)}{2^k}=\sum_{k=i}^{\infty}\frac{(\mu t)^k\,\mathrm{e}^{-2\mu t}M_{0,i}(k)}{k!} \tag{7}$$

where $M_{0,i}(k)$ is the number of allowable paths from $(0,0)$ to (k,i) in k steps.

If $\lambda\neq\mu$,

$$p_{0,i}(t)=\sum_{k=i}^{\infty}\sum_{f=0}^{k-i}\sum_{g=0}^{k-i-f}\frac{((\lambda+\mu)t)^k\,e^{-(\lambda+\mu)t}}{k!}M_{0,i,f,g}(k)\left(\frac{\lambda}{\lambda+\mu}\right)^{u+g}\left(\frac{\mu}{\lambda+\mu}\right)^{d+f} \tag{8}$$

where $M_{0,i,f,g}(k)$ is the number of allowable paths from $(0,0)$ to (k,i) in k steps with exactly f flat steps at level 0, with exactly g flat steps at level c, where u, d can be solved in terms of i,k,f,g from the restrictions $u+d+f+g=k$, $u-d=i$.

Proof. The Poisson probability of k events in time t is $(((\lambda+\mu)t)^k \mathrm{e}^{-(\lambda+\mu)t})/k!$ for $k=0,1,\ldots$. The probability of moving up (or flat from $(*,c)$) on any step is $\lambda/(\lambda+\mu)$. If $\lambda=\mu$, then the probability of moving up at each step is 1/2. The probability of moving down (or flat from $(*,0)$) on any step is $\mu/(\lambda+\mu)$. To obtain Equation (7), we condition on k and sum over all cases.

If $\lambda \neq \mu$, then the probabilities of two paths which end at i will be equal if they have the same number of flat (level 0) steps and the same number of flat (level c) steps. So we partition our summation into classes with the same number of (0 level) flat steps f and the same number of (c level) flat steps g. We condition on k, f and g and then sum over all cases to get Equation (8). □

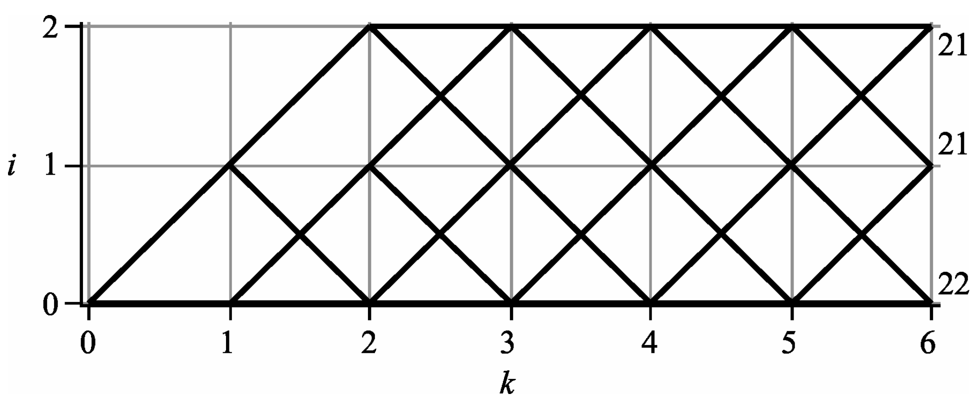
For an $M/M/1/c$ queue, the possible paths for the number of customers in the system after k steps are referred to as U-D-F-F paths since, there are two types of flat path possible. Let S_3 be the set of all lattice points at which a U-D-F-F path may terminate, that is, $S_3 = \{(k,i) \mid 0<k, 0 \le i \le \min\{k,c\}\}$. We want to find a useful bijection (one that helps us to count the paths) between U-D paths as described in Section 4 and U-D-F-F paths.

In Figure 4, we illustrate the possible U-D-F-F paths from $(0,0)$ to $(6,k)$ where $k=0,\ldots,c$ for $c=2$.

The number of paths from $(0,0)$ which end at $(6,0)$, $(6,1)$ or $(6,2)$ are indicated in the diagram by the counts appearing next to the points. We observe that $\binom{6}{0}+\binom{6}{3}+\binom{6}{6}=22$ paths end at $(6,0)$; $\binom{6}{2}+\binom{6}{5}=21$ paths end at $(6,1)$; $\binom{6}{1}+\binom{6}{4}=21$ paths end at $(6,2)$.

Next, we present our bijective map between U-D paths and U-D-F-F paths.

Figure 4 S_3 with connecting steps



5.1 U-D paths to U-D-F-F paths

In this section, we demonstrate one direction of the bijection.

To move from U-D to U-D-F-F, we use a left sided algorithm. We illustrate by using an example of an U-D path D D U U U U U D with $c = 2$ (Figure 5a).

1. Consider the first time (from the left) that an U-D path crosses the horizontal axis ($y = 0$) or level c (i.e. $y = c$). Replace that step by a flat component and reverse all subsequent steps (i.e. $U \leftrightarrow D$). For our example, we obtain

   F U D D D D D U (Figure 5b).

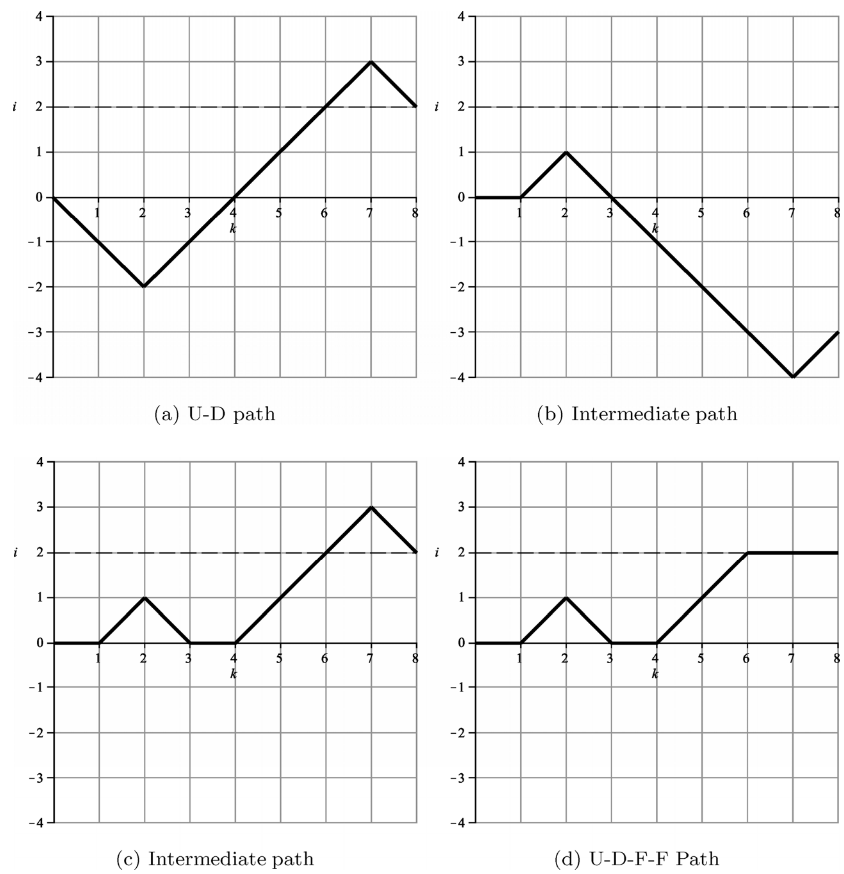
2. For the new path, find the first step that it crosses the horizontal axis or level c. Replace that first step by a flat component. Reverse all subsequent steps. For our example, we obtain

   F U D F U U U D (Figure 5c).

3. Continue until there are no more possible changes. In our example, we move to F U D F U U F U and then to

   F U D F U U F F (Figure 5d).

Figure 5 Converting an U-D path into U-D-F-F



5.2 U-D-F-F paths to U-D paths

In this section, we demonstrate the other direction of the bijection.

To move from U-D-F-F to U-D, we use a right sided algorithm. We illustrate with an example of an U-D-F-F path F U D F U U F F with $c = 2$ (Figure 5d).

1 If there is no F, do nothing to the path. Otherwise, at the F farthest to the right, if the path is at level c (respectively 0) at that point, change the F to U (respectively D). Simultaneously, any components farther to the right get reversed ($U \leftrightarrow D$). In our example, we obtain

   F U D F U U F U (Figure 5c).

2 For the new path, find the F which is now farthest to the right. At this F, if the path is at level c (respectively 0) at that point, change the F to U (respectively D). Simultaneously, any components farther to the right get reversed $(U \leftrightarrow D)$. In our example, we obtain

   F U D F U U U D (Figure 5b).

3 Continue until there are no more possible changes. In our example, we move to F U D D D D D U U and then to

   D D U U U U U D (Figure 5a).

Thus, we have a bijection between U-D and U-D-F-F paths. Moreover, this bijection has a special feature that all U-D paths which end at a particular lattice point will map to U-D-F-F paths that end at a single lattice point. The U-D paths are mapped to U-D-F-F paths with a final height that is obtained by folding over the level $c+1/2$ for heights greater than c and folding over $-1/2$ for heights less than 0.

For example, if $c = 2$ and $k = 6$, each path of U-D-F-F type with six steps will end at height 0 or 1 or 2. For $k = 6$, paths of U-D type will end at heights 6, 4, 2, 0, –2, –4, –6. There are $\binom{6}{0}$, $\binom{6}{1}$, $\binom{6}{2}$, $\binom{6}{3}$, $\binom{6}{4}$, $\binom{6}{5}$, $\binom{6}{6}$ U-D paths ending at levels 6, 4, 2, 0, –2, –4, –6, respectively. After applying the bijection, the corresponding U-D-F-F paths end at 0, 1, 2, 0, 1, 2, 0, respectively. Thus, the number of U-D-F-F paths ending at 2, 1, 0, respectively which are obtained by summing, are $\binom{6}{4}+\binom{6}{1}$, $\binom{6}{5}+\binom{6}{2}$, $\binom{6}{6}+\binom{6}{3}+\binom{6}{0}$.

5.3 Counting U-D-F-F paths

Our bijection, given in the previous sections allows us to count U-D-F-F paths by transforming them to U-D paths which are easy to count. The general result to count U-D-F-F paths requires us to consider separate cases depending on the value of k (even or odd) on the last step. The general result is as follows.

Theorem 7. Consider all U-D-F-F paths of length k beginning at level 0 and ending at level i, $(i = 0, ..., c)$ for given c. The number of such paths is

$$M_{0,i}(k) = \begin{cases} \sum_{j \equiv \frac{k+i}{2} \bmod (c+1)} \binom{k}{j} & \text{if } k+i \equiv 0 \bmod 2 \\ \sum_{j \equiv \frac{k+i+1}{2} \bmod (c+1)} \binom{k}{j} & \text{if } k+i \equiv 1 \bmod 2 \end{cases}.$$

We apply multisectioning using primitive roots of unity to sum the series in the previous result (Riordan, 1968). The following result appears in Guichard (1995). The greatest integer function appears in the upper limit of the summation.

$$\sum_{j \equiv r \bmod m} \binom{n}{j} = \frac{2^n}{m} + \frac{2^{n+1}}{m} \sum_{k=1}^{\left[\frac{m-1}{2}\right]} \cos^n \frac{\pi k}{m} \cos\left(\frac{n\pi k}{m} - \frac{2\pi r k}{m}\right).$$

We apply this result to Theorem 7 to get the following corollary.

Corollary 1. Consider all U-D-F-F paths of length k beginning at level 0 and ending at level i, $(i = 0, ..., c)$ for given c. Using the greatest integer function $[*]$, we obtain the number of U-D-F-F paths from $(0,0)$ to (k,i) with upper height limit c is

$$M_{0,i}(k) = \frac{2^k}{c+1} + \frac{2^{k+1}}{c+1} \sum_{j=1}^{\left[\frac{c}{2}\right]} \cos^k \frac{\pi j}{c+1} \cos\left(\frac{k\pi j}{c+1} - \frac{2\pi \left[\frac{k+i+1}{2}\right] j}{c+1} \right).$$

Although this result does not seem to be a large improvement, it actually is. Regardless of the size of k, the number of summands in the expression of Corollary 1 stays fixed. For small c, this is a small sum.

By using the expression in Corollary 1 together with the expression in Theorem 6, we have a convenient computational method for computing the transient probabilities. Of course, in practice, we would truncate the series when the probabilities become sufficiently small.

5.4 Finding transient probabilities for M/M/1/c queues

Example 2. Suppose, we have a similar situation as in Example 1, now with $c = 2$. Find the probability of having 0, 1 and 2 customers after 6 min (0.1 hours).

Solution. Since $\lambda = \mu = 20$ customers / hour, we can use Equation (7) from Theorem 6. Using a computer algebra system and truncating at $k = 20$, we find the probabilities (to five decimal places) to be:

$$p_{0,0}(0.1) = 0.40141$$
$$p_{0,1}(0.1) = 0.33251$$
$$p_{0,2}(0.1) = 0.26608$$

6 Conclusions

We have presented expressions for the transient probabilities for an $M/M/1/c$ queueing system. For the case when $\lambda = \mu$, the expressions allow for considerable simplification. The expressions in this case appear to be slightly different in appearance from, but equivalent to, the usual Bessel function versions. Further, the simplicity of the analysis presented here makes the result very accessible. The $\lambda = \mu$ case is an important case. It gives good approximations for systems for which λ is close to μ. Since only transient probabilities are considered, stability conditions are not needed. Moreover, the interest in heavy traffic models (λ near μ with $\lambda < \mu$) makes our study more useful.

The special contributions of this work are the geometric bijections, presented in Sections 4 and 5 that simplify our path counting. These special types of reflection have special interesting characteristics that aid considerably in our understanding of the process. In Section 5, the counts of the number of paths ending in a particular end point surprisingly turns out to be the sum of every *k*th binomial coefficient. This allows us to use multisectioning and connects several different concepts and results together.

Although this work is currently restricted to the case $\lambda = \mu$, the bijections/ transformations given in Sections 4 and 5 may have useful extensions far beyond this case. In addition, the geometric interpretation of (repeated) folding over barriers suggests other possibilities for research, such as barriers that depend on a changing buffer size. Some of the restrictions of the model could be further examined to see what other assumptions can be relaxed.

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