M/M/1 Transient Queues and Path Counting

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Abstract

We find combinatorially the probability of having n customers in an M/M/1 queueing system at an arbitrary time t when the arrival rate is λ and the service rate is μ. If $\lambda = \mu$, then we can simplify considerably. Our method considers paths which can move Up, Down or remain Flat at each step. We count the number of U-D-F paths starting at terminating at the point (k, i). To accomplish this, it will be shown that every U-D-F path has a one-to-one correspondence with a U-D path, and the latter are easy to count.

1 Introduction

In queueing theory, the most commonly studied system is the M/M/1 system, which has exponentially distributed interarrival times, exponentially distributed service times, and a single server. The limiting probabilities of finding i customers in the system (for $i = 0, 1, 2, \ldots$) turns out to follow a geometric distribution when the arrival rate λ is less than the service rate μ. The limiting probabilities (as time tends to infinity) are easy to obtain. Details can be found in most queueing books. However, in the transient case (time is finite), results are more complex and the probabilities are most commonly derived using differential equations and the probabilities are expressed in terms of Bessel functions (see [2], for example). In the transient case, the restriction that $\lambda < \mu$ is no longer needed.

In this paper we consider a combinatorial method for finding the transient probabilities at time t. In the special case $\lambda = \mu$, the expressions can be simplified. Some combinatorial analysis of transient probabilities for queues has been considered (see [1] and [3], for example). However, our analysis is different, and gives a simple expression for the $\lambda = \mu$ case which is equal to but has a different appearance than the result based on Bessel functions. Further, our method requires the counting of certain kinds of paths, which are of interest in their own right.

2 Queueing Model

Consider a single server M/M/1 queueing system with arrivals at rate λ and service at rate μ. Assume that at time 0, there are zero customers in the system. We are interested in the probability of finding i customers in the system at time t. If we wanted to simulate such a system, after each arrival or service completion, we could generate two exponential random values, one interarrival time and one completion time, and choose the minimum to determine what the next event (an arrival or a completion) will be. But if we use this method when there were no customers in the system, then we would be allowing negative customers which we not want. Thus if there are zero customers in the system and the next event is a "departure," then we interpret this to mean that the system stays with zero customers but the clock moves forward by the minimum of the two random values. The memoryless property of the exponential distribution allows us to do this.

Let $p_{0i}(t)$ be the probability that at time t, there are i customers in the system, given that there were 0 customers at time 0. We are dealing with a Poisson process. During the time length t, there will be some number - say k events (arrivals or "departures" with the count having a Poisson distribution). The k events in time t will result in i customers in the system at time t if there is a path of k steps from $(0,0)$ to (k,i) with the following restrictions.
(*) For $m>0$, a step is a movement from an arbitrary feasible point (j,m) to $(j+1,m+1)$ or from (j,m) to $(j+1,m-1)$. If $m=0$ then a step is a movement from $(j,0)$ to $(j+1,0)$ or from $(j,0)$ to $(j,1)$.

Theorem 2.1. *If $\lambda=\mu$, then*

$$p_{0i}(t)=\sum_{k=i}^{\infty}\frac{(2\mu)^k e^{-(2\mu)t}}{k!}\frac{N_{0i}(k)}{2^k} \tag{1}$$

where $N_{0i}(k)$ is the number of paths from 0 to i in k steps as in (*).
If $\lambda\neq\mu$,

$$p_{0i}(t)=\sum_{k=i}^{\infty}\sum_{j=0}^{k-i}\frac{(\lambda+\mu)^k e^{-(\lambda+\mu)t}}{k!}N_{0ij}(k)(\frac{\lambda}{\lambda+\mu})^m(\frac{\mu}{\lambda+\mu})^n \tag{2}$$

where $N_{0ij}(k)$ is the number of paths from 0 to i in k steps with exactly j flat steps, with the restrictions that $m+n=k$, $m-(n-j)=i$.

Proof. The probability of k events in time t is $\dfrac{(\lambda+\mu)^k e^{-(\lambda+\mu)t}}{k!}$ for $k = 0, 1, \ldots$. The probability of moving up at any particular step is $\frac{\lambda}{\lambda+\mu}$. If $\lambda = \mu$, then the probability of moving up at each step is 1/2. If the movement on a step is not up, then it must be down (or flat if the start of a step is $(*, 0)$).To obtain 2.1, we condition on j and sum over all cases.
If $\lambda \neq \mu$, then the situation is more complex. In this case the probabilities of two paths will be equal only if they have the same number of flat steps. So we partition our summation into classes with the same number of flat steps. We then condition on get k and then on j and sum over all cases to get 2.1. □

For the remainder of the paper, we study properties of $N_{0,i}(k)$.

It is interesting to compare the results obtained here with the standard expression using Bessel functions (see [2]) given below.

$$p_{0i}(t) = e^{-(\lambda+\mu)t}\left(\rho^{i/2} I_i(at) + \rho^{(i-1)/2} I_{i+1}(at) + (1-\rho)\rho^i \sum_{j=i+2}^{\infty} \rho^{-j/2} I_j(at)\right)$$

where $\rho = \lambda/\mu$, $a = 2\mu\rho^{1/2}$ and $I_k(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{k+2m}}{(k+m)!m!}$. When $\lambda = \mu$, this expression reduces to

$$p_{0i}(t) = e^{-(2\mu)t}\left(I_i(2\mu t) + I_{i+1}(2\mu t)\right)$$

3 Path Counting

Definition 3.1. *An* ***upward step*** *from (a, b) is a line segment connecting the lattice points (a, b) and $(a+1, b+1)$. A* ***downward step*** *is a line segment connecting the lattice point (a, b) to $(a+1, b-1)$. A* ***flat step*** *is a line segment that joins the lattice point $(a, 0)$ to $(a+1, 0)$.*

Definition 3.2. *An* ***U-D path*** *of length k is a polygonal path composed of a sequence of upward and downward sloped steps beginning at the origin and terminating at the lattice point (k, i), where k and i are integers and $k + i \equiv 0 \mod 2$.*

Definition 3.3. *Let P be a polygonal path of length k composed of upward, downward and flat paths. If P is restricted to the first quadrant and flat paths may only occur on the horizontal axis, then P is called a* ***U-D-F path****.*

Lemma 3.1. *If k represents the length of a U-D path, 0 is the initial height, and i represents the height at which it terminates, then the number of such paths, denoted $\widehat{N}_{0,i}(k)$, is given by*

$$\widehat{N}_{0,i}(k) = \binom{k}{\frac{k+i}{2}}. \tag{3}$$

Proof. Let u and d represent the numbers of upward and downward steps, respectively, contained in an U-D path of length k terminating at height i. Then we have $i = u - d$ and $k = u + d$. Solving for u gives $u = \frac{k+i}{2}$. Since there are only two choices for each segment, upward or downward, the number of such paths is

$$\widehat{N}_{0,i}(k) = \binom{k}{u} = \binom{k}{\frac{k+i}{2}}.$$

□

Let S_1 be the set of all lattice points at which a U-D path may terminate, that is, $S_1 = \{\ (k,i) \mid k > 0,\ -k \le i \le k,\ \ k+i \equiv 0 \mod 2\}$. If each such lattice point is connected to its neighbours only by an upward or a downward step and is assigned a label α, where α is the number of U-D paths terminating at that point, then we have a representation of Pascal's triangle. (See Figure 1).

Thus for example, after step 6, we have

$\widehat{N}_{0,-6}(6)$	$\widehat{N}_{0,-4}(6)$	$\widehat{N}_{0,-2}(6)$	$\widehat{N}_{0,0}(6)$	$\widehat{N}_{0,2}(6)$	$\widehat{N}_{0,4}(6)$	$\widehat{N}_{0,6}(6)$
1	6	15	20	15	6	1

Now suppose that S_1 is folded along the line $i = \frac{1}{2}$, so that the each point (k,i) below the k-axis is mapped to the odd point $(k, -i-1)$. The result is a set of points, $S_2 = \{\ (k,i) \mid k > 0,\ 0 \le i \le k,\ \ k,i \in \mathbb{Z}\}$, which is the set of all lattice points at which an U-D-F path may terminate. Figure 2 shows S_2 connected using the permissible steps. If each Pascal label α from S_1 is carried over to S_2, then α corresponds to the number of U-D-F paths terminating at that point.

Thus for example, if $N_{0,i}(6)$ represents the number of U-D-F paths from 0 to i after 6 steps, the counts can be shown to be

$N_{0,0}(6)$	$N_{0,1}(6)$	$N_{0,2}(6)$	$N_{0,3}(6)$	$N_{0,4}(6)$	$N_{0,5}(6)$	$N_{0,6}(6)$
20	15	15	6	6	1	1

Compare these values with $\widehat{N}_{0,i}(6)$ and note the folding over $i = 1/2$.

This suggests that there may exist a bijective function that maps each U-D path to an U-D-F path.

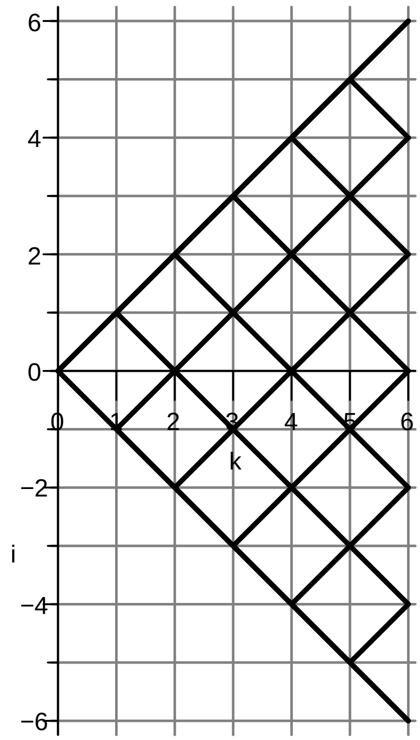

Figure 1: S_1 with connecting steps

4 Path Transformations

An U-D path can be transformed to the corresponding U-D-F path as shown below. Upward and downward steps are denoted as U and D. Flat steps may be regarded as downward steps and hence are denoted D (or $\mathbb{D}$ when the fact that they are flat needs to be emphasized). A sample path has been chosen for demonstration purposes.

4.1 U-D Paths to U-D-F Paths

1. Separate the U-D path into sections such that the divisions occur where the path touches or crosses the horizontal axis and identify the positive and negative sections. (Fig. 3)

   $^{+}$U D | $^{-}$D D U U | $^{+}$U U D D | $^{-}$D U | $^{-}$D D U D

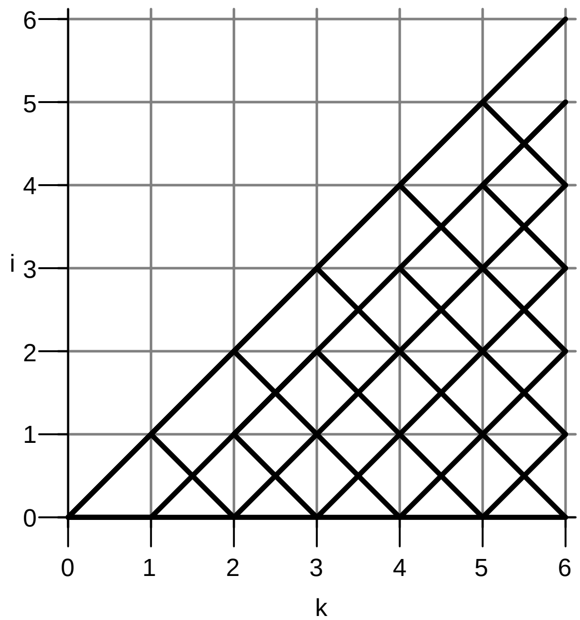

Figure 2: S_2 with connecting steps

2. Change each letter which lies within a negative section and adjacent to a division into a flat path. (Fig. 4)

   $^{+}$U D | $^{-}\mathbb{D}$ D U $\mathbb{D}$ | $^{+}$U U D D | $^{-}\mathbb{D}$ $\mathbb{D}$ | $^{-}\mathbb{D}$ D U D

3. Reverse the remaining letters in the negative sections. (Fig. 5)

   $^{+}$U D | $^{-}\mathbb{D}$ U D $\mathbb{D}$ | $^{+}$U U D D | $^{-}\mathbb{D}$ $\mathbb{D}$ | $^{-}\mathbb{D}$ U D U

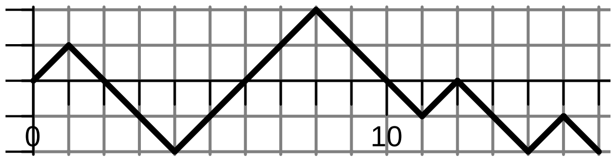

Figure 3: U-D path

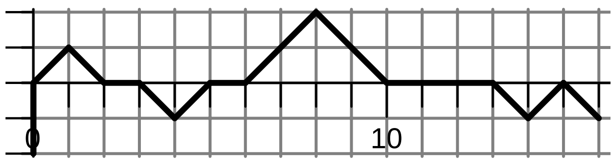

Figure 4: Intermediate path

4.2 U-D-F Paths to U-D Paths

1. Identify the flat paths and group them in pairs from left to right. (Fig. 5)

   U D [$\mathbb{D}$ U D $\mathbb{D}$] U U D D [$\mathbb{D}$ $\mathbb{D}$] [$\mathbb{D}$ U D U

2. Reverse the letters contained between each pair of flat paths. (Fig. 4)

   U D [$\mathbb{D}$ D U $\mathbb{D}$] U U D D [$\mathbb{D}$ $\mathbb{D}$] [$\mathbb{D}$ D U D

3. Change the first flat path of each pair to a downward path and the second one to an upward path. (Fig. 3)

   U D [D D U U] U U D D [D U] [D D U D

Since there exists an invertible function that converts between U-D paths and U-D-F paths, we have a bijection.

If the U-D path terminates at the point (k, i) where $k \geq 0$ then the corresponding U-D-F path also terminates at (k, i). However, an U-D path which ends at the point (k, i) with $k < 0$ corresponds to a U-D-F path that

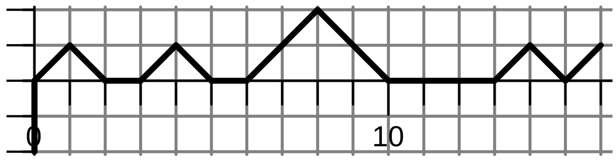

Figure 5: U-D-F path

ends at the point $(k, -i-1)$.

If instead we begin with a U-D-F path that ends at (k, i) such that $k+i \equiv 0 \mod 2$, then the corresponding U-D path also terminates at the point (k, i). Similarly, a U-D-F path which terminates at (k, i) such that $k + i (\equiv 1 \mod 2)$ is transformed into a U-D path that ends at the point $(k, -i-1)$.

5 Counting U-D-F Paths

Theorem 5.1. *Let k represent the length of the U-D-F path and i represent the height at which the path terminates. Then the number of such U-D-F paths, denoted $N_{0i}(k)$, is given by*

$$N_{0i}(k) = \binom{k}{\left[\frac{k+i+1}{2}\right]}, \tag{4}$$

where $[\bullet]$ is the greatest integer function.

Proof. Since there exists a bijection between the U-D-F paths and the U-D paths as described in section 4, there are the same number of U-D-F paths as there are corresponding U-D paths.

$$N_{0i}(k) = \begin{cases} \widehat{N}_{0,i}(k) & \text{if } k+i \equiv 0 \mod 2 \\ \widehat{N}_{0,-i-1}(k) & \text{if } k+i \equiv 1 \mod 2. \end{cases}$$

Then by Lemma 3.1,we have

$$N_{0i}(k) = \begin{cases} \binom{k}{\frac{k+i}{2}} & \text{if } k+i \equiv 0 \mod 2 \\ \binom{k}{\frac{k-i-1}{2}} & \text{if } k+i \equiv 1 \mod 2. \end{cases}$$

The property $\binom{n}{r} = \binom{n}{n-r}$ is employed to simplify the expression.

For $k+i \equiv 1 \mod 2$,

$$N_{0i}(k) = \binom{k}{\frac{k-i-1}{2}} = \binom{k}{k - \frac{k-i-1}{2}} = \binom{k}{\frac{k+i+1}{2}}.$$

Since

$$[\frac{k+i+1}{2}] = \begin{cases} \frac{k+i}{2} & \text{if } k+i \equiv 0 \mod 2 \\ \frac{k+i+1}{2} & \text{if } k+i \equiv 1 \mod 2, \end{cases}$$

then in general we have

$$N_{0i}(k) = \binom{k}{[\frac{k+i+1}{2}]}.$$

□

5.1 Counting U-D-F Paths Starting from (0,m)

In an M/M/1 queueing system, we might wish to know the probability of having i customers in the system if we start with m customers at time 0 when the arrival rate λ equals the service rate μ. Let $N_{mi}(j)$ be the number of paths from $(0,m)$ to (j,i). The analog to Theorem 2.1 is the following.

Theorem 5.2. *With the conditions above, assume $\lambda = \mu$. Let $p_{mi}(t)$ be the probability of having i customers in the system at time t if there are m customers at time* 0. *Then*

$$p_{mi}(t) = \sum_{k=i}^{\infty} \frac{(\lambda+\mu)^k e^{-(\lambda+\mu)t}}{k!} \frac{N_{mi}(k)}{2^k}. \tag{5}$$

The count of the number of paths $N_{mi}(k)$ uses the same folding at -1/2 as was used previously. For example suppose we consider paths of length $k=6$ starting from $(0,m)$ for $m=2$. The the counts of the U-D paths are

$\widehat{N}_{0,-4}(6)$	$\widehat{N}_{0,-2}(6)$	$\widehat{N}_{0,2}(6)$	$\widehat{N}_{0,2}(6)$	$\widehat{N}_{0,4}(6)$	$\widehat{N}_{0,6}(6)$	$\widehat{N}_{0,8}(6)$
1	6	15	20	15	6	1

Folding over $i=-1/2$, the U-D-F counts become

$N_{0,0}(6)$	$N_{0,1}(6)$	$N_{0,2}(6)$	$N_{0,3}(6)$	$N_{0,4}(6)$	$N_{0,6}(6)$	$N_{0,8}(6)$
15	6	20	1	15	6	1

Thus we get our binomial counts, but in a strange order! The analog of Theorem 5.1 is the following.

Theorem 5.3. *Consider all U-D-F paths from $(0, m)$ to (k, i). Then the number of such U-D-F paths, denoted $N_{mi}(k)$, is given by*

$$N_{mi}(k) = \begin{cases} \binom{k}{\frac{k+i-m}{2}} \textit{ if } k+i-m \equiv 0 \mod 2 \\ \binom{k}{\frac{k+i+m+1}{2}} \textit{ if } k+i-m \equiv 1 \mod 2. \end{cases} \tag{6}$$

6 Conclusion and Acknowledgment

We have presented expressions for the transient probabilities for an M/M/1 queueing system. For the case when $\lambda = \mu$, the expressions allowed for considerable simplification. These expressions appear to be slightly different in appearance from the Bessel function versions. Numerical work confirms that these new expressions give the same values as the Bessel functions function versions. Further, the simplicity of the analysis presented here is unmatched in any other presentation. The $\lambda = \mu$ case is an important case. It gives good approximations for systems for which λ is close to μ. Since only transient probabilities are considered, the standard stability condition $\lambda < \mu$ is not needed. However, the interest in heavy traffic models (λ is near μ with $\lambda < \mu$) makes our study more useful.

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