Busy Period M/M/*/ Laplace Transforms

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Probabilistic interpretation of Laplace Transforms

DEFINITION: The Laplace transform $L(s)$ of a function $f(x)$ with positive support is given by

$$L_X(s) = \int_0^\infty e^{-sx} f(x) dx \text{ where } s > 0.$$ 

THEOREM: Let $X$ be a r.v. with positive support and with pdf $f(x)$. Let $Y$ be a r.v. independent of $X$, such that $Y \sim$ exponential with rate $s$. Then

$$L_X(s) = P(X < Y).$$
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- The exponential random variable $Y$ is called the catastrophe.
- The Laplace transform of a p.d.f of a random variable $X$ is the probability that $X$ occurs before the catastrophe.
Define an $i$ channel busy period for an $M/M/c$ system ($1 \leq i \leq c$) to begin with an arrival to a system with $i - 1$ and end at the next point in time when the system dips to $i - 1$. Let $T_{b,i}$ be the time length of the $i$-channel busy period.

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MH: The most interesting cases are $i = 1$ and $i = c$. We focus on the $i = 1$ case.
M/M/c case

$M/M/c$ represents a system where arrivals follow a Poisson process at rate $\lambda$ and form a single queue, there are $c$ servers and service times per server are exponentially distributed at rate $\mu$. 

$L_1(s) =$ 

$\mu \frac{s}{\lambda + \mu + s}$

$L_1(s) = \frac{\lambda + \mu}{\lambda + \mu + s}$

$L_1(s) = \left( \frac{\lambda + \mu}{\lambda + \mu + s} \right)^2$
M/M/c case

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**M/M/1 model**

Let $L_1(s)$ be the Laplace transform for the busy period of an M/M/1 queueing system. Then

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} (L_1(s))^2$$
Let $L_1(s)$ be the Laplace transform for the busy period of an $M/M/2$ queueing system. Let $L_2(s)$ be the probability that a busy period of an $M/M/2$ system, which begins with two customers, will end (reach 0) before a catastrophe. Let $M_{2,1}(s)$ be the probability that the $M/M/2$ system drops from 2 customers to 1 customer before a catastrophe. Let $M_{3,2}(s)$ be the probability that the $M/M/2$ system drops from 3 customers to 2 customer before a catastrophe. Let $\lambda$ and $\mu$ be the arrival and service rates. Then
Let $L_1(s)$ be the Laplace transform for the busy period of an $M/M/2$ queueing system. Let $L_2(s)$ be the probability that a busy period of an $M/M/2$ system, which begins with two customers, will end (reach 0) before a catastrophe. Let $M_{2,1}(s)$ be the probability that the $M/M/2$ system drops from 2 customers to 1 customer before a catastrophe. Let $M_{3,2}(s)$ be the probability that the $M/M/2$ system drops from 3 customers to 2 customer before a catastrophe. Let $\lambda$ and $\mu$ be the arrival and service rates. Then

$$L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_2(s)$$

(1)

$$L_2(s) = M_{2,1}(s)L_1(s)$$

(2)
M/M/2 continued

\[ M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{3,2}(s) M_{2,1}(s) \]  
\[ M_{3,2}(s) = M_{2,1}(s) \]  

So we can get:

\[ M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{2,1}(s)^2 \]  
\[ L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{21}(s) L_1(s) \]
From above we can find the Laplace transform of the $M/M/c$ busy period:

\[ L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L_2(s) \quad (7) \]
\[ L_2(s) = M_{2,1}(s)L_1(s) \quad (8) \]
\[ M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{3,2}(s)M_{2,1}(s) \quad (9) \]
\[ M_{3,2}(s) = M_{2,1}(s) \text{ is now false for } c \geq 3 \quad (10) \]
From above we can find the Laplace transform of the $M/M/c$ busy period:

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(7)

$$L_2(s) = M_{2,1}(s)L_1(s)$$

(8)

$$M_{2,1}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s}M_{3,2}(s)M_{2,1}(s)$$

(9)

$$M_{3,2}(s) = M_{2,1}(s)$$ is now false for $c \geq 3$

(10)

$$M_{3,2}(s) = \frac{3\mu}{\lambda + 3\mu + s} + \frac{\lambda}{\lambda + 3\mu + s}M_{4,3}(s)M_{3,2}(s)$$

(11)
M/M/c, c > 2

\[ M_{c+1,c}(s) = M_{c,c-1}(s) \]

\[ M_{c,c-1}(s) = \frac{c \mu}{\lambda + c \mu + s} + \frac{\lambda}{\lambda + c \mu + s} M_{c+1,c}(s) M_{c,c-1}(s) \]

\[ M_{c-1,c-2}(s) = \frac{(c - 1) \mu}{\lambda + (c - 1) \mu + s} + \frac{\lambda}{\lambda + (c - 1) \mu + s} M_{c,c-1}(s) M_{c-1,c-2}(s) \]

\[ \ldots \]
M/M/c, $c > 2$

\[
M_{c,c-1}(s) = \frac{\lambda + c\mu + s - \sqrt{(\lambda + c\mu + s)^2 - 4c\lambda\mu}}{2\lambda}
\]

\[
M_{c-1,c-2}(s) = \frac{(c-1)\mu}{\lambda + (c-1)\mu + s - \lambda M_{c,c-1}(s)}
\]

\[
M_{c-2,c-3}(s) = \frac{(c-2)\mu}{\lambda + (c-2)\mu + s - \lambda M_{c-1,c-2}(s)}
\]

\[\vdots\]

\[
L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)}
\]
LT plots, lambda=5, mu=6, c=1,2,3
LT derivative, lambda=5, mu=6, c=1
The slope of LT at 0 gives the negative of the expected busy period time length. $E(B)$ The slope of the derivative of LT at 0 gives the second moment $E(B^2)$. 
For M/M/1/k, arrivals follow a Poisson process at rate $\lambda$, there is one server, service times are exponentially distributed with rate $\mu$, and the maximum number of customers in the system, including customer in service, is $k$. We seek the LT of the busy period.

M/M/1/1

$L_B(s) = P(BusyPeriodEndsBeforeCatastrophe) = P(ServiceEndsBeforeCatastrophe) = \frac{\mu}{(\mu + s)}.$
\[ L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{21}(s)L_1(s) \]

\[ M_{21}(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{32}(s)M_{21}(s) \]

\[ \vdots \]

\[ M_{k-1,k-2}(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{k,k-1}(s)M_{k-1,k-2}(s) \]

\[ M_{k,k-1}(s) = \frac{\mu}{\mu + s} \]
Solving

\[ M_{k,k-1}(s) = \frac{\mu}{\mu + s} \]

\[ M_{k-1,k-2}(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{k,k-1}(s)} \]

\[ \vdots \]

\[ M_{21}(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{3,2}(s)} \]

\[ L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)} \]
LT Busy Period M/M/1/4, \( \lambda = 5 \), \( \mu = 6 \)
For M/M/c/c, arrivals follow a Poisson process at rate $\lambda$, there are $c$ servers, service times are exponentially distributed with rate $\mu$ per server, and the maximum number of customers in the system, including customer in service, is $c$. We seek the LT of the busy period.

M/M/1/1

$L_B(s) = \mu/(\mu + s)$.
M/M/c/c, \( c \geq 3 \)

\[
L_1(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} M_{21}(s)L_1(s)
\]

\[
M_{21}(s) = \frac{2\mu}{\lambda + 2\mu + s} + \frac{\lambda}{\lambda + 2\mu + s} M_{32}(s)M_{21}(s)
\]

\[\vdots\]

\[
M_{c-1,c-2}(s) = \frac{(c-1)\mu}{\lambda + (c-1)\mu + s} + \frac{\lambda}{\lambda + (c-1)\mu + s} M_{c,c-1}(s)M_{c-1,c-2}(s)
\]

\[
M_{c,c-1}(s) = \frac{c\mu}{c\mu + s}
\]
Solving

\[ M_{c,c-1}(s) = \frac{c\mu}{c\mu + s} \]

\[ M_{c-1,c-2}(s) = \frac{(c-1)\mu}{\lambda + (c-1)\mu + s - \lambda M_{c,c-1}(s)} \]

\[ \vdots \]

\[ M_{21}(s) = \frac{2\mu}{\lambda + 2\mu + s - \lambda M_{3,2}(s)} \]

\[ L_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda M_{2,1}(s)} \]
LT Busy Period M/M/4/4, lambda=5, mu=6
The End.
Thank you!