Scheduling a Rescue

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Outline

Problem Description

In June and July of 2018, a group of thirteen persons on a soccer team was trapped in flooded caves in Thailand. One statement that appeared in the media stated that rescuers chose to take the strongest boys first. This was argued to give the best chance of survival. Later statements said that the weaker boys actually were removed first.

Problem Description

From wikipedia,
"For the first part of the extraction, eighteen rescue divers consisting of thirteen international cave divers and five Thai Navy SEALs were sent into the caves to retrieve the boys, with one diver to accompany each boy on the dive out. There were conflicting reports that the boys were rescued with the weakest first or strongest first. In fact, the order was which boy volunteered first."

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We consider different models and criteria for which removing the stronger persons first may or may not be the best strategy. The models to be presented do not fit precisely into standard queueing or perishable inventory or survival analysis type models, and the objective here is different than in other settings.

Symbols

Symbol	Definition
n	number of items
i	label of an item according to its order of being processed
$P_0(i)$	initial probability of success for i-th item
t_i	the times of start of processing for the i-th item
T_i	the interval between individual processing time starts ($T_i = t_{i+1} - t_i$)
X_i	$=$ 1 if item i is successfully processed, 0 else
Y	The total number of successes $Y = \sum X_i$

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We consider those n items to be processed one at a time, and assume $T_i = T$ for all i.
For the Thailand situation, we have $n = 13$.

Measures of Success

- The expected number of successfully processed items (expected number of rescued people).
- The probability that all items are successfully processed (probability that all people are rescued).

Analysis Outline

- Constant Probabilities $P_0(k)$
- Variable Probabilities $P_1(t_k)$
  - Additive Decrease
    $P_1(t_k) = P_0(k) + f(t_k)$
  - Multiplicative Decrease
    $P_1(t_k) = P_0(k) \cdot f(t_k)$

Constant Probabilities

Theorem

For the generalized binomial model, both $E(Y)$ and $P(\text{all successes})$ are constant regardless of the order in which items occur.

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Proof.

Note that $E(X_i) = 1p_i + 0(1 - p_i) = p_i$.

$$E(Y) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p_i$$

$$P(\text{all successes}) = \prod_{i=1}^{n} p_i$$

Since the expressions for $E(Y)$ and $P(\text{all successes})$ do not change when we change the order in which the items are processed, the result follows. □

Variable Probabilities

Assume that as time goes by, the probabilities change from P_0 to P_1, where $P_1(t_k)$ is the probability of success of processing the k-th item.

$$E(\text{number of successes}) = \sum_{i=1}^{n} E(X_i) = \sum_{k=1}^{n} P_1(t_k)$$

$$P(\text{all successes}) = \prod_{i=1}^{n} P_1(t_k)$$

Variable Probabilities

Additive Decrease

With constant interval T, the ordered processing start times for the items are $\vec{v} = (0, T, 2T, \ldots, (n-1)T) = (t_1, t_2, \ldots, t_n)$.

According to additive method, $P_1(t_k) = P_0(k) + f(t_k)$ where $f(t_k) < 0$.

Variable Probabilities

Additive Decrease

Theorem

If $P_0(i) + f(t_i) > 0, \forall i, k,$ *then*

$$E(\#successes) = \sum_{i=1}^{n} P_1(t_i)$$

$$= \sum_{i=1}^{n} (P_0(i) + f(t_i))$$

$$= \sum_{i=1}^{n} P_0(i) + \sum_{i=1}^{n} f((i-1)T)$$

$$P(all\ successes) = \prod_{i=1}^{n} P(i\text{-}th\ item\ is\ success)$$

$$= \prod_{i=1}^{n} (P_0(i) + f((i-1)T))$$

Variable Probabilities

Additive Decrease

According to the arithmetic/geometric mean inequality, with $\sum_{i=1}^{n} P_1(t_i)$ fixed, the product of n terms would be larger when terms are close(r) to each other.

That is to say, the probability that all items succeed depends on the order of service while the expected number of successes does not. In this case, it is better to attend to the weaker items first.

Variable Probabilities

Additive Decrease

The former result would change if the assumption $P_0(i) + f(t_i) > 0$ does not hold.

Variable Probabilities

Additive Decrease

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Theorem

If for some i we have $P_0(i) + f(t_i) \leq 0$, then

$$E(\#successes) = \sum_{i=1}^{n} P_1(t_i) = \sum_{i=1}^{n} (P_0(i) + f(t_i))^+$$

$$P(all\ successes) = \prod_{i=1}^{n} P_1(t_i) = \prod_{i=1}^{n} (P_0(i) + f(t_i))^+ = 0$$

Variable Probabilities

Additive Decrease

The expected number of successes could be written as

$$
\begin{aligned}
E(\#successes) &= \sum_{i=1}^{n} P_1(t_i) \\
&= \sum_{i=1}^{n}(P_0(i)+f(t_i)) - \sum_{P_0(i)+f(t_i)<0}(P_0(i)+f(t_i)) \\
&= \sum_{i=1}^{n}(P_0(i)+f(t_i)) + \sum_{P_0(i)<-f(t_i)}|P_0(i)+f(t_i)|
\end{aligned}
$$

If $\frac{\mathrm{d}}{\mathrm{d}t}f(t) < 0$, then items with bigger P_0 should be processed earlier, so that the number of terms $s.t.\ P_0(i) < -f(t_i)$ and values of those terms will all increase, which would lead to a higher expected number of successes. In this case, it is better to attend to the stronger items first, assuming that the expected number is more important than P(all Successes).

Simple Example

$P_0 = (.9, .8, .7)$. $t = (0, 1, 2)$. $f(t) = -.1 * t = (0, -.1, -.2)$
Then $P_1 = P_0 - f(t) = (.9 - 0, .8 - .1, .7 - .2) = (.9, .7, .5)$
$E(\#successes) = .7$. $P(allsuccess) = .9 * .7 * .5 = .315$
===
$P_0 = (.7, .8, .9)$. $t = (0, 1, 2)$. $f(t) = -.1 * t = (0, -.1, -.2)$
Then $P_1 = (.7 - 0, .8 - .1, .9 - .2) = (.7, .7, .7)$
$E(\#successes) = .7$. $P(allsuccess) = .7 * .7 * .7 = .343$
Choose weaker items first.

Simple Example

$P_0 = (.9, .1)$. $t = (0, 1)$. $f(t) = -.2 * t = (0, -.2)$
Then $P_1 = P_0 - f(t) = (.9 - 0, .1. - .2) = (.9, -.1) \leftarrow (.9, 0)$
$E(\#successes) = .45$. $P(allsuccess) = 0$
===
$P_0 = (.1, .9)$. $t = (0, 1)$. $f(t) = -.2 * t = (0, -.2)$
Then $P_1 = (.1, .9 - .2) = (.1, .7)$
$E(\#successes) = .4$. $P(allsuccess) = .1 * .7 = .07$
Choose stronger items first to maximize total expected success.

Variable Probabilities

Multiplicative Decrease

According to multiplicative method,

$$P_1(t_k) = P_0(k) \cdot f(t_k)$$

where $f(t_k) \in (0, 1)$.

Variable Probabilities

Multiplicative Decrease

Theorem

With multiplicative decrease, there are

$$E(\#successes) = \sum_{i=1}^{n} P_1(t_i) = \sum_{i=1}^{n} (P_0(i) \cdot f(t_i))$$

$$P(all\ successes) = \prod_{i=1}^{n} P_1(t_i) = \prod_{i=1}^{n} (P_0(i) \cdot f(t_i))$$

$$= \prod_{i=1}^{n} P_0(i) \cdot \prod_{i=1}^{n} f(t_i)$$

Corollary

$E(\#successes)$ is maximized if higher prob. of success items are processed first.
$P(all\ successes)$ is independent of the order of service.

Numerical Example

The Thailand rescue serves as a numerical example of the model we have built.

Considering people who were trapped in the cave would get weaker over time, we set $\frac{\mathrm{d}}{\mathrm{d}t} f(t) < 0$ for both additive and multiplicative situation.

Numerical Example

Probabilities with additive decrease

For example, set $T = 1$ and choose $f(t) = -0.06t$ for additive decrease, generate 13 random values uniformly on (0.5,1) to represent the probabilities $P_0(j)$ of success for person j at time 0.

As $t_k = \sum_{i=1}^{k-1} T = k - 1$,
we have $P_1(t_k) = (P_0(k) - 0.06(k-1))^+$, $k = 1, 2, \ldots, 13$.

Numerical Example

Probabilities with additive decrease

The probabilities are presented in a matrix shown below, where the square at *i*-th row and *j*-th column corresponds to the *j*-th person's probability of survival if he is saved at the *i*-th stage. The bigger the square is and the darker its color, the bigger the probability.

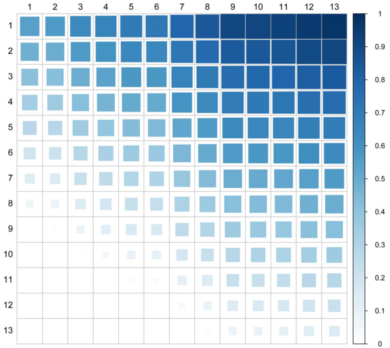

Figure 1: Prob that customer *j* survives if served at stage *i*

Numerical Example

Probabilities with additive decrease

Set $P_0 = (0.5337, 0.5897, 0.6306, 0.6336, 0.6916, 0.7068, 0.7229, 0.7593, 0.7649, 0.8395, 0.9046, 0.9339, 0.9733)$, the expected number of survivors and the probability that all those trapped survive are shown in table 1.

Table 1: $E(\#survivors)$ and $P(\text{all survive})$

	Stronger First	Weaker First
$E(\#survivors)$	5.2612	5.0046
$P(\text{all survive})$	0	2.6720×10^{-6}

Numerical Example

Probabilities with multiplicative decrease

Choose $f(t) = 0.95^t$ for this condition. With the same vector P_0, the expected number of survivors and the probability that all those trapped survive are shown in table 2.

Table 2: $E(\#survivors)$ and $P(\text{all survive})$

	Stronger First	Weaker First
$E(\#survivors)$	7.4918	7.0136
$P(\text{all survive})$	0.0003	0.0003

Summary Table

		MaxExp		MaxP(ALL)	
		WeakFirst	StrFirst	WeakFirst	StrFirst
Add Decr	All large p	Tie	Tie	Win	Lose
	Some small p	Lose	Win	Win	Lose
Mult Decr	All p	Lose	Win	Tie	Tie
Add Incr	Some large p	Lose	Win	Lose	Win
	No large p	Tie	Tie	Lose	Win
Mult Incr	Some large p	Lose	Win	Lose	Win
	No large p	Win	Lose	Tie	Tie

The End

Thank you for your attention!