Batch Arrivals and Delays CanQueue2012

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September 4, 2012

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- Review of Probabilistic Interpretation of Laplace Transforms

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- Use for Busy period of an M/M/1 system

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Probabilistic interpretation of Laplace Transforms

DEFINITION: The Laplace transform $L(s)$ of a function $f(x)$ with positive support is given by

$$L_X(s) = \int_0^\infty e^{-sx} f(x) dx$$

where $s > 0$.

Catastrophe Process

THEOREM: Let X be a r.v. with positive support and with pdf $f(x)$. Let Y be a r.v. independent of X, such that $Y \sim$ exponential with rate s. Then

$$L_X(s) = P(X < Y).$$

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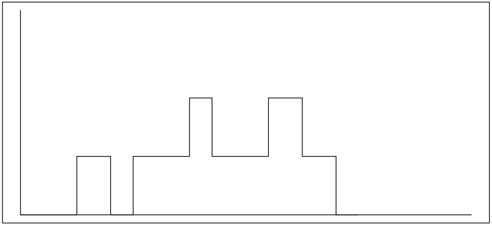
- The exponential random variable Y is called the catastrophe.
- The Laplace transform of a p.d.f of a random variable X is the probability that X occurs before the catastrophe.
- More precisely, the Laplace transform of a probability density function $f(x)$ of a random variable X can be interpreted as the probability that X precedes a catastrophe where the time to the catastrophe is an exponentially distributed random variable Y with rate s, independent of X.

Application to busy periood of an M/M/1 system



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- If the next event is an arrival, then the "game" is still on but we are now two steps away from the target. So we need to complete essentially two busy periods before the catastrophe.
- Thus we get the equation:

Application to busy periood of an M/M/1 system

- $L(s) = \frac{\mu}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu + s} L(s)^2$

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Model with Bulk Arrival and Delay

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- Assume customers arrive as singles or in pairs.
- Assume interarrival times are exponential – rates $k\lambda$ and $(1-k)\lambda$.
- Customers receive exponentially distributed service one at a time (rate μ).

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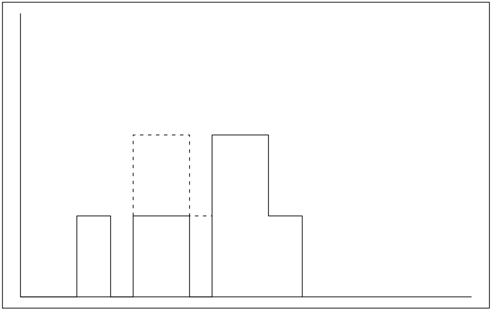
Model with Bulk Arrival and Delay

- If customers arrive in pairs, then one of the customers is the primary customer and joins any existing queue for service.
- The other customer of the pair is a secondary customer and will not join the queue for service until another customer arrives (single or pair). If the new arrival is a pair, then the former seondary customer behaves like a regular customer, but one of the new arrivals takes the role of secondary customer.

Model with Bulk Arrival and Delay



Model with Bulk Arrival and Delay

We define the states of the system as pairs
(0,0), (1,0), (2,0), (3,0), ...
(0,1), (1,1), (2,1), (3,1), ...
where the first coordinate is number of regular customers in the system, and the second coordinate is the number of secondary customers in the system. The following diagram shows the possible transitions.

Model with Bulk Arrival and Delay

Let $L_{Busy}(s)$ be the Laplace Transform of the busy period.
Let $L_{10}(s)$ be the Laplace Transform of the time to empty the system i.e. move to (0,0) beginning in state $(1,0)$.
Let $L_{11}(s)$ be the Laplace Transform of the time to empty the system i.e. move to (0,0) beginning in state $(1,1)$.

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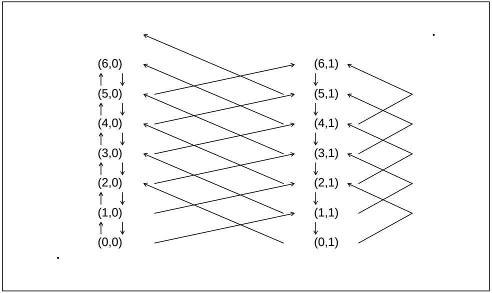
  - $p =$ Pr((i,0) to (i-1,0) first of $\{$(i-1,0),(i-1,1),C$\}$
  - $q =$ Pr((i,0) to (i-1,1) first of $\{$(i-1,0),(i-1,1),C$\}$
  - $r =$ Pr((i,1) to (i-1,0) first of $\{$(i-1,0),(i-1,1),C$\}$
  - $t =$ Pr((i,1) to (i-1,1) first of $\{$(i-1,0),(i-1,1),C$\}$
- Note that p, q, r, t are all functions of s

Model with Bulk Arrival and Delay



Model with Bulk Arrival and Delay

- More Notation:
- Let
- $\alpha = \dfrac{\lambda k}{\lambda + \mu + s}$; $\qquad \alpha^* = \dfrac{\lambda k}{\lambda + s}$;
- $\beta = \dfrac{\lambda(1-k)}{\lambda + \mu + s}$; $\qquad \beta^* = \dfrac{\lambda(1-k)}{\lambda + s}$;
- $\gamma = \dfrac{\mu}{\lambda + \mu + s}$

Model with Bulk Arrival and Delay

Then

$$\begin{aligned} L_{10}(s) &= \gamma + \alpha p L_{10}(s) + \alpha q L_{11}(s) + \beta r L_{10}(s) + \beta t L_{11}(s) \\ L_{11}(s) &= \gamma\alpha^* p L_{10}(s) + \gamma\alpha^* q L_{11}(s) + \gamma\beta^* r L_{10}(s) + \gamma\beta^* t L_{11}(s) \\ &\quad + \alpha p p L_{10}(s) + \alpha p q L_{11}(s) + \alpha q r L_{10}(s) + \alpha q t L_{11}(s) \\ &\quad + \beta r p L_{10}(s) + \beta r q L_{11}(s) + \beta t r L_{10}(s) + \beta t t L_{11}(s) \end{aligned}$$

These equations are linear in $L_{10}(s)$ and $L_{11}(s)$.

Model with Bulk Arrival and Delay

where

$$
\begin{aligned}
p &= \gamma + \alpha pp + \alpha qr + \beta rp + \beta tr \\
q &= \alpha pq + \alpha qt + \beta rq + \beta tt \\
r &= \alpha ppp + \alpha pqr + \alpha qrp + \alpha qtr + \beta rpp + \beta rqr + \beta trp + \beta ttr \\
t &= \gamma + \alpha ppq + \alpha pqt + \alpha qrq + \alpha qtt + \beta rpq + \beta rqt + \beta trq + \beta ttt
\end{aligned}
$$

These nonlinear equations allows us to find p, q, r, t through iteration methods.

Model with Bulk Arrival and Delay

Finally

$$L_{Busy}(s) = kL_{10}(s) + (1 - k)L_{11}(s)$$

Thank you!
THE END