Probabilistic Inversion of Laplace Transforms

Myron Hlynka
Zishan Huang
Richard Caron

Department of Mathematics and Statistics
University of Windsor
Windsor, ON, Canada.

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Outline

- Definition of Laplace Transform
- Probabilistic Interpretation
- Approximation Technique for Inversion
- Solving linear equations and difficulties
- Resolution

Definition of Laplace Transform

DEFINITION: The Laplace transform $L(s)$ of a pdf $f(x)$ with positive support is given by

$$L_X(s) = \int_0^{\infty} e^{-sx} f(x) dx$$

where $s > 0$.

Catastrophe Process

THEOREM: Let X be a r.v. with positive support and with pdf $f(x)$. Let Y be a r.v. independent of X, such that $Y \sim$ exponential with rate s. Then

$$L_X(s) = P(X < Y).$$

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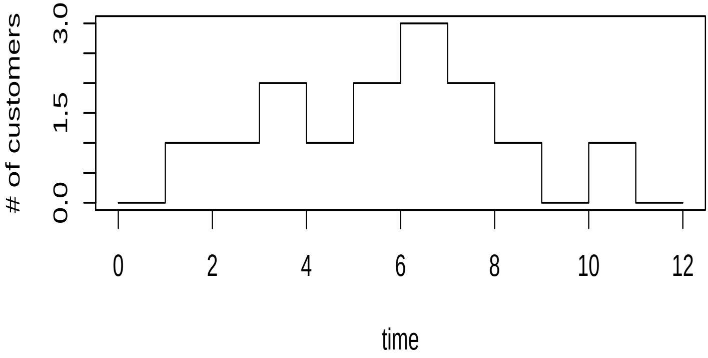
- The exponential random variable Y is called the catastrophe.
- The Laplace transform of a p.d.f of a random variable X is the probability that X occurs before the catastrophe.
- More precisely, the Laplace transform of a probability density function $f(x)$ of a random variable X can be interpreted as the probability that X precedes a catastrophe where the time to the catastrophe is an exponentially distributed random variable Y with rate s, independent of X.

Busy Period Plot

A busy period begins with a single customer and ends when the customer count returns to zero.



LT of M/M/1 Busy period

$$L(s) = P(BP < catastrophe) = \frac{\mu}{\lambda+\mu+s} + \frac{\lambda}{\lambda+\mu+s} L(s)^2$$

Solving this quadratic gives

$$L(s) = \frac{\lambda+\mu+s-\sqrt{(\lambda+\mu+s)^2-4\lambda\mu}}{2\lambda}$$

For $\lambda = 1$,$\mu = 2$, we get

$$L(s) = \frac{3+s-\sqrt{(1+6s+s^2)}}{2}$$

Procedure for Approximate Inversion

STEP 1: Given $L_X(s)$, we can compute $E(X) = -L'(0)$ and $E(X^2) = L''(0)$ and hence find the variance.
If we do this numerically (check the slope at 0 and the change of slope at zero), we do not need to work with increasingly complicated derivative expressions for $L(s)$.

Example

Example (using R)
$L = function(s)\{(3 + s - sqrt(1 + 6 * s + s^2))/2\}$
$d1 = (L(.0001) - L(0))/.0001 = -0.9998001$
$d2 = (L(.0002) - 2 * L(.0001) + L(0))/.0001^2 = 3.996403$
Thus $E(X) = 1, E(X^2) = 4, var(X) = 4 - 1^2 = 3, sd = \sqrt{3} = 1.732$
Using Chebyshev's inequality, at least 3/4 of the data are between 0 and 4.5

Procedure Steps 2 and 3

STEP 2: Choose the values for s. There are lots of possible choices, and lots of choices for the number of values of s to choose. Small values of s near zero can give lots of information. In our example we choose $s = 0, .1, .2, \ldots, 1.0$ (11 values of s)
STEP 3: Choose our x values in the region suggested by Chebyshev's Inequality. It is our intention to approximate the pdf by a discrete mass function. We choose 11 values for X, to match the 11 values that we chose for s. Our choice is .3, .9, 1.5, 2.1, 2.7, 3.3, 3.9, 4.5, 5.1, 5.7, 6.3. These values will be the centres of our intervals. The corresponding intervals are $[0, .6), [.6, 1.2), \ldots, [6.0, \infty)$

Step 4

STEP 4. Evaluate $L(s)$ for each value of s from Step 2.

The values of $L(0), L(.1), \ldots, L(1)$ are computed to be
1.0000000 0.9155711 0.8516685 0.8000000 0.7566019
0.7192236 0.6864471 0.6573140 0.6311422 0.6074278 0.5857864

Step 5

STEP 5: Corresponding to the 11 values of X, we would like to attach probabilities $p_1, \ldots, p_{11}$. If we can estimate these values, we will have a discrete approximation of the pdf.

We have
$.91557 = L(.1) = P(X < Y)$ for Y distributed as exponential at rate $s = .1$. Since $P(Y > y) = e^{-sy}$ we get

$$
\begin{aligned}
L(.1) &= P(X < Y) \\
&= P(X = .3)P(Y > .3) + \cdots + P(X = 6.3)P(Y > 6.3) \\
&= p_1 e^{-s(.3)} + p_2 e^{-s(.9)} + \cdots + p_{11} e^{-s(6.3)} \\
&= p_1 e^{-.1(.3)} + p_2 e^{-.1(.9)} + \cdots + p_{11} e^{-.1(6.3)}
\end{aligned}
$$

Repeat for other values of s.
We get a system of equations

$$
\begin{aligned}
1.00000 &= p_1 e^{-0(.3)} + p_2 e^{0(.9)} + \cdots + p_{11} e^{-0(6.3)} \\
.91557 &= p_1 e^{-.1(.3)} + p_2 e^{.1(.9)} + \cdots + p_{11} e^{-.1(6.3)} \\
&\vdots \\
.58579 &= p_1 e^{-1.0(.3)} + p_2 e^{-1.0(.9)} + \cdots + p_{11} e^{-1.0(6.3)} \qquad (1)
\end{aligned}
$$

Step 6

Step 6. This is a linear system of equations $L(s) = Dp$, where $L(s)$ and p are 11×1 vectors and D is an 11×11 matrix. The first equation insures that the probability vector p sums to 1. Solve.

Solution

The solution for the probability vector p is
$(18.7, -253.0, 1611.4, -6079.9, 15036.0, -25454.0,$
$29870.3, -23995.1, 12630.3, -3934.8, 551.28)$

Something is horribly wrong. We have some huge negative probabilities. It would be worth noting that for other choices of the 11 values of s, we received an error message that our matrix D is almost singlular.

Resolution

A solution to this is to solve the system of equations by restricting the values of p to be nonnegative. This turns the problem into a quadratic programming problem in which the sum of squares of the differences between $(L(0), L(.1), \ldots, L(1.0)) = (1, .91557, \ldots, .58579)$ and the expression on the RHS of (1) given by the constrained unknown vector p, is minimized. The vector p is is easily computable using EXCEL's package SOLVER.
Our new result is $p = (0.66, 0.23, 0.0019, 0.0020, 0.0017,$
$0.0014, 0.00079, 0.014366, 0.036466, 0.055656)$
Much better!

An additional constraint

This last result also has some difficulty. The function drops near zero and then starts to increase. This is strange behaviour, perhaps due to the discrete approximation and perhaps due to the fact that we collapsed values from $(6, \infty)$ into a single point. It is easy in SOLVER to add inequality constraints, So, following the ideas used in Ridge regression and Lasso, we add a condition that the sum of the ratios of consecutive p_i's must be less than 50.

Our new result for p is (0.413, 0.267, 0.173,0.113, 0.073, 0.0466, 0.029, 0.017, 0.009, 0.004, 0)

The last entry does not satisfy the ratio criterion, but it fits the deceasing pattern. The probabilities no longer sum to 1, but the sum is close to 1. Since the interval size is .6, we must adjust these numbers by dividing by .6 to get equivalent pdf values. We can plot these values using R.

We can plot adjusted p vs x to get

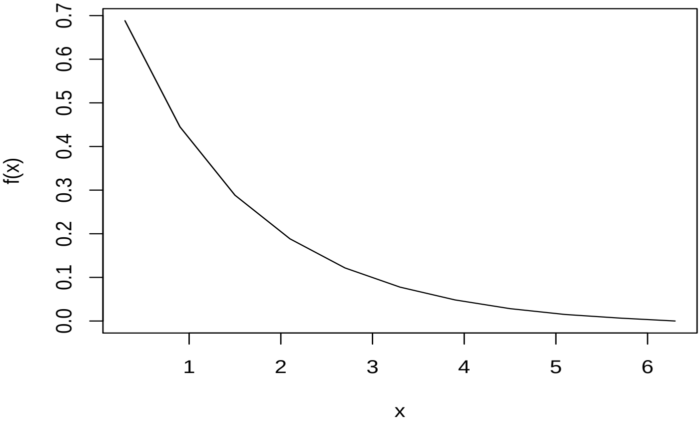


Figure: p vs x

The actual result, using R, function invlap, in package pracma gives

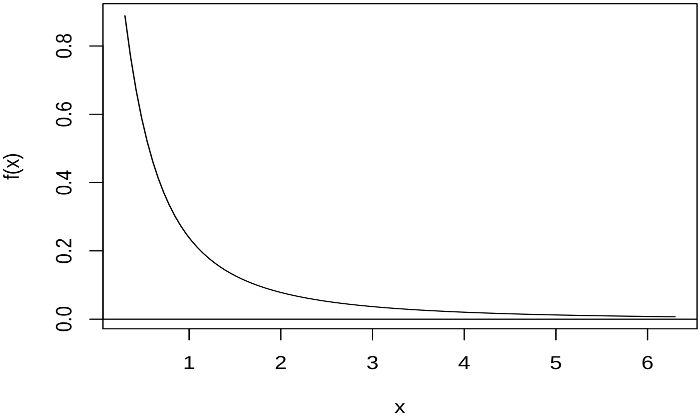


Figure: prob density function

The End.

Thank you.