INVARIANTS
December 2nd

(1) Is it possible for a knight to start at one corner of a 5×5 chessboard and reach the opposite corner passing once through all squares?
(2) If we remove two opposite corners from an 8×8 chessboard, is it possible to cover what remains with 2×1 dominoes?
(3) Let $a_1, a_2, \ldots, a_{2009}$ be an arbitrary permutation of the numbers $1, 2, \ldots, 1999$. Prove that $(a_1 - 1)(a_2 - 2)\cdots(a_{1999} - 1999)$ must be an even number.
(4) Start with the positive integers $1, 2, \ldots, 4n - 1$. In one move, you may take any ordered pair of integers from the list and replace it with the difference of the two integers. Prove that after $4n - 2$ moves, the list consists of one even integer.
(5) On every square of a 2009×2009 chessboard is written either $+1$ or -1. For every row (and column) we compute the product R_i (resp. C_i) of all entries in that row (resp. column). Prove that $\sum_{i=1}^{2009}(R_i + C_i) \neq 0$.
(6) There are 5 red marbles and 6 green marbles in a jar. Thaddeus plays the following game: he removes two marbles from the jar at a time.
   (a) If both marbles are green, he puts one green marble back.
   (b) If the marbles are different coloured, he puts one red marble back.
   (c) If the marbles are both red, he puts one green marble back (if he has one).

   Show that eventually there is only one marble left. What colour is it?
(7) (a) There are 2 red marbles and 4 blue marbles in a jar. Two people play a game, where on each turn they can remove any number of marbles of *one* colour. (They must remove at least one marble in each turn.) The players take turns moving, and the loser of the game is the player who leaves the same number of red marbles and blue marbles in the jar. Describe a winning strategy for the *first* player (i.e. explain how the first player can always win regardless of what the second player does).
   (b) Suppose now that there are 2000 red marbles and 2001 blue marbles. Show that the *second* player has a winning strategy.
   (c) Generalize the problem: if there are x red marbles and y blue marbles (with $x \neq y$), under what conditions does the second player have a winning strategy? Justify your answer.
(8) In parliament, each member has at most three enemies (clearly the author of this problem is an idealist). Prove that parliament can be separated into two parites, so that each member has at most one enemy in his or her own party.
(9) A pile contains 30 pebbles. Jason and Jimmy alternate turns, removing either one or two pebbles from the pile at each turn. Jason goes first. The player who removes the last pebble wins. Who has a winning strategy?
(10) White and black checkers occupy the leftmost and rightmost cells of a 1×20 chessboard. White and black take turns moving their pieces either one or two steps in either direction. Neither jumping over the other piece nor landing on an occupied square are permitted. Who has a winning strategy?
(11) A game starts with four heaps of beans containing $3, 4, 5, 6$ beans. Katrina and Wmoloki move alternately, in that order. A move consists of removing:
   (a) one bean from a heap provided that there are at least two beans left behind in that heap
   (b) OR a complete heap of two or three beans.

   The player who takes the last heap wins. Who has a winning strategy? (1995 Putnam)
(12) Aaron and Betty take turns entering a symbol in an empty cell of a $1 \times n$ chessboard, where n is an integer greater than 1. Aaron always enters the symbol X and Betty always enters the symbol O. Two identical symbols may not occupy adjacent cells. A player without a move loses the game. If Aaron goes first, who has a winning strategy? (Fall 2007 Junior A Level Tournament of the Towns.)
(13) The Game of Nim: There are an arbitary number of piles of sticks. Each pile can have any number of sticks. Two players take turns moving. A move consists of picking a pile and removing at least one stick from that pile. The player who takes the last stick wins. Who has the winning strategy? (The winning strategy is easy to describe but hard to find.)