University of Windsor Problem Solving December 2, 2008

1 Invariants

An invariant is a quantity that does not change when the system is transformed in some way. Sometimes the parity (odd/even measure) of some expression is the invariant. Here are some current websites that have invariant problems.

Example: Every person on earth has shaken hands with a certain number of people. Prove that the number of people who have shaken and odd number of hands is even.
SOLUTION: We can view people as nodes where an arc between two nodes means that the nodes have shaken hands with each other. Let N_{old} be the number of people who have shaken an odd number of hands. If we remove an arc between nodes A and B, there are three cases.
Case1: Before removal of the arc, both A and B have both shaken an odd number of hands. After removing the arc, A and B become even hand shakers so $N_{new} = N_{old} - 2$ and has the same parity.
Case2: Before removal of the arc, A and B have both shaken an even number of hands. After removing the arc, both A and B become odd hand shakers and $N_{new} = N_{old} + 2$ and has the same parity.
Case3. A has shaken hands with an odd number of people and B has shaken hands with an even number of people (or vice versa). Then the parity of A and B both change and the value of $N_{new} = N_{old}$ with the same parity.
In all three cases the parity of N_{new} equals that of N_{old}. Thus we can remove all arcs and have the same parity as N_{old}. But with no arcs, the number of people who have shaken an odd number of hands is $N_{new} = 0$, which is even. Thus N_{old} is even.

Example: An 8×8 has two diagonally opposite corners removed so has 62 squares left. Can the 62 remaining squares be covered by 31 2×1 dominoes.
SOLUTION: Suppose the chess board is colored white and black. Both of the two diagonally opposite corners have the same color, say white. Each domino covers one white square and one black square and since there are 30 black square and 32 white square, the covering is impossible.
The invariant is the excess of uncovered blacks over whites after each placement of a domino. This is always 2. At the end there are two black squares so the covering is impossible.

http://math.arizona.edu/∼savitt/teaching/math294/F06invariants.pdf

1. In the mathematics department, every faculty member has at most three enemies. (Assume that being an enemy is symmetric: if Professor X is an enemy of Professor Y, then Professor Y is an enemy of Professor X.) Prove that the faculty can be divided into two committees such that each faculty member has at most one enemy on their committee.

2. Is it possible for two different powers of 2 to have the same digits (in a different order)?

3. Suppose that not all four integers a, b, c, d are equal. Start with (a, b, c, d) and repeatedly replace (a, b, c, d) by $(a-b, b-c, c-d, d-a)$. Show that eventually at least one number of the quadruple will become arbitrarily large.

4. Start with the positive integers $1, \ldots, 4n - 1$. In one move you may replace any two integers by their difference. Prove that after $4n - 2$ steps, the final remaining integer will be even.

5. 64 coins are arranged in an 8-by-8 square. Initially half the coins are heads-up and half the coins are tails-up, in an alternating pattern. At each step there are three possible moves: flip all eight coins in a single row; flip all eight coins in a single column; flip all four coins in a 2-by-2 square. Is it possible to reach a configuration where exactly one coin is heads-up and the rest are tails-up?

6. A circle is divided into six sectors. The numbers 1, 0, 1, 0, 0, 0 are written into the sectors (in counterclockwise order). At each step you may increase two neighboring numbers by 1. After a sequence of such steps is it possible for all the numbers to be equal?

7. Three bugs are crawling on the coordinate plane. They move one at a time, and each bug will only crawl in a direction parallel to the line between the other two. If the bugs start out at $(0, 0), (3, 0), (0, 3)$, can the bugs end up at $(1, 2), (2, 5), (2, 3)$? Is it possible that after some time the first bug will end up back where it started, while the other two bugs have switched places?

8. Start with the set $\{3, 4, 12\}$ at each step you may choose two of the numbers a, b and replace them by $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you reach the set $\{4, 6, 12\}$? Can you reach any set $\{x; y; z\}$ with $|x - 4|, |y - 6|, |z - 12|$ all less than $1/\sqrt{3}$?

9. In a certain small town there live n fickle dwarves, each of whom lives in a red house or a blue house. Every day, one dwarf looks at all the houses other than his own; if he discovers that the color of his own house is less popular among the $n - 1$ other houses than the other color, he repaints his house in the other color. Prove that eventually there will be no more repainting.

10. The vertices of an n-gon are labeled by real numbers $x_1, X_2, \ldots, x_n$. Let a, b, c, d be four successive labels. If $(a - d)(b - c) < 0$, we may swap b with c. Can this switching operation be performed infinitely many times?

11. Start with the integer 72006. At each step, delete the leading digit, and add it to the remaining number. This is repeated until a number with exactly 10 digits remains. Prove that this number has two equal digits.

12. There is a checker at the point $(1,1)$ in the plane. At each step, you may move the checker in one of two ways: by doubling one of the two coordinates, or (if the coordinates are unequal) by subtracting the smaller coordinate from the larger one. Is it possible for the checker to end up at the point $(3,3)$?

13. n numbers are written on a blackboard. At each step, two numbers a, b are erased, and $(a+b)/4$ is written on the board. If the initial n numbers are all equal to 1, prove that after $n-1$ steps, the one remaining number is at least $1/n$.

14. There is a row of 1000 integers. A second row of integers is constructed underneath it, as follows: underneath the integer a, write the number of times that a occurs in the first row. In the same way, we produce a third row by counting occurrences in the second row, and so forth. Prove that eventually one of the rows is identical to the subsequent row.

15. Nine squares in a 10-by-10 grid are infected. In one time unit, the squares which share an edge with at least two infected squares also become infected. Can the infection spread to the whole square?

Other Websites:
1. http://www.math.uwaterloo.ca/∼snew/Contests/ProblemSessions/soln1.pdf (with solutions)
2. http://www.math.hmc.edu/∼ajb/PCMI/pcmi03_b.pdf