University of Windsor Problem Solving October 28, 2008

1 Pigeonhole Principle

Introduction
A. If $n > m$ pigeons are placed into m boxes, then there exists (at least) one box with at least two pigeons.
B. If $n > m$, then any function $f : [n] \to [m]$ is not injective.($[n] = \{1, 2, \ldots, n\}$).
C. If $m(k-1)+1$ pigeons are placed in m boxes, then there exists (at least) one box that contains at least k pigeons.
D. If the average of m numbers $x_1, x_2, \ldots, x_m$ is greater(less) than a, then at least one x_i is greater(less) than a.

Problems
1. Among any 13 U. of Windsor students, there are two born in the same month.
2. Among all the people in Canada, there are two that have the same number of hairs.
3. There are 25 students in the Putnam class and the sum of their ages is 514 years.
Show that there are 17 students such that the sum of their ages is at least 350 years.
4. Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one half unit apart. What if the equilateral triangle is replaced by a square whose side has length of one unit?
5. Is there a pattern in the final digits of the Fibonacci numbers? Recall that the Fibonacci numbers are defined as $F_1 = 1$, $F_2 = 1$ and $F_{i+1} = F_i + F_{i-1}$ for $i = 2, 3, \ldots$. Thus the Fibonacci numbers are 1,1,2,3,5,8,13,21,34,... and the final digits are 1,1,2,3,4,8,3,1,4,....
6. Given any n+2 integers, show that there exist two of them whose sum, or else whose difference, is divisible by 2n?
7. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, , 100. Prove that there must be two distinct integers in A whose sum is 104. [Actually, 20 can be replaced by 19.]
8. Given 4 points on the circumference of a circle, show that at least three of them are in the same semicircle.
9. (Putnam 2002, A2) Prove that given any five points on a sphere, there exists a closed hemisphere which contains four of the points.

Some of the above material is from
http://www.math.wm.edu/∼shij/math410-problem-solving/pigeon-hole.pdf
See also
http://www.mast.queensu.ca/∼mikeroth/putnam/handouts/week01.pdf