Problem Solving October 21, 2008

1 Difference of Squares

Result: $a^2 - b^2 = (a-b)(a+b)$

Example 1.1 $96 \times 104 =?$

Solution; $96 \times 104 = (100-4)(100+4) = 100^2 - 4^2 = 10000 - 16 = 9984.$

Example 1.2 *Express* $S = \dfrac{1}{(1+x)(1+x^2)(1+x^4)(1+x^8)}$ *as a power series.*

Solution:

Recall $a + ar + ar^2 + \cdots = \dfrac{a}{1-r}$ *for* $|r| < 1$.

$$S = \frac{(1-x)}{(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)} = \frac{1-x}{(1-x^2)(1+x^2)(1+x^4)(1+x^8)}$$

$$= \cdots = (1-x)\frac{1}{1-x^{16}} = (1-x)(1+x^{16}+x^{32}+x^{48}+\dots)$$

$$= 1 - x + x^{16} - x^{17} + x^{32} - x^{33} + \dots.$$

Example 1.3 *Show the product of 4 consecutive integers cannot be a perfect square.*

Solution:

$$S = n(n+1)(n+2)(n+3) = n(n+3)(n+1)(n+2) = (n^2+3)(n^2+3n+2)$$

$$= (n^2+3n+1-1)(n^2+3n+1+1) = (n^2+3n+1)^2 - 1.$$

So S *cannot be a perfect square.*

Example 1.4 *:*

Observe that $3^2 + 4^2 = 5^2$ *so* $5^2 - 4^2 = 3^2$ *so* $(5-4)(5+4) = 3^2$ *so* $1(9) = 3^2$.

Observe that $5^2 + 12^2 = 13^2$ *so* $13^2 - 12^2 = 5^2$ *so* $(13-12)(13+12) = 5^2$ *so* $1(25) = 5^2$.

$(3, 4, 5)$ *and* $(5, 12, 13)$ *are Pythagorean triples.*

Find another set of triples using the technique suggested here.

SOLUTION:

Working backward from the next odd number 7, we have

$$7^2 = 1(49) = (25-24)(25+24) = 25^2 - 24^2$$

so $24^2 + 7^2 = 25^2$.

Exercise 1.1 *Solve for* M *and* N *(positive integers).*

$M^2 - N^2 = 12$

Exercise 1.2 *Solve for* x *if* $(1 - \frac{1}{3^2})(1 - \frac{1}{4^2})(1 - \frac{1}{5^2}) \ldots (1 - \frac{1}{1991^2}) = \frac{x}{1991}$.

Exercise 1.3 *Factor* $x^4 + x^2 + 1$.

Exercise 1.4 *Define the Fermat numbers as* $F_n = 2^{(2^n)} + 1$ *for* $n = 0, 1, 2, , \ldots$.

Show that F_m *is relatively prime to* F_n *for* $m \neq n$.

Reference: http://tiger.uofs.edu/staff/LEONGT2/mathblog/article06.html