DEPARTMENT OF MATHEMATICS AND STATISTICS
Theory of Interest 62-392 Test 2 SOLUTIONS M. Hlynka
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Directions: You must show your work in ALL questions, including multiple choice. !

1. (Jerry Veeh problem) A 15 year 1000 par bond has 7% semiannual coupons and is callable at par after 10 years. What is the price of the bond to yield 5% nominal semiannual.
SOLUTION: Since yield is less than the coupon rate, the purchaser paid a premium for the bond. The issuer wants to end the coupon payments as soon as possible. so we assume the bond will be called after 10 years. Take a period to be 6 months. Thus

$$\begin{aligned} P &= Cv^{20} + Fra_{\overline{20}|} = 1000(1.025^{20}) + 35a_{\overline{20}|.025} \\ &= 1000(1.025^{-20}) + 35\frac{1-1.025^{-20}}{.025} = 1155.892 \end{aligned}$$

2. (old SOA problem) A 20-year loan of 20000 may be repaid under the following two methods: i) amortization method with equal annual payments at an annual effective rate of 6.5% ii) sinking fund method in which the lender receives an annual payment of 1600 and a lump sum of 20000 after 20 years accumulated through a sinking fund which earns an annual effective rate of j. Both methods require a payment of X to be made at the end of each year for 20 years. Set up an equation for j and check which of the following values of j is best.
(A) .118 (B) .130 (C) .142
SOLUTION:
$20000 = Xa_{\overline{20}|.065}$ so $X = \dfrac{20000(.065)}{1-1.065^{-20}} = 1815.13$
We also need $20000 = (1815.13 - 1600)s_{\overline{20}|j}$ or $s_{\overline{20}|j} = 92.97$ Check each of the three answers to get C is best.

3. An investor makes a payment of \$200 immediately and a payment of \$264 at the end of two years in exchange for a payment of \$460 at the end of one year.
(a) Find the equation of value for the interest rate i.
(b) Find i.
SOLUTION:
The equation of value (at the end of year 2) is

$$200(1+i)^2 + 264 = 460(1+i).$$

This is a quadratic in i.
$200 + 400i + 200i^2 + 264 = 460 + 460i$.
so $4 - 60i + 200i^2 = 0$ so $1 - 15i + 50i^2 = 0$. We get $i = .1$ and $i = .2$. So the

interest rate is either 10% or 20%. We cannot eliminate either answer. They are both possible.

4. On January 1, 2005, an amount 100 was invested into a fund. X was deposited or withdrawn (you don't yet know which) on April 1, 2005. The balance in the fund on December 31, 2005 was 105. The dollar-weighted rate of return was 12%. Find X.
(A) 6.43 (B) -6.43 (C) 5.91 (D) -5.91 (E) -5.00
SOLUTION:

$$.12 = \frac{B - (A + \sum C_i)}{A + \sum C_i(1 - t_i)} = \frac{105 - (100 + X)}{100 + X(3/4)}.$$

Thus $12 + X(.09) = 5 - X$ so $X = \dfrac{-7}{1.09} = -6.42$. (The negative value indicates a withdrawal.)

5. (old SOA Problem, simplified) Donald takes a loan to be paid with annual payments of 500 at the end of each year for 2n years. The annual effective interest rate is 4.94%.
The sum of the interest paid in year 1 plus the interest paid in year $n + 1$ is equal to 720. Calculate the amount of interest paid in year 1.
SOLUTION
Both n and L (loan size) are unknown.
$500(1 - v^{2n}) + 500(1 - v^n) = 720$ (check formulas in the amortization table).
So $-500v^{2n} - 500v^n + 280 = 0$. This is a quadratic in v^n so from the quadratic formula, $v^n = .4$ (the other root is negative).
We want the amount of interest paid in year 1, namely $500(1 - v^{2n}) = 500(1 - (.4)^2) = 500(.84) = 420$.

6. (Broverman exercise 4.1.2) A 6% bond maturing in 8 years with semiannual coupons to yield 5% convertible semiannually is to be replaced by a 5.5% bond yielding the same return. In how many years should the new bond mature? (Both bonds have the same price, yield rate and face amount.
SOLUTION:
Let P =price of bond 1 (= price of bond 2). 8 years means 16 periods. Bond 1 gives $P = Fv^{16} + Fra_{\overline{16|}} = F(1.025)^{-16} + F(.03)a_{\overline{16|}.025}$.
For bond 2, let n be the number of years. There are $2n$ periods. Bond 2 gives $P = F(1.025)^{-2n} + F(.0275)a_{\overline{2n|}.025}$. Equating the two expressions and cancelling F gives
$1.025^{-16} + .03(1 - 1.025^{-16})/.025 = 1.025^{-2n} + .0275(1 - 1.025^{-2n})/.025$. Thus $1.025^{-2n}/10 = 1.1 - (1.025^{-16} + .03(1 - 1.025^{-16})/.025) = 0.03472499$. Thus $-2n(ln(1.025)) = ln(.3472499)$ so
$2n = -.5 * ln(.3472499)/ln(1.025)$ and $n = 21.41755$ years. We round this to 21.5 years to get the final coupon.

7. (Odufrey problem) The amount of principal repaid in the 3rd payment of a 5-year level payment loan at 9% is 300. What is the original loan value?

SOLUTION: Let L be loan amount and K be annual payment.
Repayment amount in third year is $300 = Kv^3 = K(1.09)^{-3}$. Thus $K = 300(1.09)^3 = 388.51$. The loan amount is $L = Ka_{\overline{5}|} = 1511.16$.

8. (Ostaszewski problem) Chuck puts 100 into a fund on Jan. 1, 2007. On July 1, 2007, Chuck puts an additional amount X into the fund. On January 1, 2008, the fund balance is 200. The dollar-weighted return rate for 2007 is 4.4%, and the time-weighted return rate is 5%. Find the effective rate of return earned in the first half of 2007.
A. 3.39% B. 3.45% C. 3.81% D. 4.12% E. 4.76%
SOLUTION:
$.044 = \dfrac{B - A - C}{A + C(.5)} = \dfrac{200 - 100 - X}{100 + X(.5)}$.
Solve for X to get $X = 95.6/1.022$.
Let i_1,i_2 be interest rates for the first and second halves of the year. Then

$$1.05 = (1 + i_1)(1 + i_2). \tag{1}$$

Thus $200 = 100(1 + i_1)(1 + i_2) + X(1 + i_2) = 100(1.05) + X(1 + i_2) = 105 + X(1 + i_2)$.
So $1 + i_2 = \dfrac{95}{X}$. Then from (1),
$1 + i_1 = \dfrac{1.05}{1 + i_2} = \dfrac{1.05(X)}{95} = \dfrac{1.05(95.6)}{95(1.022)} = 1.03886$.
Finally $i_1 = .033886$. A

9. A loan of X has an effective annual rate 8%. It is to be repaid by the end of 10 years. If the loan is repaid with a single payment at time $t = 10$, the total interest paid is \$468.05 more than if the loan were repaid by ten equal annual payments. Find the value of X.
SOLUTION:
TotalInterest1$= X(1.08)^{10} - X = X(1.08^{10} - 1) = 1.158925X$
TotalInterest2$= \dfrac{X}{a_{\overline{10}|}}(10 - a_{\overline{10}|}) = 0.4902949X$
$468.05 = 1.158925X - 0.4902949X = 0.6686301X$ so $X = 700.01$