Dr. M. Hlynka. Math 62-392.01 Test 2. April 3, 2006. 75 minutes.
Calculators allowed. Theory of Interest
Show all work. Justify all answers. All questions are worth 5 points.

1. A man buys a home for 250000. He pays 30000 in cash. The balance will be paid with a 25 year mortgage with [nominal annual] interest at 8% compounded semiannually. Find the level payment required under the mortgage at the end of each month.
A. 1550 B 1430 C 1270 D 1680 E 1720
1. SOLUTION 1: Interest rate is .04 per six months so $(1+j)^6 = 1.04$. Thus $j = .006558$. Then $220000 = Xa_{\overline{300}|j}$ so $X = \dfrac{220000}{a_{\overline{300}|j}} = \dfrac{220000}{131.0276} = 1679.04$. Answer is D.
SOLUTION 2: Interest rate is $i = .04$ for a period of 6 months. There are 25(2)=50 periods.
$a_{\overline{50}}^{(6)}$ means PV of payments of 1/6 per month.
$6a_{\overline{50}}^{(6)}$ means PV of payments of 1 per month.
$Xa_{\overline{50}}^{(6)}$means PV of payments of X per month. Then $220000 = 6Xa_{\overline{50}|}^{(6)}$ so $X = 220000/6a_{\overline{50}|}^{(6)}$.
Now $6a_{\overline{50}|}^{(6)} = 6\dfrac{1-v^{50}}{i^{(6)}} = 6\dfrac{1-1.04^{-50}}{6[1.04^{1/6}-1]} = .859287385/.0006558197 = 131.0249$ so $X = 220000/131.0249 = 1679.07$. Answer is D.

2. On January 1, 2005 100 was invested into a fund. X was deposited or withdrawn (you don't yet know which) on April 1, 2005. The balance in the fund on December 31, 2005 was 105. The dollar-weighted rate of return was 12%. Find X.
(A) 6.43 (B) -6.43 (C) 5.91 (D) -5.91 (E) -5.00
2. SOLUTION:

$$.12 = \frac{B-(A+\sum C_i)}{A+\sum C_i(1-t_i)} = \frac{105-(100+X)}{100+X(3/4)}.$$

Thus $12 + X(.09) = 5 - X$ so $X = \dfrac{-7}{1.09} = -6.42$. (The negative value indicates a withdrawal.)

3. Find the present value of a perpetuity with payments every two years beginning immediately where the payments have the form $p,\ p+q,\ p+2q,\ p+3q,\ \ldots$.
3. SOLUTION:

$$PV = p + (p+1q)v^2 + (p+2q)v^4 + (p+3q)v^6 + \cdots = p(1+v^2+v^4+\ldots) + qv^2(1+2v^2+3v^4+\ldots)$$
$$= \frac{p}{1-v^2} + qv^2(1+v^2+v^4+\ldots)(1+v^2+v^4+\ldots) = \frac{p}{1-v^2} + \frac{qv^2}{(1-v^2)^2}$$

4. A large investment corporation provides the following financial figures for the year ending December 31, 1996:

Assets,	December 31, 1995		470million
Assets,	December 31, 1996		590million
Revenue	Statement		
− − −	− − −	− − −	− − −
	Deposit Income		290million
	Interest Income		60million
	Withdrawals		180million
	Expenses		50million

Find the interest rate earned on investments during 1996.
A. 5% B. 8% C. 12% D. 15% E. 10% F. 6%
4. Solution: $I = B - A - C = 590 - 470 - (290 - 180 - 50) = 590 - 470 - 60 = 60$ as given. Then $i \approx \dfrac{I}{A+.5C} = \dfrac{60}{470+.5(60)} = \dfrac{60}{500} = .12$ So interest rate is $= 12\%$.

5. The accumulated amount of an annuity-immediate of R per year payable quarterly for seven years is \$3317.25. Find R if $i = 0.05$ is the annual effective interest rate.
5. SOLUTION. Work with periods of one quarter year. Let j be the interest per quarter. Then $(1+j)^4 = 1+i = 1.05$ so $j = .012272234$. The accumulated (future) value is $FV = 3317.25 = (R/4)s_{\overline{28}|j} = (R/4)(33.1725)$ so $R = 400$.

6. A fund earns interest at a rate equivalent to 5% per annum effective. A person makes continuous deposits to the Fund for 10 years The rate at which deposits are made is $1000+100t$ per annum at time t (in years). How much will the person have in the fund at the end of the ten year period. (Set up the proper integral but do not evaluate. The letters i, v, δ should not appear in your integral.)
6. SOLUTION:
$1.05 = 1+i = v^{-1} = e^{\delta}$ so $\delta = \ln(1.05)$ and $e^{\delta t} = 1.05^t$
$PV = \int_0^n f(t)v^t\,dt = \int_0^{10}(1000+100t)e^{-\delta t}dt$.
Thus after 10 years, the accumulated value is

$$(1.05)^{10}PV = 1.05^{10}\int_0^{10}(1000+100t)e^{-\delta t}dt = 1.05^{10}\int_0^{10}(1000+100t)(1.05^{-t})dt$$

7. A loan of 5000 with interest at 5% per annum effective, will be repaid by payments of 1000 each made at the end of each of the first, second, third, and fourth years and a larger amount sufficient to retire the loan at the end of the fifth year. Find the amount payable at the end of the fifth year.
A. 1856 B. 1213 C. 1617 D. 1315 E. 1380
7. SOLUTION:
The outstanding balance at the end of the fourth year is
$B = 5000(1.05)^4 - 1000s_{\overline{4}|} = 1767.40625$ The payment in the fifth year must be $B(1+i) = 1767.40625(1.05) = 1855.78$ Answer is A.

8. Mary purchases an increasing annuity-immediate for 50,000 that makes twenty annual payments as follows:
(i) $P, 2P, \ldots, 10P$ in years 1 through 10, and
(ii) $10P(1.05), 10P(1.05)^2, \ldots, 10P(1.05)^{10}$ in years 11 through 20.
The annual effective interest rate is 7% for the first 10 years and 5% thereafter. Calculate P.
(A) 564, (B) 574, (C) 584, (D) 594, (E) 604
8. SOLUTION:
Let $v = 1/(1+.07)$ and let $w = 1/(1+.05)$.

$$\begin{aligned}50000 &= (Pv + 2Pv^2 + \cdots + 10Pv^{10}) + v^{10}(10P(1.05)w + \cdots + 10P(1.05)^{10}w^{10})\\ &= Pv(1 + 2v + \cdots + 10v^9) + v^{10}P(10 + 10 + \cdots + 10)\end{aligned}$$

$$\begin{aligned}S &= 1 + 2v + 3v^2 + \cdots + 10v^9\\ vS &= \quad 1v + 2v^2 + \cdots + 9v^9 + 10v^{10}\\ (1-v)S &= 1 + \ v + \cdots + v^9 - 10v^{10}\end{aligned}$$

So $S = \dfrac{1-v^{10}}{(1-v)^2} - \dfrac{10v^{10}}{1-v} = \dfrac{1-v^{10}-10v^{10}+10v^{11}}{(1-v)^2} = \dfrac{1-5.59184+4.75093}{.00427985} = 37.170874$
so $50000 = P(1.07)^{-1}(37.170874) + (1.07)^{-10}P(100)$
so $P = \dfrac{50000}{34.739135 + 50.8349292} = 584.29$. C

For the last three questions, 9, 10, 11, ONLY the answer is needed. You do not need to show any work for these questions (and any work shown will not be graded.)

9. Donald takes a loan to be paid with annual payments of 500 at the end of each year for 2n years. The annual effective interest rate is 4.94%.
The sum of the interest paid in year 1 plus the interest paid in year $n+1$ is equal to 720. Calculate the amount of interest paid in year 10.

9. SOLUTION (taken from Actuarial Problems for Chapter 6.)
Both n and L are unknown
$500(1-v^{2n})+500(1-v^n)=720$ (check formulas in the amortization table).
So $-500v^{2n}-500v^n+280=0$. This is a quadratic in v^n so from the quadratic formula, $v^n=.4$ (the other root is negative).
We want the amount of interest paid in year 10, namely $500(1-v^{2n-9})$.
Also $v^9=\dfrac{1}{1.0494^9}=.64793$. Hence

$$InterestPaidInTenthYear=500(1-v^{2n-9})=500(1-(.4)^2(.64793^{-1}))=376.53.$$

10. A 30-year \$500,000 loan is paid off via the accumulation of a sinking fund. Sinking fund deposits are made at the beginning of each year. If the effective annual interest rate earned on the sinking fund is 10%, determine the amount in the sinking fund immediately prior to the 10th deposit.
10. SOLUTION.
Determine the sinking fund deposit by using a 30-year annuity-due accumulation factor (the sinking fund must accumulate to \$500,000 after 30 years). The 10th deposit will be made at time 9, so you want the sinking fund balance at time 9. Use the sinking fund deposit and a 9-year annuity-due accumulation factor to determine the balance.
$500000=D\ddot{s}_{\overline{30}|.1}$ so $D=500000/\ddot{s}_{\overline{30}|}=500000/180.943425=2763.29466$
The amount immediately prior to the tenth deposit will be the accumulated amount including the tenth deposit less the tenth deposit. i.e.

$$Ds_{\overline{10}|i=.10}-D=D(s_{\overline{10}|i=.10}-1)=2763.29466(15.93742-1)=41276.51.$$

11. Eric deposits 12 into a fund at time 0 and an additional 12 later into the same fund at time 10. The fund credits interest at an annual effective rate of i. Interest is payable annually and reinvested at an annual effective rate of $0.75i$. At time 20, the accumulated amount of the reinvested interest payments is equal to 64. Calculate i, $i>0$.
11. SOLUTION

0	$12i$	$12i$	$\ldots$	$12i$	$24i$	$24i\ldots.24i$	(Interest Payment)
|–	|–	|–…–		|–	|–	|…|	
0	1	2	$\ldots$	10	11	$12\ldots20$	(Time)

$$64=12is_{\overline{20}|.75i}+12is_{\overline{10}|.75i}=12i\frac{(1+.75i)^{20}-1}{.75i}+12i\frac{(1+.75i)^{10}-1}{.75i}=16x^2+16x-32$$

where $x=(1+.75i)^{10}$.
So $16x^2+16x-96=0$ so $x^2+x-6=0$ so $(x+3)(x-2)=0$ so $x=2$. Then $(1+.75i)^{10}=2$ so $i=.095698$